On the asymptotics of exit problem for controlled Markov diffusion processes with random jumps and vanishing diffusion terms

CONTROL SYSTEMS AND THE QUEST FOR AUTONOMY
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Outline

- Introduction
  - General objectives
- Part I - On the asymptotic estimates for exit probabilities
  - Exit probabilities
  - Connection with stochastic control problem
- Part II - Minimum exit rate problem for prescription opioid epidemic models
  - On the minimum exit rate problem
  - Connection with principal eigenvalue problem
- Further remarks

Introduction

Consider the following n-dimensional process $x^\epsilon(t)$ defined by

$$dx^\epsilon(t) = F(t, x^\epsilon(t), y^\epsilon(t))dt \tag{1}$$

and an m-dimensional diffusion process $y^\epsilon(t)$ obeying the following SDE

$$\begin{aligned} dy^\epsilon(t) =& f(t, y^\epsilon(t))dt + \sqrt{\epsilon}\sigma(t, y^\epsilon(t))dw(t), \\ & (x^\epsilon(s), y^\epsilon(s)) = (x, y), \quad t \in [s, T], \end{aligned} \tag{2}$$

where

- $(x^\epsilon(t), y^\epsilon(t))$ jointly defined an $\mathbb{R}^{(n+m)}$-valued Markov diffusion process,
- $w(t)$ is a standard Wiener process in $\mathbb{R}^m$,
- the functions F and f are uniformly Lipschitz, with bounded first derivatives,

Introduction . . .

- $\sigma(t, y)$ is an $\mathbb{R}^{m\times m}$-valued Lipschitz continuous function such that $a(t, y) = \sigma(t, y)\,\sigma^T(t, y)$ is uniformly elliptic, i.e.,

$$a_{min}|p|^2 < p \cdot a(t, y)p < a_{max}|p|^2, \quad p, y \in \mathbb{R}^m, \quad \forall t > 0,$$

  for some $a_{max} > a_{min} > 0$, and
- ϵ is a small positive number representing the level of random perturbation.

Remark (1)

Note that the small random perturbation enters only in the second system and then passes to other system. As a result, the diffusion process $(x^\epsilon(t), y^\epsilon(t))$ is degenerate, i.e., the associated backward operator is degenerate.

Introduction . . .

Here, we distinguish two general problems:

- **A direct problem**: the study of asymptotic behavior for the diffusion process $(x^{\epsilon}(t), y^{\epsilon}(t))$, as $\epsilon \to 0$, provided that some information about the deterministic coupled dynamical systems, i.e.,

$$\dot{x}^0(t) = F(t, x^0(t), y^0(t)), \qquad \dot{y}^0(t) = f(t, y^0(t))$$

  and the type of perturbation are known.
- **An indirect problem**: the study of the deterministic coupled dynamical systems, when the asymptotic behavior of the diffusion process $(x^{\epsilon}(t), y^{\epsilon}(t))$ is known.

General objectives

- To provide a framework that exploits three way connections between:[1]
  (i) boundary value problems associated with certain second order linear PDEs,
  (ii) stochastic optimal control problems, and
  (iii) probabilistic interpretation of controlled principal eigenvalue problems.
- To provide additional results for stochastically perturbed dynamical systems with randomly varying intensities.

Typical applications include: climate modeling [*Benzi et al. (1983); Berglund & Gentz (2002, 2006)*], electrical engineering [*Bobrovsky, Zakai & Zeitouni (1988); Zeitouni and Zakai (1992)*], molecular and cellular biology [*Holcman & Schuss (2015)*], mathematical finance [*Feng et al. (2010)*], and stochastic resonance [*Häggi et al. (1998); Moss (1994)*]. **General works include**: [*Berglund & Gentz (2006); Freidlin & Wentzell (1998); Olivieri & Vares (2005)*].

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[1] **G. K. Befekadu & P. J. Antsaklis**, On the asymptotic estimates for exit probabilities and minimum exit rates of diffusion processes pertaining to a chain of distributed control systems, *SIAM J. Contr. Opt.*, 53 (2015) 2297-2318.

