Discrete Topological Invariants of Quasi-Projective Varieties: A Quick Trip

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> *To understand coverings of X ramified along Z.*

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Then,

$$U \cap H \hookrightarrow U,$$

is an $(m-1)$-equivalence.

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- If codim $Z > 1$, then $\pi_1(U) = \pi_1(X)$.

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- $\pi_1(U^m) = \ldots = \pi_1(U^2)$.

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- If X is smooth and $Z = \emptyset$, then condition on H is trivial.
- The *genericity* condition is important.

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Example

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H_1 NOT transversal to $\mathcal{A}$, since it contains A_{03},

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$$\begin{array}{rcl} \mathbb{C}^* \times \mathbb{C}^2 & \to & \mathbb{C}^2 \\ (t, (x, y)) & \mapsto & t * (x, y) = (t^q x, t^p y) \end{array}$$

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$$\mathbb{S}_{p,q} := \{(x, y) \in \mathbb{C}^2 \mid \|x\|_p + \|y\|_q = 1\}$$

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$$\begin{array}{rrcl} F: & I \times U & \to & U \\ & (t,(x,y)) & \mapsto & \mu * (x,y) := (\mu^q x, \mu^p y), \end{array}$$

where

$$\mu(t,x,y) := (1-t) + \frac{t}{\sqrt[pq]{\|x\|_p + \|y\|_q}}.$$

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Note that $\mu(0,x,y)=1$ and $\mu(1,x,y)=\frac{1}{\sqrt[pq]{\|x\|_p+\|y\|_q}}$. Hence $F(0,x,y)=(x,y)$ and $F(1,x,y)=\mu(1,x,y)*(x,y)$, where

$$\|\mu(1,x,y)^q x\|_p+\|\mu(1,x,y)^p y\|_q=1.$$

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$$\pi_1(U) = \pi_1(\mathbb{S}^3 \setminus T_{rp_1, rq_1})$$

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  $$\pi_1(U) = \langle m_1, \ldots, m_r, \alpha, \beta, \ell : \prod m_i = \alpha^a \beta^b, [\ell, m_i] = 1, \ell = \alpha^{p_1} = \beta^{q_1} \rangle$$

  where $p = rp_1$, $q = rq_1$, $\gcd(p_1, q_1) = 1$, and $ap_1 + bq_1 = 1$.

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- $U = \mathbb{C}^2 \setminus \mathcal{C}$ where $\mathcal{C} = \{(x,y) \in \mathbb{C}^2 \mid y^q = x^p\}$.

  $$\pi_1(U) = \langle m_1, \ldots, m_r, \alpha, \beta, \ell : \prod m_i = \alpha^a \beta^b, [\ell, m_i] = 1, \ell = \alpha^{p_1} = \beta^{q_1} \rangle$$

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  - If $p = q = r$, then $\pi_1(U) = \langle m_1, m_2, \ldots, m_r : [\prod_{i=1}^r m_i, m_j] = 1 \rangle$.

Homology

Proposition

Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_r \subset \mathbb{P}^2$ *irreducible decomposition of a plane curve. Then*

$$H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^r \times \mathbb{Z}/d\mathbb{Z},$$

where $d := \gcd(d_0, \ldots, d_r)$, $d_i := \deg(\mathcal{C}_i)$.

Milnor Fiber

Assume $\mathcal{C} = V(f) \subset \mathbb{P}^2$ irreducible, f homogeneous polynomial of degree d in $\mathbb{C}[x, y, z]$.

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$$\begin{array}{rccc} h: & F & \to & F \\ & (x, y, z) & \mapsto & \xi_d(x, y, z) \end{array}$$

where $\xi_d^d = 1$ primitive.

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$$\begin{array}{rcc} F & \xrightarrow{j} & U \\ p = (x, y, z) & \mapsto & [x : y : z] \end{array}$$

and $j(p_1) = j(p_2)$ iff $p_2 = h^k(p_1)$ for some $k = 0, ..., d - 1$.

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Hence,

$$1 \to \pi_1(F) \to \pi_1(U) \to \mathbb{Z}/d\mathbb{Z} = H_1(U) \to 0$$

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Corollary

If $\mathcal{C} \subset \mathbb{P}^2$ irreducible, then $\pi_1(F) = [\pi_1(U), \pi_1(U)]$.

Meridians

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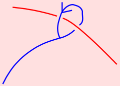

Meridians

Lemma (Generation by Meridians)

If X is simply connected, then $\pi_1(U)$ is generated by meridians around the irreducible components of $\mathcal{C}$.

Meridians

Lemma (Conjugation class of Meridians)

If γ_1, γ_2 are meridians around the same irreducible component in Z, then γ_1 and γ_2 are conjugated in $\pi_1(U)$.

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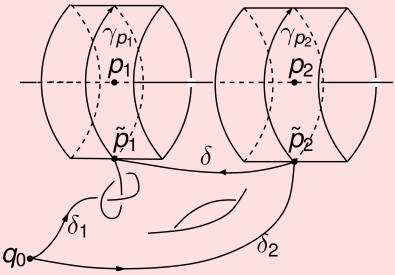

Figure: Conjugate meridians

Meridians

Lemma (Component deletion)

If γ_1 meridian around the irreducible component Z_1 and $Z = Z_1 \cup Z_2$ ($Z_1 \not\subset Z_2$), $U := X \setminus Z$, $U_2 := X \setminus Z_2$, then

$$\begin{array}{ccccccccc} 1 & \to & \langle \gamma_1 \rangle & \to & \pi_1(U) & \xrightarrow{j_*} & \pi_1(U_2) & \to & 1 \\ & & & & \gamma & \mapsto & \begin{cases} 1 & \text{if } \gamma \text{ meridian around } Z_1 \\ \gamma & \text{otherwise} \end{cases} & & \end{array}$$

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Exercise

If $\mathcal{C}$ has a locally irreducible singular point of multiplicity $d-1$, then $\pi_1(U) = \mathbb{Z}/d\mathbb{Z}$.

Local Fundamental Group

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$p \in \mathcal{C} \subset \mathbb{P}^2$.

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Study $(\mathbb{B}^4_\varepsilon, \mathcal{C}, p)$

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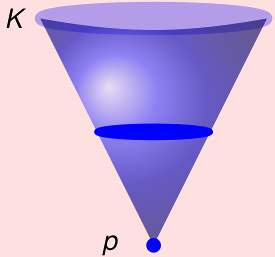

Figure: $(\mathcal{C}, p)$ at a smooth point.

