Schubert Calculus in the Grassmannian

Goal: understand linear intersection problems

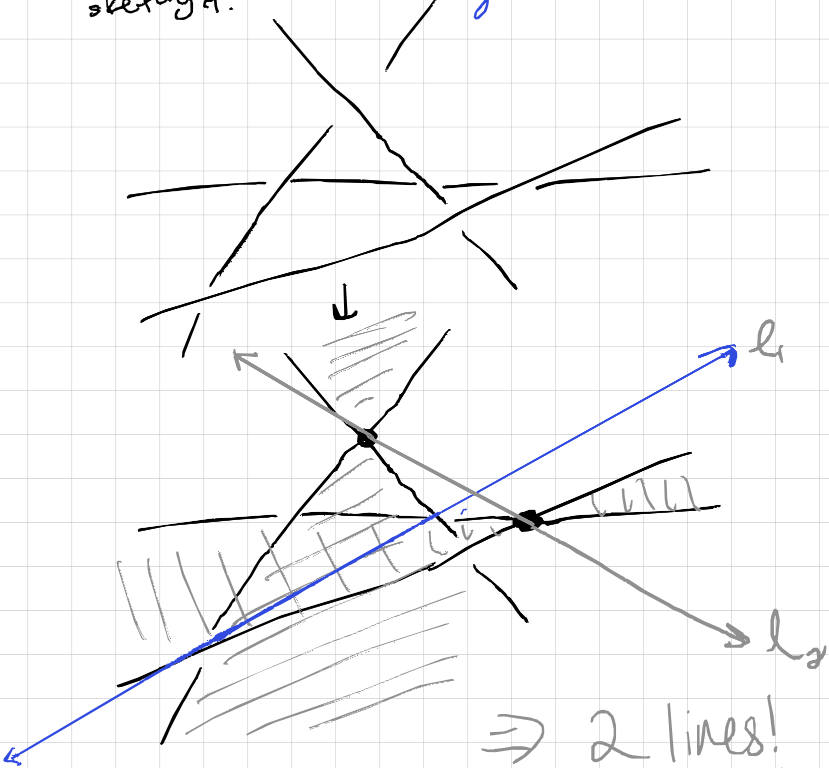
Easy: Q: how many lines pass through 2 given points in the plane?
A: 1 (as long as the points are distinct)

Q: how many points lie on 2 given lines in the plane?
A: 1 (as long as the lines aren't parallel or equal)

Note: dimension matters here!

Harder: Q: how many lines pass through 4 given lines in 3-space?
↑ generic

sketchy A:



$\Rightarrow$ 2 lines! for this degenerate version

Claim: this preserves # of solutions

But how do we actually know this?

In late 1800's, early 1900's geometers were doing these types of calculations. In particular Schubert proposed this type of soln to the problem, arguing "conservation of number" + "special position"

In response to these efforts, in his 15th Problem, Hilbert calls for

1) rigorous development of Schubert's calculus, +
2) a way to predict # of solns prior to doing algebraic elimination

How to formalize these problems + solve them rigorously? How do we know there is a general solution?

(any generic choice yields same intersection #)

First road block? How to avoid parallel issue? What if we say those parallel lines just meet at ∞?

$\boxed{\text{Def}}$ n-dim projective space (over $\mathbb{C}$) $\mathbb{P}^n$ is the set of equivalence classes in $\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}$ where

$$\underbrace{(x_0 : x_1 : \dots : x_n)}_{\text{point in } \mathbb{P}^n} \sim \overbrace{(y_0 : y_1 : \dots : y_n)}^{\text{point in } \mathbb{P}^n} \text{ if}$$

$\exists \alpha \in \mathbb{C}-0$ s.t. $x_i = \alpha y_i \ \forall i \in [n]$

point in $\mathbb{P}^n$ $\longleftrightarrow$ line in $\mathbb{C}^{n+1}$ through origin

line in $\mathbb{R}^3$ $\to$ line in $\mathbb{P}^3$ $\to$ plane in $\mathbb{C}^4$

We'll view these problems in projective space

lines in $\mathbb{P}^3$ $\longleftrightarrow$ 2-planes in $\mathbb{C}^4$

Def The Grassmannian $Gr(n,k)$ is the set of all k-dim subspaces in $\mathbb{C}^n$.

We can associate any point in $Gr(n,k)$ with the span of k LI vectors in $\mathbb{C}^n$

$\uparrow$

represent point in $Gr(n,k)$ with a $k \times n$ matrix whose rows are these vectors

<u>Exercise</u>: each point in $Gr(n,k)$ is the rows span of a unique full-rank $k \times n$ matrix in RREF

we know each matrix will have a well-def RREF. Are these families of matrices helpful to study?