Part I - Asymptotic estimates for exit probabilities

Let $D \subset \mathbb{R}^n$ be a bounded open domain with smooth boundary ∂D. Let τ_D^ϵ be the exit time for the process $x^\epsilon(t)$ from D

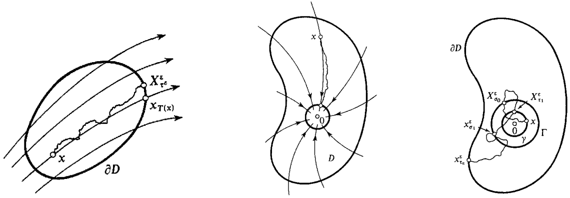
$$\tau_D^\epsilon = \inf\Big\{ t > s \,\Big|\, x^\epsilon(t) \in \partial D \Big\}.$$

For a given $T > 0$, define the exit probability as

$$q^\epsilon\big(s, x, y\big) = \mathbb{P}^\epsilon_{s,x,y}\Big\{ \tau_D^\epsilon \le T \Big\},$$

where the probability $\mathbb{P}^\epsilon_{s,x,y}$ is conditioned on $(x, y) \in D \times \mathbb{R}^m$.
Important: Note that the solution $q^\epsilon\big(s, x, y\big)$, as $\epsilon \to 0$, strongly depends on the behavior of the trajectories for the corresponding deterministic coupled dynamical systems, i.e.,

$$\dot{x}^0(t) = F(t, x^0(t), y^0(t))$$
$$\dot{y}^0(t) = f(t, y^0(t)), \quad (x^0(0), y^0(0)) = (x, y).$$



(a) (b) (c)

Exit probabilities ...

The backward operator for the process $\big(x^\epsilon(t), y^\epsilon(t)\big)$, when applied to a certain smooth function $\psi\big(s, x, y\big)$, is given by

$$\begin{aligned}\psi_s\big(s,x,y\big) + \mathcal{L}^\epsilon\psi\big(s,x,y\big) \triangleq \psi_s\big(s,x,y\big) &+ \frac{\epsilon}{2}\operatorname{tr}\Big\{a\big(s,y\big)\,\psi_{yy}\big(s,x,y\big)\Big\} \\ &+ \big\langle F\big(s,x,y\big),\, \psi_x\big(s,x,y\big)\big\rangle \\ &+ \big\langle f\big(s,y\big),\, \psi_y\big(s,x,y\big)\big\rangle, \end{aligned} \tag{3}$$

where $\mathcal{L}^\epsilon$ is a second-order elliptic operator, i.e.,

$$\mathcal{L}^\epsilon\big(\cdot\big) \triangleq \frac{\epsilon}{2}\operatorname{tr}\Big\{a\big(s,y\big)\,\nabla^2_{yy}\big(\cdot\big)\Big\} + \Big\langle F\big(s,x,y\big),\, \nabla_x\big(\cdot\big)\Big\rangle + \Big\langle f\big(s,y\big),\, \nabla_y\big(\cdot\big)\Big\rangle$$

and

$$a(s,y) = \sigma(s,y)\,\sigma^T(s,y).$$

Exit probabilities ...

Let Q be an open set given by

$$Q = (0, T) \times D \times \mathbb{R}^m.$$

Assumption (1)

(a) *The function F is a bounded $C^\infty(Q_0)$-function, with bounded first derivative, where $Q_0 = (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$. Moreover, f, σ and σ^{-1} are bounded $C^\infty\big((0, \infty) \times \mathbb{R}^m\big)$-functions, with bounded first derivatives.*

(b) *The backward operator in Eq (3) is hypoelliptic in $C^\infty(Q_0)$ (which is also related to an appropriate Hörmander condition).*

(c) *Let $n(x)$ be the outer normal vector to ∂D. Furthermore, let Γ^+ and Γ^0 denote the sets of points (t, x, y), with $x \in \partial D$, such that $\big\langle F(t, x, y),\, n(x)\big\rangle$ is positive and zero, respectively.*[2]

[2] Note that

$$\mathbb{P}^{\epsilon}_{s,x,y}\Big\{\big(\tau^{\epsilon}_D, x^{\epsilon}(\tau^{\epsilon}_D), y^{\epsilon}(\tau^{\epsilon}_D)\big) \in \Gamma^+ \cup \Gamma^0 \,\Big|\, \tau^{\epsilon}_D < \infty\Big\} = 1, \quad \forall s, x, y \in Q.$$

Exit probabilities . . .