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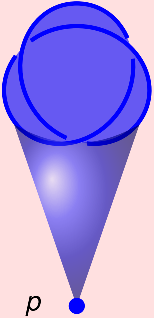

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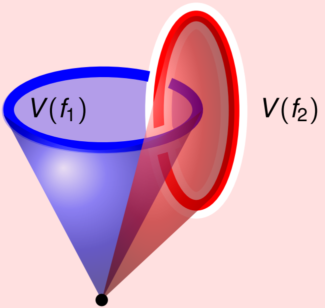

Figure: $(\mathcal{C}, p)$ at a singular point of type $f_1 f_2 = (x-y)(x+y) = x^2 - y^2 = 0$

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Definition

The invariant $\pi_1^{\text{loc}}(\mathcal{C}, p) := \pi_1(\mathbb{S}^3_\varepsilon(p) \setminus K)$ is called the *local fundamental group of* $\mathcal{C}$ *at* p.

Calculating local fundamental groups

$$K = \mathbb{S}^3 \cap \mathcal{C}$$

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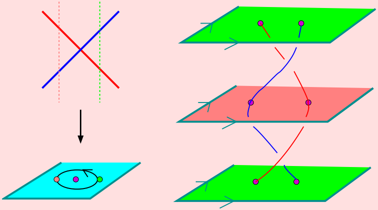

Figure: Link of a nodal point

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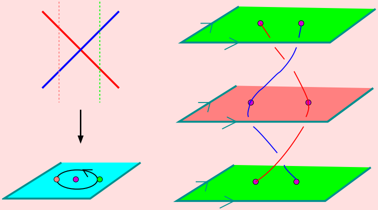

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$$\pi_1\left(\left(\mathbb{D}^2_{\delta_2} \setminus \{-\varepsilon, \varepsilon\}\right) \times [0,1]/\, g\right)$$

Calculating local fundamental groups

$$\sigma(\gamma_i) = \begin{cases} \gamma_2 & \text{if } i = 1 \\ \gamma_2\gamma_1\gamma_2^{-1} & \text{if } i = 2 \end{cases}$$

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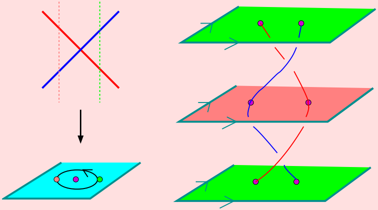

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$$\pi_1 \left(\left(\mathbb{D}^2_{\delta_2} \setminus \{-\varepsilon, \varepsilon\} \right) \times [0,1] / g \right)$$

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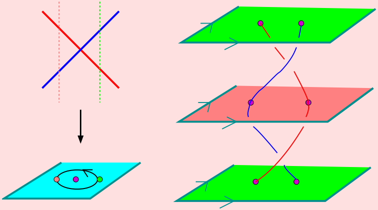

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$$\pi_1\left(\left(\mathbb{D}^2_{\delta_2} \setminus \{-\varepsilon, \varepsilon\}\right) \times [0,1]/g\right) \quad = \quad \langle \gamma_1, \gamma_2 : \gamma_i = \sigma^2(\gamma_i) \rangle$$

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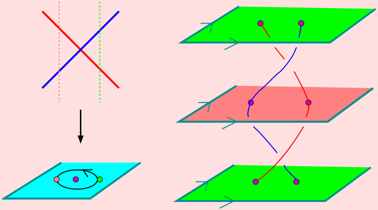

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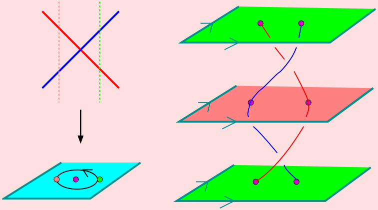

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$$\begin{aligned} \pi_1\left(\left(\mathbb{D}^2_{\delta_2} \setminus \{-\varepsilon, \varepsilon\} \right) \times [0,1] \big/ g \right) &= \langle \gamma_1, \gamma_2 : \gamma_i = \sigma^2(\gamma_i) \rangle \\ = \langle \gamma_1, \gamma_2 : \gamma_1 = \gamma_2 \gamma_1 \gamma_2^{-1} \rangle &\cong \mathbb{Z}^2. \end{aligned}$$

Calculating local fundamental groups

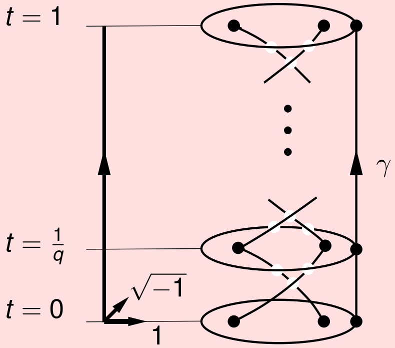

Figure: Link of a singular point of type $\{y^2 - x^q = 0\}$

Calculating local fundamental groups

In general,

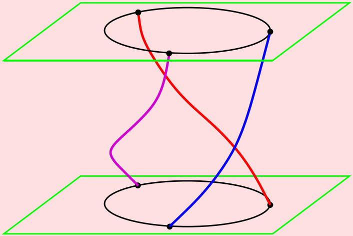

Figure: Link of a singular point of type $y^p - x^q$, $p \leq q$.

$$\pi_1\left(\left(\mathbb{D}^2_{\delta_2} \setminus \{\varepsilon\xi_p^k : k = 0, \ldots, p-1\}\right) \times [0,1]/\, g\right) = \langle \gamma_1, \ldots, \gamma_p : \gamma_i = \beta^q(\gamma_i) \rangle$$

for $\beta = \sigma_1\sigma_2 \ldots \sigma_{p-1}$.

Calculating local fundamental groups

Exercise

If $p = 2$, then

$$\pi_1(\mathbb{S}^3 \setminus K) \;=\; \langle \;\; \gamma_1, \gamma_2 : \overbrace{\gamma_1\gamma_2 \ldots}^{q} \;=\; \overbrace{\gamma_2\gamma_1 \ldots}^{q} \;\; \rangle$$

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Corollary

Let $\mathcal{C}_3 = \{[x : y : z] \in \mathbb{P}^2 \mid y^2 z = x^3\}$ *cuspidal projective cubic,* $L = \{z = 0\}$. *Then*

$$\pi_1(\mathbb{P}^2 \setminus (\mathcal{C}_3 \cup L)) \cong \langle \gamma_1, \gamma_2 : \gamma_1\gamma_2\gamma_1 = \gamma_2\gamma_1\gamma_2 \rangle.$$

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Corollary

Let $\mathcal{C}_d = \{[x : y : z] \in \mathbb{P}^2 \mid y^{d-1}z = x^d\}$, $L = \{z = 0\}$. *Then*

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Maximal Cuspidal Rational Curves

Example (Zariski '36 [85])

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- From Plücker formula (Hessian), $\kappa^* = 3d(d-2) - 6\delta = 3(d-2)$.

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Example (Zariski '36 [85])

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- If $t \mapsto [\varphi_0 : \varphi_1 : \varphi_2]$ is a parametrization of $\mathcal{C}_d$, then $\mathrm{Disc}_t(x_0\varphi_0 + x_1\varphi_1 + x_2\varphi_2)$ is the equation of $\mathcal{C}^*$.

