Goal: use $H^*(Gr(k,n))$ to solve our main problem

restate

Let's take a step back + discuss some algebraic combinatorics

Let $R = \mathbb{C}[[x_1, x_2, \ldots]]$ denote the ring of formal power series.

$\boxed{\text{Def}}$ We say a function $f \in R$ is <u>symmetric</u> if

$$f(x_1, x_2, \ldots\ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots)$$

$\forall$ permutations $\pi : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$

+ say $\deg(f) < \infty$.

Let Λ denote the ring of these sym fns.

Ex) $f = x_1 + x_2 x_3$ not symmetric

$$g = \sum_{i=1}^{\infty} x_i + \sum_{1 \le i < j < \infty} x_i x_j \text{ is symmetric}$$

Let $P_n = \{\lambda \text{ partition} \mid \ell(\lambda) \le n\}$

One nice $\mathbb{Z}$-basis in Λ are the <u>monomial symmetric functions</u>.

$$m_\lambda = \sum_{(i_1, \ldots, i_\ell) \in \mathbb{Z}_{\ge 0}^{\ell}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_\ell}^{\lambda_\ell}$$

So if $f \in \Lambda$

$f = \sum_\lambda c_\lambda m_\lambda$, $c_\lambda \in \mathbb{Z}$

By construction, m_λ are symmetric.

What about other families?

Could take

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell} \text{ where}$$

$$h_k = \sum_{i_1 \le i_2 \le \ldots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Ex) $h_3 = (x_1^3 + x_2^3 + \ldots) + (x_1^2 x_2 + x_1 x_2^2 + \ldots) + (x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots)$

$= m_3 + m_{21} + m_{111}$

A **tableau** of shape λ is a map $T: \lambda \mapsto \mathbb{Z}_{>0}$, i.e. a filling of the YD of λ w/ integers

A tableau is **semistandard** if entries weakly inc across rows & strictly inc down cols. Let $SSYT(\lambda) = \{T \text{ SSYT sh } \lambda\}$

Ex)

$$\begin{array}{cccc} 1 & 2 & 1 & 3 \\ 1 & 4 & 6 & \\ 2 & & & \end{array} \quad \text{tableau of shape } (4,3,1) \qquad \begin{array}{cccc} 1 & 1 & 2 & 3 \\ 2 & 3 & 7 & \\ 4 & & & \end{array} \quad \text{SSYT}$$

If T has μ_i i's for $i \in [\ell]$, say T has **content** $\mu = wt(T)$

Let $SSYT(\lambda,\mu) = \{T \text{ SSYT of shape } \lambda \text{ \& content } \mu\}$

Then define the Schur function

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x^{content(T)} = \sum_\mu \overbrace{\#SSYT(\lambda,\mu)}^{K_{\lambda\mu} \text{ Kostka coeff}} m_\mu$$

By 2nd equality, s_λ is symmetric.

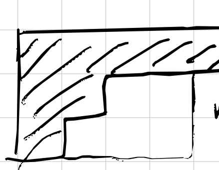
Ex) $\lambda = (2,1)$ $\quad s_\lambda = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + \ldots$

$$\begin{array}{cc} 1 & 1 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 1 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \quad \begin{array}{cc} 1 & 3 \\ 3 & \end{array} \quad \begin{array}{cc} 1 & 2 \\ 4 & \end{array} \quad \begin{array}{cc} 2 & 3 \\ 4 & \end{array} \ \cdots$$

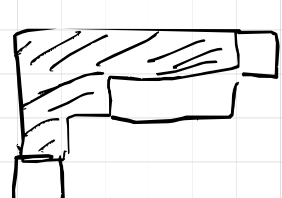
Additionally $\{s_\lambda\}$ form a $\mathbb{Z}$-basis as well

The skew shape ν/λ where $\lambda \subseteq \nu$ is the complement of λ in ν. We say ν/λ is a horiz. strip if no 2 boxes lie in same col.

Ex)



not horiz strip



But how does this connect to Lecture 1?

$H^*(Gr(k,n)) \cong \Lambda \big/ \langle s_\lambda \mid \lambda \not\subseteq k\times(n-k) \rangle$

↑ schur fns indexed by λ that doesn't fit in rectangle

$\sigma_\lambda \xmapsto{\varphi} s_\lambda$

Thus we can try to understand RHS instead

Suppose $\sigma_\lambda \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$ & $s_\lambda s_\mu = \sum_\nu d_{\lambda\mu}^\nu s_\nu$.

<u>Claim</u>: STS $c_{\lambda(a)}^\nu = d_{\lambda(a)}^\nu$ for $a \in \mathbb{Z}_{>0}$.

