# EXERCISES — FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES

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Exercises marked with a † require either some background or learning the statement of a hard theorem (e.g. Weil II). Exercises with a $*$ are difficult in a more background-independent way.

## 1. Restrictions on fundamental groups and their representations

(1) Let $H$ be the Heisenberg group, i.e. the (multiplicative) group of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that $\dim_{\mathbb{C}} W^{\mathrm{ab}} \otimes \mathbb{C}$ is even for every finite index subgroupp $W \subset H$.

(b) Show that

$$\smile : \bigwedge^2 H^1(H, \mathbb{C}) \to H^2(H, \mathbb{C})$$

is identically zero. Deduce that $H$ is not the fundamental group of a smooth projective complex variety.

(2) Let $S$ be a smooth complex variety, and $f : X \to S$ a smooth projective morphism. Let

$$\rho : \pi_1(S, s) \to GL_1(\mathbb{C}) = \mathbb{C}^\times$$

be a (one-dimensional) representation appearing as a subquotient of the natural representation of $\pi_1(S, s)$ on $H^i(X_s, \mathbb{C})$. Show that $\rho$ has finite image. (This can be done via Hodge theory, or via "spreading out" as in Grothendieck's proof of the local monodromy theorem. If you choose the latter route, you will use the Weil conjectures as an ingredient.)

(3) † Let $S$ be a smooth complex variety and $f : X \to S$ a smooth projective morphism. Show that the natural representation of $\pi_1(S, s)$ on $H^i(X_s, \mathbb{C})$ is semisimple. (Hint: After spreading out, the corresponding local system will be pure of weight $i$, in the sense of Weil II. Let $V, W$ be summands and apply Weil II to $\mathrm{Ext}^1_S(V, W)$.)

## 2. Rigid local systems

Let $\overline{X}$ be a compact Riemann surface, and $D \subset \overline{X}$ a finite collection of points; set $X = \overline{X} \setminus D$ and $j : X \hookrightarrow \overline{X}$. Let $\mathbb{V}$ be a complex representation of $\pi_1(X)$, i.e. a local system on $X$. Let $\mathrm{ad}(\mathbb{V})$ be the local system of trace zero endomorphisms of $\mathbb{V}$.

(1) (a) Construct a bijection between the space of first-order deformations of $\mathbb{V}$ (i.e. representations of $\pi_1(X)$ into $GL_n(\mathbb{C}[\epsilon]/\epsilon^2)$ which reduce to $\mathbb{V}$ mod $\epsilon$, up to conjugation by elements of
$$\ker(GL_n(\mathbb{C}[\epsilon]/\epsilon^2) \to GL_n(\mathbb{C})))$$
and $H^1(X, \mathrm{ad}(\mathbb{V}))$.

(b) Construct a bijection between the space of first-order deformations of $\mathbb{V}$ with fixed local monodromy about the points of $D$ and
$$H^1(\overline{X}, j_*\mathrm{ad}(\mathbb{V})).$$

(c) Give a formula for $\dim H^1(\overline{X}, j_*\mathrm{ad}(\mathbb{V}))$ in terms of the genus of $X$ and the local monodromies of $\mathbb{V}$, for $\mathbb{V}$ irreducible.

(2) Suppose $\mathbb{V}$ is an irreducible, physically rigid, local system on $X$. Show that
$$H^1(\overline{X}, j_*\mathrm{ad}(\mathbb{V})) = 0.$$
What can you say about the genus of $X$?

(3) Show that all irreducible rank 2 local systems on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ are rigid.

(4) Suppose $\mathbb{V}, \mathbb{W}$ are irreducible, rigid local systems on $X$ with the same local monodromies about the points of $D$. Show that $\mathbb{V} \simeq \mathbb{W}$.