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- This is a hyperplane section of $\Delta := \mathrm{Disc}_t(\sum_{i=0}^{d} x_i t^i)$!!!

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Using Lefschetz hyperplane section (in a reverse way!)

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_d^*) \cong \pi_1(\mathbb{P}^d \setminus \Delta) = \pi_1(X \setminus \Delta_X) \cong \mathbb{B}_d(\mathbb{P}^1).$$

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For $d = 3$, $\mathcal{C}_3^*$ is a tricuspidal quartic and

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_3^*) = \mathbb{B}_3(\mathbb{P}^1) = \langle x, y : xyx = yxy, xy^2x = 1 \rangle$$

Zariski's Conjecture

Theorem (Zariski '36 [85], Deligne '81 [25], Fulton '80 [34], Nori '83 [61], Zariski–Harris '86 [41])

The fundamental group of a nodal curve $\mathcal{C} \subset \mathbb{P}^2$ is abelian. In particular,

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = H_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}^{r-1} \times \mathbb{Z}/d\mathbb{Z}.$$

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Let $\mathcal{C} \subset X$ an irreducible curve in a smooth manifold X. Consider $\pi : \hat{X} \to X$ a resolution of $\mathcal{C}$ so that $\tilde{\mathcal{C}} \cup E$ is normal crossing.

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$$\pi_1(X \setminus \mathcal{C}) \to \pi_1(X)$$

is a central extension.

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Example

The projective complement of a cuspidal cubic is abelian.

Braid Monodromy

Due to Chisini '33 [19]

Problem (Chisini '47 [20])

Let S be a non-singular compact complex surface, let $\pi : S \to \mathbb{P}^2$ be a finite morphism having simple branching, and let $\mathcal{C}$ be the branch curve; then "to what extent does the pair $(\mathbb{P}^2, \mathcal{C})$ determine π"?

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Let $\pi : X \to B$ be a locally trivial fibration. We say that a morphism $s : B \to X$ is a *section* if $\pi \circ s = \mathbf{1}_B$.

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Braid Monodromy

Definition (Braid Monodromy)

If $F = \mathbb{D}^2 \setminus \{r \text{ points}\}$, then $\pi_1(F, q) = \mathbb{F}_r$ and the monodromy action is given by the action of the braid group $\mathbb{B}_d(\mathbb{D}^2)$ given by:

$$g_j^{\sigma_i} = \begin{cases} g_{i+1} & j = i \\ g_{i+1} g_i g_{i+1}^{-1} & j = i+1 \\ g_i & \text{otherwise.} \end{cases} \tag{1}$$

Braid Monodromy

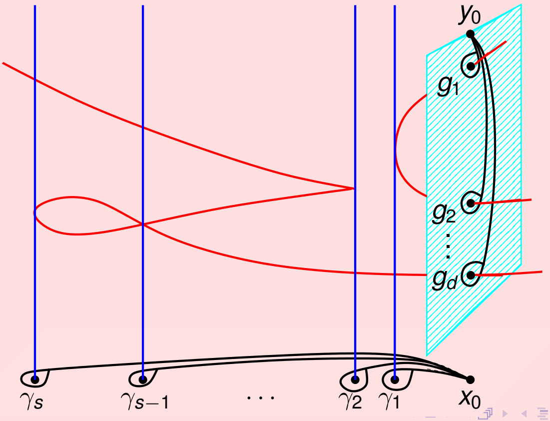
The morphism

$$\begin{array}{rcl} \pi_1(B,p) & \to & \mathbb{B}_d(\mathbb{D}^2) \\ \gamma_i & \mapsto & \beta_i \end{array}$$

is called the *braid monodromy morphism*.
If $\gamma_1,\ldots,\gamma_s$ form a geometric basis of B, then the s-uple $(\beta_1,\ldots,\beta_s)\in\mathbb{B}_d^s(\mathbb{D}^2)$ contains the topological information of the pair $(\mathbb{C}^2,\mathcal{C})$ ([48, 15]).

Zariski–Van Kampen Theorem

Theorem (Zariski–Van Kampen Theorem [81])



Zariski–Van Kampen Theorem

Theorem (Zariski–Van Kampen Theorem [81])

Consider $(\mathbb{P}^2, \mathcal{C}, L, p)$ such that $p \in L \pitchfork \mathcal{C}$ and the projection from p is generic. Then,

$$\pi_1(U) = \langle g_1, \ldots, g_d : g_i = g_i^{\beta_j}, i = 1, \ldots, d, j = 1, \ldots, s \rangle,$$

where $d := \deg \mathcal{C}$ and β_j is the braid associated with $\gamma_j \in \pi_1(\mathbb{C} \setminus \{s\, points\}, x_0) = \mathbb{F}_s$, $j = 1, \ldots, s$.

Zariski–Van Kampen Theorem

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1. $p = [0:1:0]$, $L = \{z = 0\}$, $\mathbb{P}^2 \setminus L \equiv \mathbb{C}^2$,
$\mathcal{C} \cap \mathbb{C}^2 = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$.

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2. Projection from p becomes $j(x,y) = x$. Define $L_\lambda := j^{-1}(\{\lambda\})$.

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2. Projection from p becomes $j(x, y) = x$. Define $L_\lambda := j^{-1}(\{\lambda\})$.
3. $R := \{f = \frac{\partial f}{dy} = 0\} \subset \mathcal{C}$ finite, $\Delta := f(R) = \{\lambda_1, \ldots, \lambda_s\} \subset \mathbb{C}$.

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4. $\mathcal{L} = \cup_{i=1}^{s} L_{\lambda_i}$, $W = \mathbb{C}^2 \setminus (\mathcal{C} \cup \mathcal{L})$, $j|W : W \to \mathbb{C} \setminus \Delta$ is a locally trivial fibration with generic fiber $F = L_{x_0} \cong \mathbb{C} \setminus \{d \text{ points}\}$.

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5. $1 \to \pi_1(F) = \mathbb{F}_d \to \pi_1(W) \to \pi_1(\mathbb{C} \setminus \{s \text{ points}\}) = \mathbb{F}_s \to 0$.

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6. Use the action of $\mathbb{F}_s$ on $\mathbb{F}_d$ to compute $\pi_1(W)$

$$\pi_1(W) = \langle \gamma_1, \ldots, \gamma_s, g_1, \ldots, g_d : \gamma_j^{-1} g_i \gamma_j = g_i^{\beta_j} \rangle.$$

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7. Use Lemma 4 to obtain

$$\pi_1(U) = \langle g_1, \ldots, g_d : g_i = g_i^{\beta_j}, i = 1, \ldots, d, j = 1, \ldots, s \rangle.$$

Consequences

Remark

- Since $(g_d g_{d-1} \ldots g_1)^\beta = g_d g_{d-1} \ldots g_1$, one of the relations of type $g_i = g_i^{\beta_i}$ is redundant.

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Hence a simpler presentation called Zariski's presentation.