$\hookrightarrow$ form equivalence classes & call them <u>Schubert cells</u>

Ex)

$$\begin{bmatrix} -8 & 12 & 4 & 0 & 0 & 0 & 0 \\ 1 & 8 & 2 & 10 & -3 & 3 & 0 \\ 2 & 4 & 1 & 3 & -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 3 & 1 & 0 & 0 & 0 & 0 \\ -5 & -4 & -1 & 1 & 0 & 0 & 0 \\ 7 & 4 & 1 & 3 & -1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & \boxed{1} & 0 & 0 & 0 & 0 \\ * & * & 0 & \boxed{1} & 0 & 0 & 0 \\ * & * & 0 & 0 & * & \boxed{1} & 0 \end{bmatrix}$$

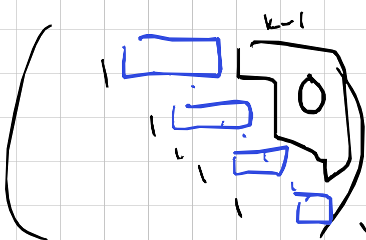
Note that $Gr(k,n)$ will be a disj union of Schubert cells

Exercise: prove it!

By structure of RREF, these look like

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 1 & 0 & 0 \\ * & 0 & * & * & 0 & * & 1 \end{bmatrix}$$

$\lambda = (2,2,1)$ $\qquad\qquad$ $\lambda = (3,1,0)$

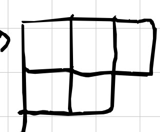


Let $\lambda_i := (\text{position of } 1 \text{ in row } i) - (k-i) - 1$

& let Ω_λ denote the Schubert cell assoc. to λ

Then $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$ is an (integer) partition.

The Young diagram of λ is the diagram of boxes s.t. have λ_i boxes in row i

$\lambda = (3,2,0) \rightsquigarrow$

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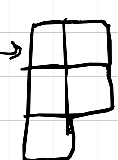


Let $|\lambda| = \sum \lambda_i$ & $\ell(\lambda) = k$

Then $$Gr(n,k) = \dot{\bigcup_{\substack{\lambda_1 \leq n-k \\ \ell(\lambda) = k}}} \Omega_\lambda$$

But we can do this wrt any basis of $\mathbb{C}^n$

To any ordered basis, we can associate a flag: $V_\bullet : \{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_{n-1} \subsetneq V_n = \mathbb{C}^n$

$\{v_1, \dots, v_n\}$

s.t. $\dim V_i = i$

& $\langle v_1, \dots, v_i \rangle = V_i$.

Then

$$\Omega_\lambda(E_\bullet) = \{ V \in Gr(n,k) \mid \dim(V \cap \langle e_1, \dots, e_r \rangle) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1} \}$$

$$\Rightarrow \dim(\Omega_\lambda) = k(n-k) - |\lambda|$$

Note $Gr(k,n)$ is $k(n-k)$-dimensional.

The Schubert variety X_λ is

$$X_\lambda(E_\bullet) = \overline{\Omega_\lambda(E_\bullet)} = \{V \in Gr(k,n) \mid \dim(V \cap \langle e_1, \dots, e_r\rangle) \geq i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}\}$$

Then we see $X_{(1,0)} = X_\square(E_\bullet) \subset Gr(4,2)$

$i=1$: $2+1-1 \leq r \leq 2+1-0$

$$= \{V \in Gr(2,4) \mid \dim(V \cap \langle e_1, e_2\rangle) \geq 1\}$$

i.e. after translation the lines in $\mathbb{P}^3$ that nontrivially intersect a given line (in at least a point)

$\therefore$ our central motivating problem is looking for

$\#$ points in $\left(X_\square(F^1_\bullet) \cap X_\square(F^2_\bullet) \cap X_\square(F^3_\bullet) \cap X_\square(F^4_\bullet)\right)$

which flags to use?

This will be well-defined for transverse flags, i.e. flags $F_\bullet$, $G_\bullet$ s.t. $\forall i,j$

if and only if $\forall F_i, G_j$

$$\dim(F_i \cap G_j) = \max(0, \dim(F_i) + \dim(G_j) - n)$$

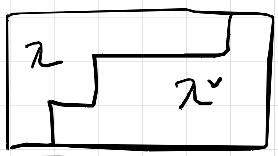
Ex) What is one such transverse pair?