Note: both rings are commutative & associative

$\boxed{\text{Thm}}$ (Comb Pieri rule) $d_{\lambda(a)}^\nu = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is horiz strip} \\ 0 & \text{else} \end{cases}$

In fact, this follows from

<u>Claim</u>: $s_\lambda \cdot h_\mu = \sum_\nu K_{\nu/\lambda,\mu}\, s_\nu$ ← exercise

Pf: $h_{(a)} = s_{(a)}$. Then $K_{\nu/\lambda,(a)} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ horiz strip} \\ 0 & \text{else} \end{cases}$

$\boxed{\text{Thm}}$ (Geom Pieri rule) $d_{\lambda(a)}^\nu = c_{\lambda(a)}^\nu$, i.e.

$$c_{\lambda(a)}^\nu = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is horiz strip} \\ 0 & \text{else} \end{cases}$$

<u>Pf</u>

If $E_\bullet, F_\bullet, G_\bullet$ generic ($E_\bullet$ std, $F_\bullet$ opp, $G_\bullet$ generic), then
$X_\lambda(E_\bullet) \cap X_{(a)}(F_\bullet) \cap X_{\nu^\vee}(G_\bullet) = \begin{cases} 1 & \text{if } \nu^\vee/\lambda \text{ horiz strip} \\ 0 & \text{else} \end{cases}$

<u>Idea</u>: Similar to Duality arg

Then $\{S_{(a)}\}$ + $\{\sigma_{(a)}\}$ generate their rings

<u>Note</u>: By Giambelli's formula, we see exactly S_λ in terms of S_k. $\left(S_\lambda = \det(S_{\lambda_i - i + j})_{i,j \le k}\right)$

$\Rightarrow$ $\{S_{(a)}\}$ generate $\{S_\lambda\}$

Since both LHS + RHS are assoc, comm rings + φ is a ring homomorphism this shows φ is a ring isomorphism

Already we can now compute

$$C = \#\left(X_\square(F_\bullet^1) \cap X_\square(F_\bullet^2) \cap X_\square(F_\bullet^3) \cap X_\square(F_\bullet^4)\right)$$

$\Rightarrow \sigma_\square^4 = C \cdot \sigma_{\substack{\square\square\\\square\square}}$ in $\begin{smallmatrix}\square&\square&\square&\square\\\square&\square&\square&\square\end{smallmatrix}$

$$(\sigma_\square \cdot \sigma_\square) \cdot \sigma_\square \cdot \sigma_\square = \left(\left(\sigma_{\square\square} + \sigma_{\substack{\square\\\square}}\right)\sigma_\square\right) \cdot \sigma_\square$$

$$= \left(\sigma_{\substack{\square\square\\\square}} + \cancel{\sigma_{\square\square\square}} + \cancel{\sigma_{\substack{\square\\\square\\\square}}} + \sigma_{\substack{\square\square\\\square}}\right)\sigma_\square$$

$$= 2\sigma_{\substack{\square\square\\\square}} \cdot \sigma_\square = 2\sigma_{\substack{\square\square\\\square\square}}$$

$\Rightarrow$ the answer must be 2.

But what if we want to compute

$\sigma_\lambda \cdot \sigma_\mu$ where λ, μ are not rows?

Since $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^{\nu} s_\nu$

want some way to 'multiply' tableaux & expand into tableaux.

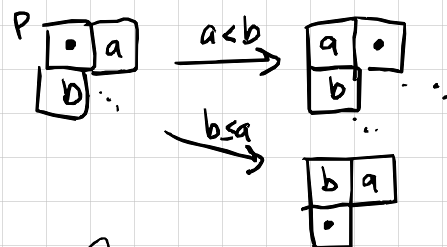
We want to map

$$\Psi: SSYT(\lambda)\times SSYT(\mu) \longrightarrow Q \in SSYT(\nu)$$

s.t. $|\Psi^{-1}(Q)| = c_{\lambda\mu}^{\nu}$ (where commutat is apparent)

Say $T \in SSYT(\lambda)$, $U \in SSYT(\mu)$.
We define $T*U$ via an iterative algorithm $Rect(V)$ where V a skew tab w/ sh p