Theorem (Zariski's Presentation)

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \langle g_1, \dots, g_d : g_i = g_i^{\beta_j}, i \in I_j^*, j = 1, \dots, s \rangle.$$

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Exercise

Show that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$ for $\mathcal{C} = \{x^d + y^d - z^d = 0\}$.

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One can use the exercise above together with

Lemma

If two curves $\mathcal{C}_1, \mathcal{C}_2$ are in a connected family of equisingular curves, then $(\mathbb{P}^2, \mathcal{C}_1) \cong (\mathbb{P}^2, \mathcal{C}_2)$.

to show that $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$ for ANY smooth curve $\mathcal{C}$.

Degeneration Theorem

Exercise

Assume $\mathcal{C} \subset \mathbb{P}^2$ has a maximal order inflection point, that is, $P \in \mathcal{C}$ regular point such that $(\mathcal{C} \cdot T_P) = d = \deg \mathcal{C}$, for T_P the tangent line of $\mathcal{C}$ at P. Then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/d\mathbb{Z}$.

Degeneration Theorem

Theorem (Dimca '92 [26])

Let $\{\mathcal{C}_t\}_{t\in(0,\varepsilon])}$ a continuous equisingular family of curves converging to a reduced curve $\mathcal{C}_0$. Then,

$$\pi_1(U_0) \twoheadrightarrow \pi_1(U_t).$$

Oka's transversality Theorem

Theorem (Oka '74 [65])

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ plane curves in $\mathbb{C}^2$ intersecting transversally in $d_1 d_2$ points, then

$$\pi_1(U) = \pi_1(U_1) \times \pi_1(U_2).$$

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Corollary

$\pi_1(\mathbb{C}^2 \setminus \mathcal{C})$ is abelian if $\pi_1(\mathbb{C}^2 \setminus \mathcal{C}_i)$ are abelian for all $\mathcal{C}_i$ irreducible component of $\mathcal{C}$ and $\mathcal{C}_i \pitchfork \mathcal{C}_j$, $i \neq j$.

Generalizations

Some of the genericity conditions in Zariski–Van Kampen Theorem 3.1 are not necessary.

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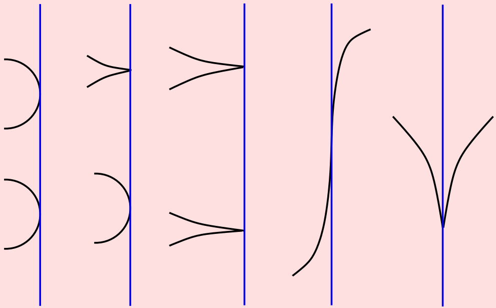


Figure: Non-generic projections

Generalizations

Example

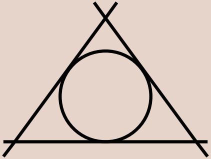


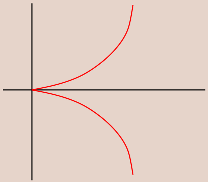
Figure: Projective curve $\mathcal{C} = \{F = 0\}$

$$\mathcal{C} = \{F(x, y, x) = xyz(x^2 + y^2 + z^2 - 2(xy + xz + yz)) = 0\}.$$

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = A(\ {}^{4}\quad{}^{4}\ ) = T(4, 4, 2).$$

Generalizations

Example



$\mathcal{C}_3 = \{y^2 = x^3\}$, $L = \{x = 0\}$.

Generalizations

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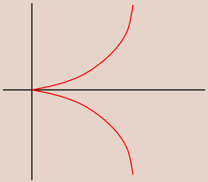
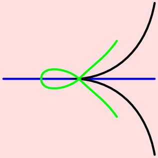


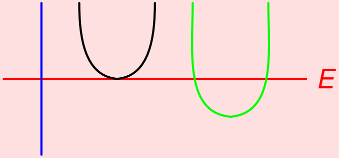
$\mathcal{C}_3 = \{y^2 = x^3\}$, $L = \{x = 0\}$.

$$\pi_1(\mathbb{C}^2 \setminus (\mathcal{C}_3 \cup L)) =$$
$$\langle x, y, z : z^{-1}xz = xyxy^{-1}x^{-1}, z^{-1}yz = yxyxy^{-1}x^{-1}y^{-1} \rangle =$$
$$\langle x, y, z : xzxy = zxyx, yzyxy = zyxyx \rangle.$$

Meridians and blow-ups



$$x = uv$$
$$y = v$$
$$\longleftarrow$$



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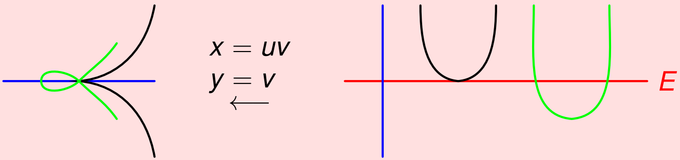
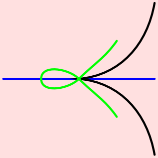


$$x = uv$$
$$y = v$$
$$\longleftarrow$$

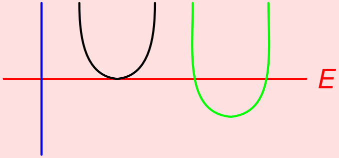
$$\gamma(t) = (0, e^{2\pi i t}) \longleftarrow \tilde{\gamma}(t) = (0, e^{2\pi i t}).$$

Hence, $\tilde{\gamma}$ is a meridian of the exceptional divisor $v = 0$.

Meridians and blow-ups



$$\begin{array}{c} x = u \\ y = uv \\ \longleftarrow \end{array}$$



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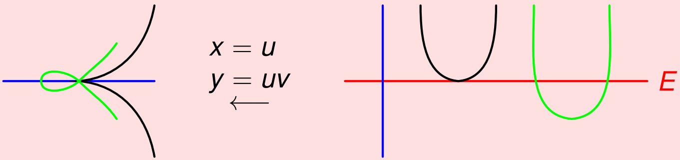

$$\gamma(t) = (0, e^{2\pi i t}) \cong (\varepsilon e^{2\pi i t}, e^{2\pi i t}) \longleftarrow \tilde{\gamma}(t) = (\varepsilon e^{2\pi i t}, \varepsilon^{-1}).$$

Meridians and blow-ups

Note that $\gamma \cong \prod_i g_i$ and $[\gamma, g_i] = 1$ for all $i \in I_P$.

Meridians and blow-ups

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Exercise

Check fundamental group of tricuspidal quartic using Cremona transformation of the tritangent conic.

(p, q)-torus type curves

Purpose: study the fundamental group of the complement of $\mathcal{C} = V(F)$

$$F = x^{aN}y^{bN} + (x^N + y^N + x^m y^m z)^d,$$

where $d = a + b$, $N = 2m + 1$, and $\gcd(a, b) = \gcd(N, d) = 1$.

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where $d = a + b$, $N = 2m + 1$, and $\gcd(a, b) = \gcd(N, d) = 1$. (Originally proved by Oka [62]. We will find an alternative proof.)