$E_\bullet$ + $F_\bullet$ opp flag $(F_1 = \langle e_n\rangle, F_2 = \langle e_n, e_{n-1}\rangle, \dots)$

For $\lambda \subseteq k \times (n-k)$ let its complement $\lambda^\vee$ be

$\lambda^\vee = (k \times (n-k)) - \lambda$, i.e. $\lambda_i + \lambda^\vee_{k+1-i} = n-k \ \forall i$

ex)



In fact, $F_\bullet, G_\bullet$ transverse $\iff \forall i,\ F_i \cap G_{n-i} = \{0\}$

Pf $\Rightarrow$ follows by def $j = n-i$

$\Leftarrow$ by induction on n. Base case $n=1$ just old transv def.

IA: assume for $n-1$. Then consider $F_\bullet/F_1$, $G_\bullet/F_1$ quotients

Then by IA, $\dim(F_i/F_1 \cap G_j/F_1) = \max(0, \dim(F_i) - 1 + \dim(G_j) - (n-1))$

$= \dim(F_i \cap G_j)$ $\qquad = \max(0, \dim(F_i) + \dim(G_j) - n)$ $\square$

$\boxed{\text{Thm}}$ Let $F_\bullet, E_\bullet$ be transverse flags & $\lambda, \mu \subseteq k \times (n-k)$.
Then if $(X_\lambda(F_\bullet) \cap X_\mu(E_\bullet)) \neq \emptyset$, then $\lambda_i + \mu_{k+1-i} \leq n-k$.
Further if $|\lambda| + |\mu| = k(n-k)$

$$\#(X_\lambda(F_\bullet) \cap X_\mu(E_\bullet)) = \begin{cases} 1 & \text{if } \mu = \lambda^\vee \\ 0 & \text{else} \end{cases}$$

But how to solve for these int #'s?

$$\begin{tikzcd}
\text{intersection theory} \arrow[r] & \boxed{\text{Cohomology } H^*(Gr(k,n))} \arrow[r] \arrow[dr] & \text{ring theory} \\
\text{Schubert variety } X_\lambda \arrow[ur] & & \text{Schubert classes } \sigma_\lambda
\end{tikzcd}$$

$H^*(Gr(k,n))$ has $\mathbb{Z}$-basis given by $\{\sigma_\lambda\}_{\lambda \subseteq k \times (n-k)}$
(↓ if I mult $\sigma_\lambda \sigma_\mu$, I expand into σ_ν w/ $\mathbb{Z}$ coeff)

Further, $H^*(Gr(k,n))$ is a graded ring
$$= \bigoplus_{i=1}^{k(n-k)} H^{2i}(Gr(k,n))$$ where $H^{2i}(Gr(k,n))$ is gen by $\{\sigma_\lambda\}_{|\lambda| = i}$

$\left(R = \bigoplus R_i \text{ st } \begin{array}{l} x \in R_i, y \in R_j \Rightarrow x \cdot y \in R_{i+j} \\ z \in R_i \quad\quad\quad\;\; x + z \in R_i \end{array} \right)$

(⊛) Further,
$$\sigma_\lambda \cdot \sigma_\mu = [X_\lambda(F_\bullet^1) \cap X_\mu(F_\bullet^2)]$$
for transverse $F_\bullet^1, F_\bullet^2$

∴ cohomology gives rigorous foundations for int. thy computations

Then if $|\lambda_1| + |\lambda_2| + \dots + |\lambda_\ell| = k(n-k)$

$\Rightarrow \sigma_{\lambda_1} \cdot \dots \cdot \sigma_{\lambda_\ell} \in H^{k(n-k)}(Gr(n,k))$

$\Rightarrow \sigma_{\lambda_1} \cdot \dots \cdot \sigma_{\lambda_\ell} = C_{\lambda_1 \dots \lambda_\ell} \, \sigma_{k \times (n-k)}$ since graded

By (⊛) it follows
$C_{\lambda_1 \dots \lambda_\ell} = \#$ points in $X_{\lambda_1}(F_\bullet^1) \cap \dots \cap X_{\lambda_\ell}(F_\bullet^\ell)$

Using this formalization,
if $C = \#$ points in $(X_\square(F_\bullet^1) \cap X_\square(F_\bullet^2) \cap X_\square(F_\bullet^3) \cap X_\square(F_\bullet^4))$

$\Leftrightarrow \sigma_\square^4 = C \cdot \sigma_{\boxplus}$

So how can we use H^* to compute c?

We know $\varphi: H^*(Gr(k,n)) \xrightarrow{\sim} R/I$?

$\sigma_\lambda \longmapsto ?$

Q: is such R/I easy to understand? s.t. we have $\varphi(\sigma_\lambda)$ easily computable? What about their products?