Consider the following boundary value problem

$$\left.\begin{array}{l} \psi_s\big(s,x,y\big) + \mathcal{L}^{\epsilon}\psi\big(s,x,y\big) = 0 \quad \text{in} \quad Q = (0,T) \times D \times \mathbb{R}^m \\ \psi\big(s,x,y\big) = 1 \quad \text{on} \quad \Gamma_T^{+} \\ \psi\big(s,x,y\big) = 0 \quad \text{on} \quad \{T\} \times D \times \mathbb{R}^m \end{array}\right\} \quad (4)$$

where $\Gamma_T^{+} = \Big\{\big(s,x,y\big) \in \Gamma^{+} \,\Big|\, 0 < s \le T\Big\}$.

Then, we have the following result for the exit probability.

Proposition (1)

Suppose that the statements (a)–(c) in the above assumption (i.e., Assumption (1)) hold true. Then, the exit probability $q^{\epsilon}(s,x,y) = \mathbb{P}_{s,x,y}^{\epsilon}\big\{\tau_D^{\epsilon} \le T\big\}$ *is a smooth solution to the above boundary value problem in Eq* (4)*. Moreover, it is a continuous function on* $Q \cup \{T\} \times D \times \mathbb{R}^m$*.*

Exit probabilities ...

Proof: Involves introducing a non-degenerate diffusion process[3]

$$dx^{\epsilon,\delta}(t) = F(t, x^{\epsilon,\delta}(t), y^{\epsilon}(t))dt + \sqrt{\delta}dV(t)$$
$$dy^{\epsilon}(t) = f(t, y^{\epsilon}(t))dt + \sqrt{\epsilon}\sigma(t, y^{\epsilon}(t))dw(t),$$

with V is a standard Wiener process in $\mathbb{R}^n$ and independent to W. Then, using the following statements

$$\left.\begin{array}{ll} (i) & \sup_{s \le r \le T} \left| x^{\epsilon,\delta}(r) - x^{\epsilon}(r) \right| \to 0 \\ (ii) & \tau_D^{\epsilon,\delta} \to \tau_D^{\epsilon} \\ (iii) & x^{\epsilon,\delta}(\tau_D^{\epsilon,\delta}) \to x^{\epsilon}(\tau_D^{\epsilon}) \end{array}\right\}, \quad \text{as} \quad \delta \to 0, \quad \mathbb{P} - \textit{almost surely.}$$

and the hypoellipticity assumption. We can relate the exit probability of the process $(x^{\epsilon,\delta}(t), y^{\epsilon}(t))$ with the boundary value problem in Eq (4).

[3] **G. K. Befekadu & P. J. Antsaklis**, On the asymptotic estimates for exit probabilities and minimum exit rates of diffusion processes pertaining to a chain of distributed control systems, SIAM J. Contr. Opt., vol. 53 (4), pp. 2297–2318, 2015.

Connection with stochastic control problems

Consider the following boundary value problem

$$\left.\begin{array}{c} g_s^\epsilon + \frac{\epsilon}{2}\,\mathrm{tr}\big\{a\, g_{yy}^\epsilon\big\} + \langle F,\, g_x^\epsilon\rangle + \langle f,\, g_y^\epsilon\rangle = 0 \quad \text{in} \quad Q \\ g^\epsilon = \mathbb{E}_{s,x,y}^\epsilon\Big\{\exp\Big(-\frac{1}{\epsilon}\Phi\Big)\Big\} \quad \text{on} \quad \partial^* Q \end{array}\right\} \qquad (5)$$