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- $P = [0 : 0 : 1] \in \mathcal{C}$, $L_t = \begin{cases} \{y = tx\} & \text{if } t \in \mathbb{C} \\ \{x = 0\} & \text{if } t = \infty \end{cases}$
- $L_0 \cap \mathcal{C} = \{P\} = L_\infty \cap \mathcal{C}$
- $\operatorname{mult}_P \mathcal{C} = 2md = (N - 1)d$.

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where $d = a + b$, $N = 2m + 1$, and $\gcd(a, b) = \gcd(N, d) = 1$.

Lemma

$(\mathcal{C}, P)$ has two branches:

- *δ_0 tangent to L_0 and singular type $(aN + md, aN + (m+1)d)$,*
- *δ_∞ tangent to L_∞ and singular type $(bN + (m+1)d, bN + md)$.*

Moreover, $L_t \cap \mathcal{C} = \{P\} \cup \{d \text{ different points}\}$ for $t \in \mathbb{C}^*$.

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Moreover, $L_t \cap \mathcal{C} = \{P\} \cup \{d \text{ different points}\}$ for $t \in \mathbb{C}^*$.

- Perform Nagata transformation to obtain Σ_N.
- $\mathbb{P}^2 \setminus (\mathcal{C} \cup L_0 \cup L_\infty) \cong \Sigma_N \setminus (\mathcal{C} \cup E \cup E_0^m \cup E_\infty^m)$
- A meridian around L_0 (reps. $L\infty$) is conjugated to a meridian around E_0^m (resp. E_∞^m).

(p, q)-torus type curves

Theorem

$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}/d\mathbb{Z} * \mathbb{Z}/N\mathbb{Z}$.

Proof.

- $\pi_1(\Sigma_N \setminus (\mathcal{C} \cup E)) = \langle g_1, \ldots, g_d : g_i = g_i^{\beta_0}, g_i = g_i^{\beta_\infty} \rangle$.
- $\beta_0 = \beta^a$, $\beta_\infty = \beta^b$, $\beta = (\sigma_1 \ldots \sigma_{d-1})^N$.
- Since $\gcd(a, b) = 1$, $g_i^{\beta_0} = g_i^{\beta_\infty} = g_i^{\beta}$.
- Since $\gcd(d, N) = 1$, relations become $\alpha_1^d = \alpha_2^N$ $(\alpha_2 = g_1 \ldots g_{d-1})$.
- A meridian around E is $(g_1 \ldots g_{d-1})^N$.

$\square$

(p, q)-torus type curves

Definition

A (p, q)*-torus type curve* $(\gcd(p, q) = 1)$ is a curve $\mathcal{C}$ that admits an equation of type $F = f_p^q + f_q^p = 0$ (f_m a homogenous polynomial of degree m in three variables). Moreover, we call $\mathcal{C}$ *generic* if $\mathcal{C}_p = \{f_p = 0\}$, $\mathcal{C}_q = \{f_q = 0\}$ are both smooth and intersect transversally (that is, at pq different points).

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Lemma

If $\mathcal{C}_0$ and $\mathcal{C}_1$ are (p,q)-torus type curves and $\mathcal{C}_1$ is generic, then there is an equisingular deformation $\{\mathcal{C}_t\}_t$, $t\in(0,\varepsilon]$ such that $\mathcal{C}_t$ is generic.

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Lemma

Under the previous conditions, any generic (p, q)-torus type curve $\mathcal{C}$ admits an epimorphism $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C})$.

(p, q)-torus type curves

Definition

An orbifold curve is a quasi-projective Riemann surface X together with a map $\varphi : X \to \mathbb{Z}_{>0}$ such that $\varphi(x) > 1$ only for a finite number of points $\Sigma = \{p_1, \ldots, p_s\}$.
Define $m_i := \varphi(p_i)$ and $\pi_1^{\text{orb}}(X \setminus \Sigma)/\langle g_j^{m_j} = 1\rangle$. A (dominant, algebraic) morphism $f : Y \to X$ is called an orbifold morphism if for all $x \in X$, $f^*(x)$ is a $\varphi(x)$-multiple.

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Lemma

If $f : Y \to X_\varphi$ is an orbifold morphism, then f induces a homomorphism

$$f_* : \pi_1(Y) \to \pi_1^{\text{orb}}(X_\varphi).$$

Moreover, if the generic fiber is connected, then f_* is surjective.

(p, q)-torus type curves

Theorem

If $\mathcal{C}$ is a generic (p, q)-torus type curve, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$.

Proof.

By Lemmas 10 and 11 once has

$$\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z} \twoheadrightarrow \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}.$$

Since $\mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$ is a Hopfian group, then $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \cong \mathbb{Z}/p\mathbb{Z} * \mathbb{Z}/q\mathbb{Z}$. $\square$

Zariski Pairs

Example

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He also claimed the existence of a sextic $\mathcal{C}_6$ with six cusps NOT on a conic whose fundamental group is abelian, that is,

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Both $\mathcal{C}_{2,3}$ and $\mathcal{C}_6$ can be constructed using Kummer covers.

Zariski Pairs

Definition

A pair of curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ with the same number of irreducible components, with the same degrees, types of singularities, and intersections are called a Zariski Pair if the pairs $(\mathbb{P}^2, \mathcal{C}_1)$ and $(\mathbb{P}^2, \mathcal{C}_2)$ are not homeomorphic.

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Remark

The curves $\mathcal{C}_{2,3}$, $\mathcal{C}_6$ form a Zariski Pair of sextics with six (ordinary) cusps (type $\mathbb{A}_2$).

Braid Monodromy

- Choose a *generic* point $P \notin \mathcal{C}$ and a transversal line $P \in L \pitchfork \mathcal{C}$.

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- Define discriminant $\Delta = \{x_1, \ldots, x_s\}$ such that $j|_W : (W, W_\mathcal{C}) \to \mathbb{C} \setminus \Delta$, $W := \mathbb{P}^2 \setminus (L \cup j^{-1}(\Delta))$, $W_\mathcal{C} := W \cap \mathcal{C}$, is a locally trivial fibration.

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- Choose a geometric basis $\gamma_1, \ldots, \gamma_s$ in $\pi_1(\mathbb{C} \setminus \Delta, x_0)$ and define $\beta_i := \mu(\gamma_i)$ the braid associated with the monodromy around γ_i.

Braid Monodromy

Definition

The s-tuple $\text{BM} := (\beta_1, \ldots, \beta_s) \in \mathbb{B}_d^s$ is the braid monodromy representation associated with $(\mathcal{C}, L, P, x_0)$.

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- Choice of a different geometric basis produces an action of $\mathbb{B}_s$.
- Choice of a different section and base point on the generic fiber produces an action of $\mathbb{B}_d$.
- Both actions commute. The action of $\mathbb{B}_s \times \mathbb{B}_d$ on each BM produces the Hurwitz class of $(\mathcal{C}, L, P)$.