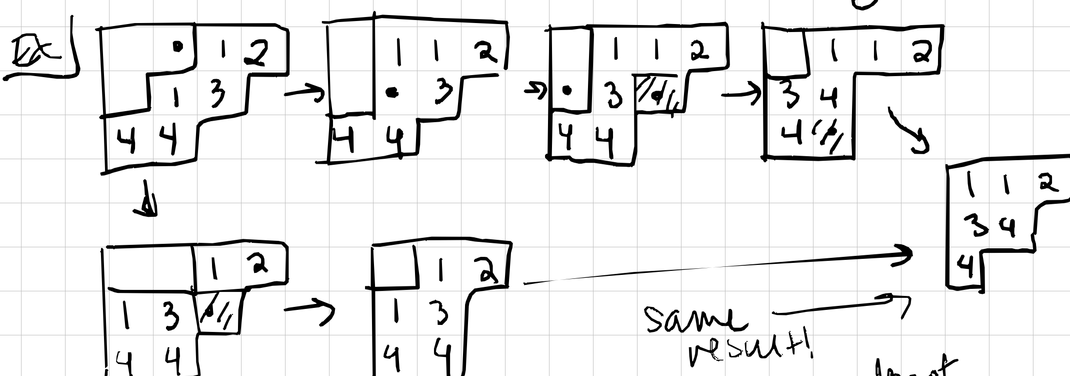
Suppose we have an inner corner of p:



continue until no boxes SW of •

Then repeat with another inner corner until none exist, i.e. the tableaux is no longer skew

Ex



same result!

Challenge: prove order of rectification doesn't matter

$\boxed{\text{Thm}}$ $c_{\lambda\mu}^{\nu} = \#\{(T,U) \in SSYT(\lambda) \times SSYT(\mu) \mid Rect(T*U) = Q\}$ for $Q \in SSYT(\nu)$ fixed

In fact, there are several combinatorial rules that allow us to compute $c_{\lambda\mu}^{\nu}$.

One way to get a clever short pf via Stembridge:

$x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Let $a_\lambda = \det(x_i^{\lambda_j}) = \sum_{w \in S_n} \varepsilon(w)\, x^{w \cdot \lambda}$

$\rho = (n-1, n-2, \ldots, 2, 1)$

$\boxed{\text{Thm}}$ For all partitions $\lambda \in P_n$, partition μ

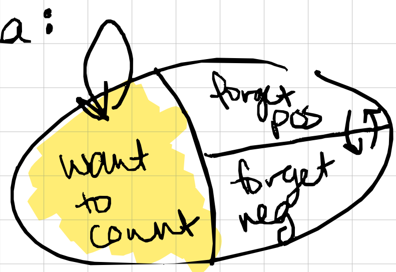
$$a_{\lambda+\rho}\, s_\mu = \sum_{\substack{T \in SSYT(\mu), \\ \lambda + wt(T_{\geq j}) \in P_n,\ \forall j}} a_{\lambda + wt(T) + \rho}$$

T restr to entries $\geq j$

In the exercises, we'll go step-by-step through a pf of this fact

The proof we'll use is by "sign-reversing involution"

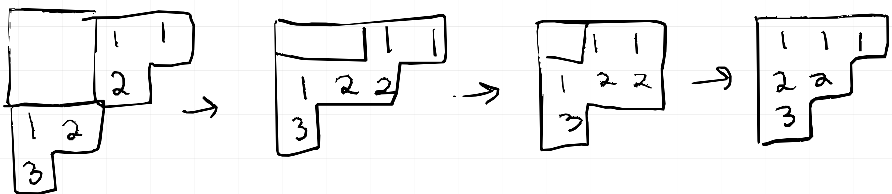
idea:

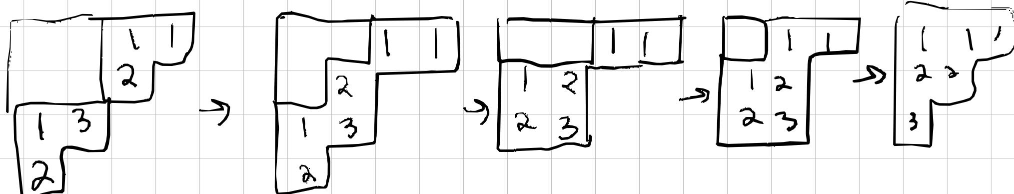


Next time: other places we see $c_{\lambda\mu}^{\nu}$

Ex What is $c^{(3,2,1)}_{(2,1),(2,1)}$?

Want $\#\left\{ T \in SSYT(\lambda * \mu) \;\middle|\; Rect(T) = \begin{array}{|ccc} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \end{array} \right\}$





only possible options

$\Rightarrow$ answer is 2

- Prove sym fns form a ring
- prove m_λ sym $\mathbb{Z}$-basis
- prove s_λ sym w/o appealing to m_λ
- prove s_λ $\mathbb{Z}$-basis
- characterize when $K_{\lambda\mu} \neq 0$
- use other schur def to prove schur pieri
- prove other schur formula
- prove $s_{(k)} = h_k$
- stembridge pf walkthrough
- lattice ~~rule~~ example
- prove equivalence of rules
- puzzle rule

- Symm fns
- Schur basis
- Isomorphism w/ $H^*(Gr(k,n))$
- Pieri?
- L-R rule
- connect to int thy & solve intro problem