where $\Phi(s,x,y)$ is bounded, nonnegative Lipschitz such that

$$\Phi(s,x,y) = 0, \quad \forall (s,x,y) \in \Gamma_T^+.$$

Introduce the following **logarithm transformation**

$$J^\epsilon(s,x,y) = -\epsilon \log g_s^\epsilon(s,x,y).$$

Then, $J^\epsilon(s,x,y)$ satisfies the following HJB equation

$$0 = J_s^\epsilon + \frac{\epsilon}{2}\,\mathrm{tr}\Big\{a\, J_{x^{\epsilon,1} x^{\epsilon,1}}^{\epsilon,\ell}\Big\} + F^T \cdot J_x^\epsilon + H\big(s,y,J_y^\epsilon\big) \ \text{ in } \ Q, \qquad (6)$$

where

$$H\big(s,y,J_y^\epsilon\big) = f^T(s,y)\cdot J_y^\epsilon - \frac{1}{2} J_y^{\epsilon\,T} \cdot a(s,y) J_y^\epsilon.$$

Connection with stochastic control problems . . .

Then, we see that $J^{\epsilon}\big(s,x,y\big)$ is a solution for the DP equation in Eq (6), which is associated to the following stochastic control problem

$$J^{\epsilon}\big(s,x,y\big) = \inf_{\hat{u}\in \hat{U}_{(s,x,y)}} \mathbb{E}^{\epsilon}_{s,x,y}\Bigg\{\int_s^{\theta} L\big(s, y^{\epsilon}(t), \hat{u}(t)\big)dt \\ + \Phi\big(\theta, x^{\epsilon}(\theta), y^{\epsilon}(\theta)\big)\Bigg\}$$

with the SDE

$$\left.\begin{aligned} dx^{\epsilon}(t) &= F\big(t, x^{\epsilon}(t), y^{\epsilon}(t)\big)dt \\ dy^{\epsilon}(t) &= \hat{u}(t)dt + \sqrt{\epsilon}\,\sigma\big(t, y^{\epsilon}(t)\big)dW(t) \\ (x^{\epsilon}(s), y^{\epsilon}(s)) &= (x,y), \quad s \le t \le T \end{aligned}\right\}$$

where $\hat{U}_{(s,x,y)}$ is a class of (non-anticipatory) continuous functions for which $\theta \le T$ and $\big(\theta, x^{\epsilon}(\theta), y^{\epsilon}(\theta)\big) \in \Gamma_T^{+}$.

Connection with stochastic control problems ...

Define

$$I^{\epsilon}\Big(\big(s,x,y\big);\,\partial D\Big) = -\epsilon\,\log \mathbb{P}^{\epsilon}_{s,x,y}\Big\{x^{\epsilon}(\theta)\in\partial D\Big\}$$
$$\triangleq -\epsilon\,\log q^{\epsilon}\big(s,x,y\big),$$

where θ (or $\theta = \tau_D^{\epsilon}\wedge T$) is the exit time of $x^{\epsilon}(t)$ from D.
Then, we have

$$I^{\epsilon}\big(s,x,y\big)\to I\big(s,x,y\big) \quad \text{as} \quad \epsilon\to 0,$$

uniformly for all $\big(s,x,y\big)$ in any compact subset Q.[4]

Further Remark: Such an asymptotic estimate is obtained based on a precise interpretation of the exit probability as a value function for a family of stochastic control problems.

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[4]**Important**: The process $\big\{x^{\epsilon}(t)\,:\,\epsilon>0\big\}$ obeys a Large deviations principle with the rate function $I^{\epsilon}\big(s,x,y\big)$, i.e., a logarithmic asymptotic for the exit position $\epsilon\to 0$,

$$\mathbb{P}^{\epsilon}_{s,x,y}\Big\{x^{\epsilon}(\theta)\in\partial D\Big\}\asymp \exp\Big\{-\tfrac{1}{\epsilon}I^{\epsilon}\big(s,x,y\big)\Big\} \quad \text{as} \quad \epsilon\to 0.$$

Part II - Minimum exit rate problem for prescription opioid epidemic models

- Recently, the United States is experiencing an epidemic of drug overdose deaths (e.g., Warner et al. *NCHS Data Brief, No* **81**, 2011; Buchanich et al. *Prev Med* **89***:317–323, 2016*; Dart et al. *N Engl J Med*, **372***:241–248, 2015*).
- In part, the opioid epidemic has been attributed due to **inappropriate physician prescribing practices** or **higher prescribing rates**, which led to an increase in substance abuse and overdose deaths (see Figure 2 below).