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Theorem

There is a one-to-one correspondence between braid monodromy representations and Hurwitz classes of $(\mathcal{C}, L, P)$.

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Theorem

There is a one-to-one correspondence between braid monodromy representations and Hurwitz classes of $(\mathcal{C}, L, P)$.

Definition

The orbit of a braid monodromy representation by the action of Hurwitz moves is called the braid monodromy class of $\mathcal{C}$.

Monodromy Factorization

Since the line L intersects transversally, the product

$$\beta_s\beta_{s-1}\cdots\beta_2\beta_1 = \prod_{i=1}^{s}\mu(\gamma_{s-i+1}) = \mu\partial\mathbb{D} = \Delta_d^2 = (\sigma_1\cdots\sigma_{d-1})^d,$$

the Garside element of $\mathbb{B}_d$.

Definition

A factorization of Δ_d^2 by quasi-positive braids is called a braid monodromy factorization.

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A factorization of Δ_d^2 by quasi-positive braids is called a braid monodromy factorization.

Open Problem

- *Which (algebraic) factorizations are realizable in the algebraic category? (Moishezon [56])*
- *Problem solved for smooth curves (Itzhak-Teicher [11])*

Monodromy Factorization

Theorem (Carmona [15])

The braid monodromy class of $\mathcal{C}$ fully determines the topology of the pair $(\mathbb{P}^2, \mathcal{C})$. In other words, if two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same braid monodromy class, then there is a homeomorphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\varphi(\mathcal{C}_1) = \mathcal{C}_2$.

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Proof.

Retraction to a tubular neighborhood of $\mathcal{C} \cup (L \cup j^{-1}(\Delta))$ and reconstruct this tubular neighborhood using the braid monodromy. $\square$

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Theorem (Libgober [50])

The 2-dimensional complex associated with the Zariski presentation has the homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}$.

Monodromy Factorization

Theorem (Carmona [15])

The braid monodromy class of $\mathcal{C}$ fully determines the topology of the pair $(\mathbb{P}^2, \mathcal{C})$. In other words, if two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same braid monodromy class, then there is a homeomorphism $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\varphi(\mathcal{C}_1) = \mathcal{C}_2$.

Theorem (Libgober [50])

The 2-dimensional complex associated with the Zariski presentation has the homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}$.

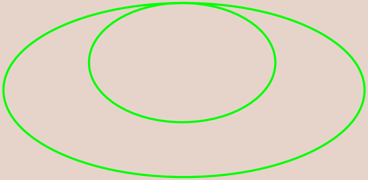
Proof.

The 2-dimensional complex associated with the Artin presentation of a link $K \subset \mathbb{S}^3$ has the homotopy type of $\mathbb{S}^3 \setminus K$. □

Monodromy Factorization: a non-trivial example

Example

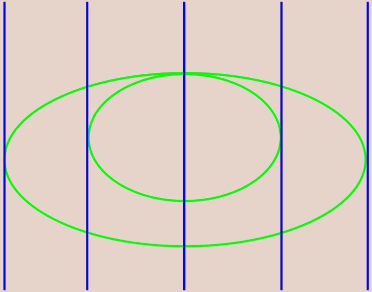
Consider the following quartic:



Monodromy Factorization: a non-trivial example

Example

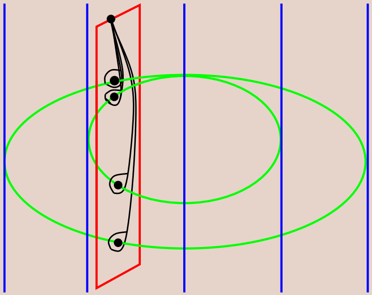
Projecting from $[0:1:0]$:



Monodromy Factorization: a non-trivial example

Example

Computing the braid monodromy:

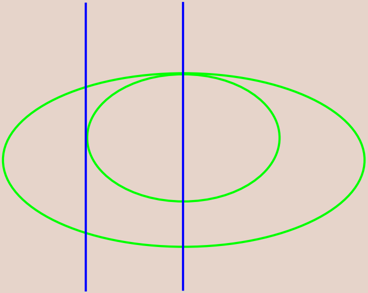


Geometric basis: $\gamma_5\gamma_4\gamma_3\gamma_1\gamma_2 = \partial\mathbb{D}$.

Monodromy Factorization: a non-trivial example

Example

Braids β_2 and β_3 look like local braids:

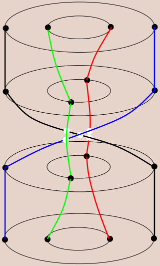


$\beta_2 = \sigma_2$ and $\beta_3 = \sigma_1^8$.

Monodromy Factorization: a non-trivial example

Example

However, the remaining braids β_4 and β_5 depend on global monodromy:

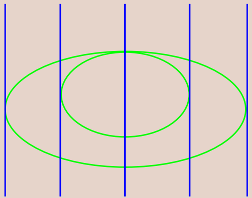


here $\beta_1 = \sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3$.

Monodromy Factorization: a non-trivial example

Example

There is a *half-turn* around the tacnode affecting the second tangency. The braid becomes $\beta_4 = \sigma_1^4 \cdot \sigma_2 \cdot \sigma_1^{-4}$, and hence $\beta_5 = (\gamma_4\gamma_3\gamma_1\gamma_2)^{-1}\Delta_4^2$.



Monodromy Factorization: a non-trivial example

Example

Hence the braid monodromy factorization is

$$(\beta_5, \sigma_1^4\sigma_2\sigma_1^{-4}, \sigma_1^8, \sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3, \sigma_2),$$

where $\beta_5 = \sigma_2^{-1} \cdot \sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_3 \cdot \sigma_1^{-8} \cdot \sigma_1^4\sigma_2\sigma_1^{-4} \cdot (\sigma_1\sigma_2\sigma_3)^4$.

Monodromy Factorization: a non-trivial example

Example

$$
\begin{aligned}
(r_1)\quad & g_1 = g_1^{\sigma_1^8} = (g_2g_1)^4 g_1 (g_2g_1)^{-4} && \Rightarrow [(g_2g_1)^4, g_1] = 1 \\
& g_2 = g_2^{\sigma_1^8} = (g_2g_1)^4 g_2 (g_2g_1)^{-4} && \Rightarrow [(g_2g_1)^4, g_2] = 1 \\
& g_3 = g_3^{\sigma_1^8} = g_3 \\
& g_4 = g_4^{\sigma_1^8} = g_4
\end{aligned} \tag{1}
$$

Monodromy Factorization: a non-trivial example

Example

$$
(r_2) \quad \begin{array}{llll} g_1 = g_1^{\sigma_2} = g_1 & & & \\ g_2 = g_2^{\sigma_2} = g_3 & & \Rightarrow & g_2 = g_3 \\ g_3 = g_3^{\sigma_2} = g_3 g_2 g_3^{-1} & & \Rightarrow & g_2 = g_3 \\ g_4 = g_4^{\sigma_2} = g_4 & & & \end{array} \tag{1}
$$