(5) *† Suppose now that $\overline{X} \simeq \mathbb{P}^1$ and that $\infty \in D$. Recall the definition of middle convolution; we have a diagram as follows:

$$\begin{tikzcd}
 & X \times X \setminus \Delta \arrow[rr, hook, "i"] \arrow[dl, "\pi_1"'] \arrow[dr, "\alpha"] & & \mathbb{P}^1 \times X \arrow[dr, "\pi_2"] & \\
X & & \mathbb{A}^1 \setminus \{0\} & & X
\end{tikzcd}$$

where the $\pi_i$ are the two projections, $\Delta$ is the diagonal, $i$ is the natural inclusion, and $\alpha$ is the map $(x, y) \mapsto x - y$. Let $\chi_\lambda$ be the local system of rank 1 on $\mathbb{A}^1 \setminus \{0\}$ with local monodromy $\lambda$ about 0. For $\mathbb{V}$ a local system on $X$, we define
$$MC_\lambda(\mathbb{V}) := R^1\pi_{2*}i_*(\pi_1^*\mathbb{V} \otimes \alpha^*\chi_\lambda).$$
Show that if $\lambda \neq 1$, then there is a natural isomorphism
$$MC_{-\lambda}(MC_\lambda(\mathbb{V})) \simeq \mathbb{V}.$$
(Hint: Katz does this using the $\ell$-adic Fourier transform, but one can do it directly. I would suggest starting by ignoring the $i_*$ and trying to write the various functors involved as "Fourier-Mukai kernels.")

(6) Show that if $\lambda \neq 1$, then $MC_\lambda$ sends irreducible local systems to irreducible local systems.
(7) Compute the local monodromy of $MC_\lambda(\mathbb{V})$ in terms of $\lambda$ and the local monodromy of $\mathbb{V}$.
(8) Compute the dimension of $\mathrm{MC}_\lambda(\mathbb{V})$ in terms of the local monodromy of $\mathbb{V}$ about $D$.
(9) * Show that if $\mathbb{V}$ is irreducible, of rank at least two, and rigid, there exists a rank one local system $\mathbb{L}$ on $X$ and a constant $\lambda \neq 1$ such that

$$\dim MC_\lambda(\mathbb{V} \otimes \mathbb{L}) < \dim \mathbb{V}.$$

Deduce Katz's classification of rigid local systems on $X$. Conclude that if a rigid local system has quasi-unipotent local monodromy about $D$, then it appears in the cohomology of a family of smooth projective varieties over $X$.
(10) Suppose $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and that $\mathbb{V}$ has rank 2 and quasi-unipotent local monodromy. Write down an explicit family of varieties in whose cohomology $\mathbb{V}$ appears. (This was known to Riemann.)

## 3. Mapping class group dynamics and surface group representations

Let $\Sigma_{g,n}$ be a compact orientable surface of genus $g$ with $n$ punctures. Let $\mathrm{Mod}_{g,n}$ be the mapping class group of $\Sigma_{g,n}$.

(1) Let $g > 0$. Show that the only finite orbits of the action of the mapping class group $\mathrm{Mod}_{g,n}$ on $\mathrm{Hom}(\pi_1(\Sigma_{g,n}), \mathbb{C}^\times)$ are the homomorphisms with finite image. (You can and should do this explicitly—maybe start with $n = 0$, in which case it boils down to understanding the action of $\mathrm{Sp}_{2g}$ on $(\mathbb{C}^\times)^{2g}$, but it could also be fun to try to do it with Hodge theory.)
(2) * Construct an irreducible representation of $\pi_1(\Sigma_{g,n})$ with infinite image, which is fixed (up to conjugacy) by $\mathrm{Mod}_{g,n}$. Can you write such a representation down explicitly?
(3) * Construct a non-unitary representation $\pi_1(\Sigma_{0,n}) \to SL_2(\mathbb{C})$ whose orbit in

$$\mathrm{Hom}(\pi_1(\Sigma_{0,n}), SL_2(\mathbb{C}))/\sim$$

under $\mathrm{Mod}_{0,n}$ has compact closure.
(4) * Construct an irreducible representation

$$\pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$

with infinite image such that every simple closed loop in $\Sigma_{g,n}$ maps to a quasi-unipotent matrix. Can you construct such a representation so that every simple closed loops maps to a finite order matrix? (The latter was done by Koberda-Santharoubane via TQFT methods. I don't know an algebro-geometric construction.)