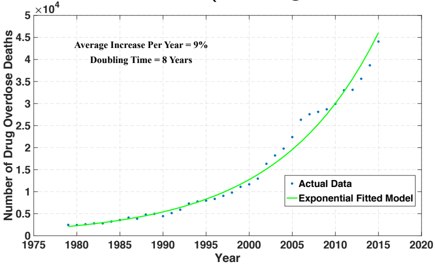
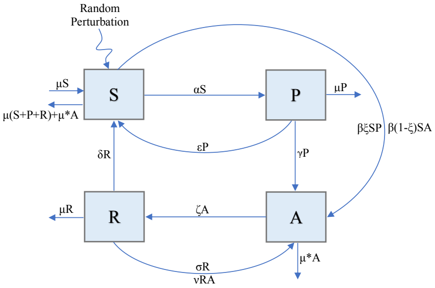


Figure 1: Number of drug overdose deaths per year in US from 1979 to 2015 (**Source**: MOIRA Death Record Repository, University of Pittsburgh).

Part II - Minimum exit rate problem . . .

Consider the following prescription opioid epidemic dynamical model



For a normalized population, denote the susceptible **S**, addicted **A** and recovered **R** by $X_1(t)$, $X_2(t)$ and $X_3(t)$, respectively. Then, we can be written the opioid epidemics as follows

$$d\mathbf{X}(t) = \mathbf{F}(\mathbf{X}(t))dt + \sqrt{\epsilon}BdW(t), \tag{7}$$

Minimum exit rate problem . . .

Consider the following controlled-version of SDE

$$d\mathbf{X}_{0,\mathbf{x}}^{u,\epsilon}(t) = \left[\mathbf{F}(\mathbf{X}_{0,\mathbf{x}}^{u,\epsilon}(t)) + \tilde{B}u(t)\right]dt + \sqrt{\epsilon}BdW(t),\ \mathbf{X}_{0,\mathbf{x}}^{u,\epsilon}(0) = \mathbf{x},$$

where u is a progressively measurable process such that

$$\mathbb{E}\int_0^\infty |u(t)|^2 dt < \infty.$$

Let τ_D^ϵ be the exit time for $\mathbf{X}_{0,\mathbf{x}}^{u,\epsilon}(t)$ from the domain D, with smooth boundary ∂D, i.e.,

$$\tau_D^\epsilon = \inf\left\{t > 0 \,\middle|\, \mathbf{X}_{0,\mathbf{x}}^{u,\epsilon}(t) \in \partial D\right\}. \tag{8}$$

Connection with principal eigenvalue problem

Typical problem: Involves maximizing the mean exit time, which is equivalent to minimizing the principal eigenvalue λ_u^ϵ

$$\lambda_u^\epsilon = -\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\mathbf{x}}^{u,\epsilon}\bigl\{\tau_D^\epsilon > t\bigr\},$$

with respect to a certain class of admissible controls.

Connection with controlled-eigenvalue problem

$$\left.\begin{array}{c} -\mathcal{L}_u^\epsilon \psi_u(\mathbf{x}) = \lambda_u^\epsilon \psi_u(\mathbf{x}) \quad \text{in} \quad D \\ \psi_u(\mathbf{x}) = 0 \quad \text{on} \quad \partial D \end{array}\right\} \tag{9}$$

where the admissible optimal control u^* can be determined by any measurable selector of

$$\arg\max\bigl\{\mathcal{L}_u^\epsilon \psi(\mathbf{x}, \cdot)\bigr\}, \quad \mathbf{x} \in D.$$

Simulation results

Table: Literature based parameter values

Parameter	Numerical value	Parameter	Numerical value
α	0.15	δ	0.1
ε	0.8 - 8	ν	0.2
β	0.0036	σ	0.7
ξ	0.74	μ	0.007288
γ	0.00744	μ^*	0.01155
ζ	0.2 - 2	-	-

For an **addiction-free equilibrium**

$$X_1^* = \frac{\varepsilon + \mu}{\alpha + \varepsilon + \mu}, \quad X_2^* = 0, \quad X_3^* = 0 \text{ and } Z^* = \frac{\alpha}{\alpha + \varepsilon + \mu}.$$

Domain of interest,

$$D \subset \left\{ \begin{array}{cc} X_i(t) \geq 0, & i = 1,2,3 \\ X_1(t) + X_2(t) + X_3(t) \leq 1, & \forall t \geq 0 \end{array} \right\},$$

with smooth boundary ∂D.