Monodromy Factorization: a non-trivial example

Example

$$
(r_3) \quad \begin{aligned}
g_1 &= g_1^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_2^{-1} g_4 g_2 && \Rightarrow g_4 = g_2 g_1 g_2^{-1} \\
g_2 &= g_2^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_2 && \\
g_3 &= g_3^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_3 && \\
g_4 &= g_4^{(\sigma_3^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_3)} = g_4 g_2 g_1 g_2^{-1} g_4^{-1} && \Rightarrow g_4 = g_2 g_1 g_2^{-1}
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Monodromy Factorization: a non-trivial example

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$$
\begin{aligned}
(r_5) \quad & g_1 = g_1^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = (g_3(g_2g_1)^{-2}(g_1g_2)^2g_1g_3(g_2g_1)^{-2}) * g_1 \\
(r_6) \quad & g_2 = g_2^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = (g_3(g_2g_1)^{-2}(g_1g_2)^2g_1g_3(g_2g_1)^{-2}(g_1g_2)^2g_1) * g_3 \\
(r_7) \quad & g_3 = g_3^{(\sigma_1^4\sigma_2\sigma_1^{-4})} = g_1^{-1}g_2^{-1}g_1^{-1}g_2g_1g_2g_1 \\
& g_4 = g_4^{\sigma_1^4\sigma_2\sigma_1^{-4}} = g_4, \\
(r_8) \quad & g_4g_3g_2g_1 = 1 \equiv (g_2g_1g_2^{-1})g_2g_2g_1 = (g_2g_1)^2 = 1
\end{aligned}
\tag{1}
$$

Monodromy Factorization: a non-trivial example

Example

$$\pi_1(\mathbb{C}^2 \setminus \mathcal{C}) = \langle g_1, g_2 : [(g_2 g_1)^4, g_1] = 1 \rangle.$$

Monodromy Factorization: a non-trivial example

Example

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Adding the relation $g_4 g_3 g_2 g_1 = 1$ one obtains

$$\pi_1(\mathbb{P}^2 \setminus \mathcal{C}) = \langle g_1, g_2 : (g_2 g_1)^2 = 1 \rangle = \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}.$$

Monodromy Factorization: a non-trivial example

Example

The homotopy type of $\mathbb{C}^2 \setminus \mathcal{C}$ is given by the 2-complex associated with a Zariski's Presentation 3.3

$$\left\langle g_1, g_2, g_3, g_4 : \begin{array}{l} [(g_2g_1)^4, g_1] = 1, \\ g_2 = g_3, \\ g_4 = g_2g_1g_2^{-1}, \\ g_2 = g_3, \\ g_4 = g_2g_1g_2^{-1} \end{array} \right\rangle$$

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Hence by Libgober's Theorem 46 on homotopy type

$$\mathbb{C}^2 \setminus \mathcal{C} \overset{\text{h.t.}}{\cong} (\mathbb{S}^3 \setminus K_{2,8}) \vee \mathbb{S}^2 \vee \mathbb{S}^2.$$

Homotopy Type

Open Problem

- *If* $U = \mathbb{C}^2 \setminus \mathcal{C}$. *Is it true that*

$$\pi_1(U) + \chi(\mathcal{C}) \Rightarrow \textit{homotopy type?}$$

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That is, given two affine curves $\mathcal{C}_1$ *and* $\mathcal{C}_2$

$$\left.\begin{array}{c} \pi_1(U_1) \cong \pi_1(U_2) \\ \chi(\mathcal{C}_1) = \chi(\mathcal{C}_2) \end{array}\right\} \Rightarrow U_1 \overset{h.t.}{\cong} U_2\,?$$

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(Note that this problem has a negative answer in the general case of 2-dimensional complexes, Dunwoody [29]).

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- *Does Libgober's Theorem 46 on homotopy type also hold for projective curves for some preferred presentation?*

Geometric Morphism Problem

Problem

Let $U \subset X$ compact Kähler manifold, $\psi : \pi_1(U) \twoheadrightarrow G$ f.g. kernel, $G \cong \pi_1(C)$.
Determine if/when there exists an admissible map $F : U \to C$ to a smooth complex algebraic curve C realizing ψ, that is, such that ψ and

$$F_* : \pi_1(U, u) \to \pi_1(C, F(u))$$

coincide up to isomorphism in the target.

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Geometric Morphism Problem

Theorem

Let U be a compact Kähler manifold (resp. a proper Zariski open set in a compact Kähler manifold) and let $\psi : \pi_1(U) \to G$ an epimorphism with finitely generated kernel, where G is the fundamental group of a smooth compact complex curve of genus $g \geq 2$ (resp. a free group $\mathbb{F}_s$ with $s \geq 2$). Then there exists an admissible map $F : U \to C_g$, where C_g is a smooth compact complex curve of genus g (resp. $F : U \to C_{g,r}$, where $C_{g,r}$ is a smooth compact complex curve of genus g with $r = s + 1 - 2g$ points removed) with no multiple fibers such that ψ coincides with F_* up to isomorphism in the target.

Orbifolds and Orbifold Fundamental Groups

Definition (Orbifold)

An *orbifold* curve $S_{\bar{m}}$ is a Riemann surface S with a function $\bar{m} : S \to \mathbb{N}$ whose value is 1 outside a finite number of points. A point $P \in S$ for which $\bar{m}(P) > 1$ is called an *orbifold point*.

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Definition (Orbifold Fundamental Group)

For an orbifold $S_{\bar{m}}$, let $P_1, \ldots, P_n$ be the orbifold points, $m_j := \bar{m}(P_j) > 1$. Then, the *orbifold fundamental group* of $S_{\bar{m}}$ is

$$\pi_1^{\text{orb}}(S_{\bar{m}}) := \pi_1(S \setminus \{P_1, \ldots, P_n\})/\langle \mu_j^{m_j} = 1 \rangle,$$

where μ_j is a meridian of P_j. We will denote $S_{\bar{m}}$ simply by $S_{m_1,\ldots,m_n}$.

Orbifold Morphisms

Definition

A dominant algebraic morphism $\varphi : X \to S$ defines an *orbifold morphism* $X \to S_{\bar{m}}$ if for all $P \in S$, the divisor $\varphi^*(P)$ is a $\bar{m}(P)$-multiple.

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Proposition ([5, Proposition 1.5])

Let $\rho : X \to S$ define an orbifold morphism $X \to S_{\bar{m}}$. Then φ induces a morphism $\varphi_* : \pi_1(X) \to \pi_1^{\text{orb}}(S_{\bar{m}})$.
Moreover, if the generic fiber is connected, then φ_* is surjective.

