EXERCISES – DISTRIBUTION OF CLASS GROUPS AND THEIR NONABELIAN GENERALIZATIONS

YUAN LIU

Problem 1: Class group of quadratic fields.
Let K be a quadratic extension of $\mathbb{Q}$.

(1) Prove: if L is an unramified abelian extension of K, then $L/\mathbb{Q}$ is Galois.
(2) If L is an unramified abelian extension of K, then can you determine the Galois group of $L/\mathbb{Q}$?
(3) Write K as $\mathbb{Q}(\sqrt{d})$ with $d = (-1)^r p_1 p_2 \cdots p_s$, where $r = 0$ or 1 and p_i are distinct prime numbers. Determine the maximal extension L of K such that L/K is unramified and $\mathrm{Gal}(L/\mathbb{Q})$ is a 2-elementary abelian group.

Problem 2: For a global field K and a set S of primes of K, let K_S denote the maximal extension of K that is unramified outside S. Similarly, given another set T of primes of K, let K_S^T be the maximal extension in K_S such that K_S^T/K is split completely at all primes in T.

(1) Prove that K_S and K_S^T are Galois over K. Then denote $G_S(K) := \mathrm{Gal}(K_S/K)$ and $G_S^T(K) := \mathrm{Gal}(K_S^T/K)$.
(2) What is $G_\emptyset(\mathbb{F}_q(t))$? What is $G_\emptyset^{\{\infty\}}(\mathbb{F}_q(t))$? What is $G_\emptyset(\mathbb{Q})$?
(3) When K is a number field, let ∞ denote the set of archimedean primes of K. Show that $G_\emptyset^\infty(K) = G_\emptyset(K)$.

Problem 3: Use the Grothendieck-Lefschetz trace formula to compute $\#\mathbb{P}^1(\mathbb{F}_q)$ and $\#\mathbb{A}^1(\mathbb{F}_q)$.

Problem 4: Let G be a finite p-group or a finitely generated pro-p group.

(1) Prove that if M is a maximal subgroup of G, then $G/M \simeq \mathbb{Z}/p\mathbb{Z}$.
(2) The Frattini subgroup $\Phi(G)$ of G is defined to by the intersection of all maximal subgroups of G. Prove that $\Phi(G)$ is a normal subgroup of G. Let π denote the surjection $G \twoheadrightarrow G/\Phi(G)$. Prove that a set $\{g_1, \dots, g_d\} \in G$ generates G if and only if $\{\pi(g_1), \dots, \pi(g_d)\}$ generates $G/\Phi(G)$.
(3) Prove that $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ is the minimal number of generators of G.
(4) Furthermore, assume G is finitely presented, and assume $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p) = d$. Let F_d denote the free pro-p group on d generators. Then there exists a surjection $\rho : F_d \twoheadrightarrow G$. The minimal number of relators of G is the smallest integer r so that there exists a set $\{x_1, \dots, x_r\} \in \ker\rho$ satisfying that $\ker\rho$ is the smallest normal subgroup of F_d containing $\{x_1, \dots, x_r\}$. Prove that $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p) = r$.

Problem 5: Use the estimation of $\dim_{\mathbb{F}_p} H^i(G_\varnothing(K), \mathbb{F}_p)$ for $i = 1, 2$ [NSW08, Theorem 10.7.10] and the Golod-Shafarevich Theorem [NSW08, Theorem 3.9.7] to find a number field K and a prime p such that the p-class tower group of K is an infinite pro-p group.

Problem 6: Let $x_1, \ldots, x_n$ be random elements in $\mathbb{Z}_p^{\oplus n}$ with respect to the Haar measure. Prove the following formulas:

$$\lim_{n\to\infty} \operatorname{Prob}\left(\mathbb{Z}_p^{\oplus n}/[x_1, \ldots, x_n] \simeq H\right) = \frac{1}{|\operatorname{Aut}(H)|} \prod_{i=1}^{\infty} \left(1 - \frac{1}{p^i}\right),$$

$$\lim_{n\to\infty} \mathbb{E}\left(\# \operatorname{Sur}(\mathbb{Z}_p^{\oplus n}/[x_1, \ldots, x_n] \to H)\right) = 1.$$

References

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, Second, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR2392026

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W Green St, Urbana, IL 61801 USA

Email address: yyyliu@illinois.edu