Simulation results . . .

The Jacobian matrix $J(\mathbf{X})$ is given by

$$J(\mathbf{X})\big|_{\mathbf{X}=\mathbf{X}^*} = \left[\frac{\partial f_i(\mathbf{X})}{\partial X_j}\right]_{ij}\Bigg|_{\mathbf{X}=\mathbf{X}^*}, \qquad i,j \in \{1,2,3\}$$

$$= \begin{bmatrix} -(\alpha+\varepsilon+\mu) & \dfrac{\beta(\varepsilon+\mu)}{\alpha+\varepsilon+\mu} - (\varepsilon+\mu) + \mu^* & \delta - \varepsilon \\ 0 & \dfrac{\beta(\varepsilon+\mu)}{\alpha+\varepsilon+\mu} - (\zeta+\mu^*) & \sigma \\ 0 & \zeta & -(\delta+\sigma+\mu) \end{bmatrix}$$

The corresponding eigenvalues for $J(\mathbf{X}^*)$, that is, $\big\{-3.1573, -0.0323, -1.0331\big\}$, are all strictly negative and, hence, the addiction-free equilibrium is asymptotically stable, with a **reproduction number** $\mathcal{R}_o = 0.0766$.

Simulation results . . .

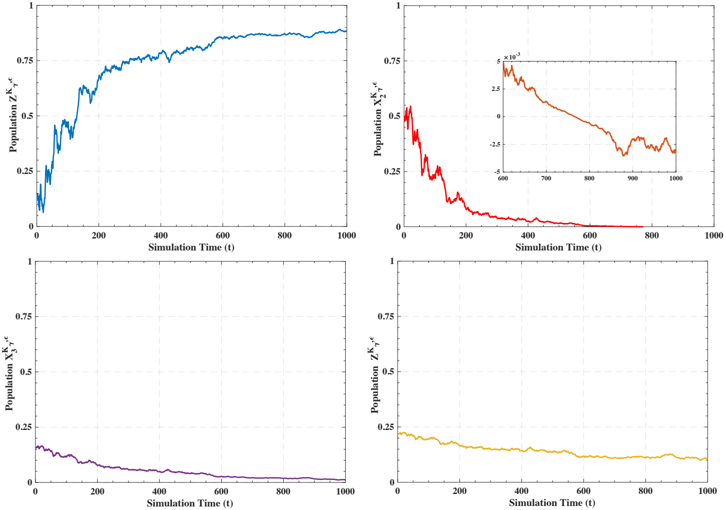


Figure: Population trajectory for small randomly perturbing noise, with an intensity level of $\epsilon = 0.01$.

Some relevant publications

G. K. Befekadu & P. J. Antsaklis, *On the asymptotic estimates for exit probabilities and minimum exit rates of diffusion processes pertaining to a chain of distributed control systems*, SIAM J. Control & Opt., vol. 53 (4), pp. 2297–2318, 2015.

G. K. Befekadu & P. J. Antsaklis, *On the problem of minimum asymptotic exit rate for stochastically perturbed multi-channel dynamical systems*, IEEE Trans. Automat. Contr., vol. 60 (12), pp. 3391–3395, 2015.

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G. K. Befekadu, *Large deviation principle for dynamical systems coupled with diffusion-transmutation processes*, Accepted to Syst. & Contr. Lett., 2018.

G. K. Befekadu, *On the controlled eigenvalue problem for stochastically perturbed multi-channel systems*, Preprint arXiv:1501.01256 [math.OC], 9 pages, October 2016.

G.K. Befekadu & Q. Zhu, *Optimal control of diffusion processes pertaining to an opioid epidemic dynamical model with random perturbations*, Submitted to the J. Math. Biology.