Applications

Example

Consider F equation of $\mathcal{C}_{2,3}$ in Zariski's Example. Since F fits in a functional equation of type

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Then (2) induces a rational map

$$\begin{array}{cccc} \varphi: & \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^1 \\ & [x:y:z] & \mapsto & [h_2^3 : h_3^2] \end{array}$$

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Then (2) induces a morphism

$$\hat{\varphi} : \quad \widehat{\mathbb{P}}^2 \quad \to \quad \mathbb{P}^1$$

such that $\hat{\varphi} = \varphi \circ \varepsilon$.

Applications

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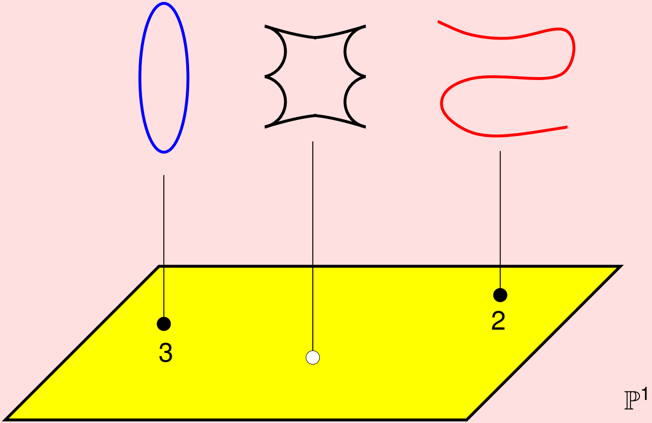
Consider F equation of $\mathcal{C}_{2,3}$ in Zariski's Example. Since F fits in a functional equation of type

$$h_3^2 + h_2^3 + F = 0, \tag{2}$$

- $\hat{\varphi}|_{\mathbb{P}^2\setminus\mathcal{C}}$ has two multiple fibers (over $[0:1]$, $[1:0]$).
- $\bar{m}([0:1]) = 2$, $\bar{m}([1:0]) = 3$
- $\hat{\varphi}_{2,3} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{2,3} \setminus \{[1:-1]\}$ orbifold morphism.
- Since the pencil is primitive, there is an epimorphism

$$\hat{\varphi}_{2,3} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{2,3} \setminus \{[1:-1]\}) = \mathbb{Z}_2 * \mathbb{Z}_3.$$


3
2
$\mathbb{P}^1$

Applications

Example

In general, suppose F fits in a functional equation of type

$$F_1 h_1^p + F_2 h_2^q + F_3 h_3^r = 0, \tag{3}$$

- Then (3) induces a morphism $\hat{\varphi} : \widehat{\mathbb{P}}^2 \to \mathbb{P}^1$ given by $\varphi([x:y:z]) = [F_1 h_1^p : F_2 h_2^q]$.
- $\hat{\varphi}|_{\mathbb{P}^2 \setminus \mathcal{C}}$ has three multiple fibers (over $[0:1]$, $[1:0]$, and $[1:-1]$).
- $\bar{m}([0:1]) = p$, $\bar{m}([1:0]) = q$, and $\bar{m}([1:-1]) = r$.
- $\hat{\varphi}_{p,q,r} : \mathbb{P}^2 \setminus \mathcal{C} \to \mathbb{P}^1_{p,q,r} \setminus \hat{\varphi}(\{F_1 F_2 F_3 = 0\})$ orbifold morphism.
- If the pencil is primitive, there is an epimorphism

$$\hat{\varphi}_{p,q,r} : \pi_1(\mathbb{P}^2 \setminus \mathcal{C}) \to \pi_1^{\text{orb}}(\mathbb{P}^1_{p,q,r} \setminus \hat{\varphi}(\{F_1 F_2 F_3 = 0\})) = \frac{\alpha\mathbb{Z}_p * \beta\mathbb{Z}_q}{(\alpha\beta)^r}.$$

Another Application

Corollary

The number of multiple members in a (primitive) pencil of plane curves (with no base components) is at most two.

Geometric Orbifold Morphism Problem

Problem

Let $U \subset X$ open in a compact Kähler manifold and let $\psi : \pi_1(U) \twoheadrightarrow G$ with f.g. kernel where $G \cong \pi_1^{\text{orb}}(C)$, C smooth complex algebraic curve with an extra orbifold structure.
Determine if/when there exists an admissible map $F : U \to C$ to a smooth complex curve C realizing ψ, that is, such that, if C is endowed with its maximal orbifold structure with respect to F, ψ and $F_* : \pi_1(U) \to G = \pi_1^{\text{orb}}(C)$ coincide up to isomorphism in the target.

Curve orbifold group

Definition (Curve orbifold group)

A curve orbifold group is a group which admits a presentation of the form

$$\mathbb{G}_{g,(r,\bar{m})} := \left\langle \; a_i, b_i, x_j, y_k, \; \begin{smallmatrix} i=1,\dots,g \\ j=1,\dots,s \\ k=1,\dots,r \end{smallmatrix} \;\middle|\; \begin{array}{c} \prod[a_i,b_i] = \prod x_j \prod y_k, \\ x_1^{m_1} = \ldots = x_n^{m_s} = 1 \end{array} \right\rangle \tag{4}$$

for some $g, s, r \geq 0$ and $m_1, \dots, m_s \geq 2$.

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for some $g, s, r \geq 0$ and $m_1, \dots, m_s \geq 2$.

$$\chi_{g,(r,\bar{m})} := 2 - 2g - r - \sum_{i=1}^{n} \left(1 - \frac{1}{m_i}\right).$$

Geometric Orbifold Morphism Theorem

Theorem

Let $U \subset X$ open in a compact Kähler manifold, $\psi : \pi_1(U) \twoheadrightarrow \mathbb{G}_{g,(r,\bar{m})}$ with f.g. kernel K, and $\chi_{g,(r,\bar{m})} < 0$. Then, $\exists$ sm. qproj. curve C and an admissible map $F : U \to C$ such that:

1. *F induces an orbifold morphism $F : U \to C_{\bar{m}}$, where $C_{\bar{m}}$ is maximal with respect to F.*
2. *$F_* : \pi_1(U) \to \pi_1^{\text{orb}}(C_{\bar{m}})$ coincides with ψ up to isomorphism in the target.*
3. *C is projective if and only if $r = 0$.*
4. *If $r = 0$, C has genus g. If $r \geq 1$, C is a genus g' curve with $r' \geq 1$ points removed, where $2g' + r' = 2g + r$.*

Moreover, one such F is unique up to isomorphism of algebraic varieties in the target.

Examples

- if $\pi_1(\mathbb{P}^2 \setminus \mathcal{C}_{p,q}) \cong \mathbb{Z}_p * \mathbb{Z}_q$, then $\mathcal{C}_{p,q} = \{f_p^q + f_q^p = 0\}$

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- What if $\chi_{g,(r,\bar{m})} \geq 0$?
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  - $\mathbb{Z}_2 * \mathbb{Z}_2$.
- Case $\chi_{g,(r,\bar{m})} > 0$ with finite kernel also true.

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