CMND 2026 Schubert Calculus

Lecture 1 Problem Set: Grassmannian Schubert varieties

Lecturer: Colleen Robichaux

1. Let $\mathsf{P}_{n,k}$ denote the space of block upper-triangular matrices of the form
$$\left(\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right), \text{where } A \in \mathsf{GL}_k(\mathbb{C}) \text{ and } C \in \mathsf{GL}_{n-k}(\mathbb{C}).$$
   (a) Prove $\mathsf{P}_{n,k}$ is a subgroup of $\mathsf{GL}_n(\mathbb{C})$.
   (b) Prove the isomorphism $\mathsf{Gr}(k,n) \cong \mathsf{GL}_n(\mathbb{C})/\mathsf{P}_{n,k}$.
   (c) Prove that $\dim \mathsf{Gr}(k,n) = k(n-k)$.

2. Describe $\Omega_\emptyset(E_\bullet)$ and $\Omega_{k\times(n-k)}(E_\bullet)$, where $E_\bullet$ is the standard flag.

3. Determine the partition λ such that the vector space corresponding to the matrix
$$\begin{pmatrix} -2 & 0 & 3 & -5 & -6 & 3 & -2 & 1 \\ 5 & 1 & 2 & 3 & 5 & 4 & -1 & 1 \\ 5 & 1 & 5 & -2 & -1 & 7 & -1 & 1 \end{pmatrix}$$
   lies in $\Omega_\lambda(E_\bullet)$, where $E_\bullet$ is the standard flag.

4. Prove that $\dim X_\lambda(E_\bullet) = \dim \Omega_\lambda(E_\bullet) = k(n-k) - |\lambda|$.

5. Fix a partition $\lambda \subseteq k \times (n-k)$.
   (a) Characterize the partitions μ for which we have $\Omega_\mu(E_\bullet) \subseteq X_\lambda(E_\bullet)$.
   (b) Characterize the partitions μ for which we have $\Omega_\mu(E_\bullet) \subseteq X_\lambda(E_\bullet) - \Omega_\lambda(E_\bullet)$.

6. Prove that two complete flags $E_\bullet$ and $F_\bullet$ in $\mathbb{C}^n$ are transverse if and only if $F_{n-i} \cap E_i = 0$ for each $i \in [n]$.

7. Prove that for any pair of transverse flags $E'_\bullet$ and $F'_\bullet$, there exists some $g \in \mathsf{GL}_n(\mathbb{C})$ such that $gE'_\bullet = E_\bullet$ and $gF'_\bullet = F_\bullet$, where $E_\bullet$ is the standard flag and $F_\bullet$ is the opposite flag. (That is, $F_j = \langle e_{n-j+1}, \ldots, e_n \rangle$.)

8. Take $E_\bullet$ as the standard flag and $F_\bullet$ as the opposite flag in $\mathbb{C}^n$. Suppose $\lambda, \mu \subseteq k \times (n-k)$. Consider the flags $A_\bullet$ and $B_\bullet$, where $A_i = E_{n-k+i-\lambda_i}$ and $B_\bullet$, where $B_i = F_{n-k+i-\mu_i}$. Suppose $G_i = A_i \cap B_{k+1-i}$ for $i \in [k]$.
   (a) Which standard basis vectors generate G_i?

(b) When is $G_i = \{0\}$?

9. Prove the duality property: i.e. that for $\lambda, \mu \subseteq k \times (n-k)$ such that $|\lambda| + |\mu| = k \cdot (n-k)$ and transverse flags $E_\bullet$ and $F_\bullet$, we have

$$X_\lambda(E_\bullet) \cap X_\mu(F_\bullet) \neq \emptyset \quad \text{if and only if} \quad \lambda^\vee = \mu.$$

10. Take $E_\bullet$ as the standard flag and $F_\bullet$ as the opposite flag in $\mathbb{C}^n$. Suppose $\lambda, \mu \subseteq k \times (n-k)$. Consider the flags $A_\bullet$ and $B_\bullet$, where $A_i = E_{n-k+i-\lambda_i}$ and $B_\bullet$, where $B_i = F_{n-k+i-\mu_i}$. Suppose $G_i = A_i \cap B_{k+1-i}$ for $i \in [k]$.

    (a) Let $G = G_1 + \ldots + G_k$. Prove $G = \bigcap_{i=0}^k (A_i + B_{k-i})$.

    (b) Let $a \in \mathbb{Z}_{>0}$. Suppose $|\lambda| + |\mu| = k(n-k) - a$. Prove $\sum_{i=1}^k \dim G_i = k + a$.

    (c) Describe when G is a direct sum of the G_i's.

11. *If you know some cohomology*: Prove the following:

    (a) the cells $\Omega_\lambda(E_\bullet)$ form a cellular decomposition of $\mathsf{Gr}(k,n)$.

    (b) the classes $[X_\lambda(E_\bullet)]$ form a basis for $H^*(\mathsf{Gr}(k,n))$.

12. *If you know some more algebra*: Suppose $V \in \mathsf{Gr}(k,n)$. Then we may consider the map to the projectivization of the kth-exterior power

$$\phi : \mathsf{Gr}(k,n) \to \mathbb{P}(\wedge^k \mathbb{C}^n),$$

where the coordinates of $\phi(V)$ are the $k \times k$-minors $P_{i_1,\ldots,i_k}$ of the matrix representation of V. We call these minors the Plücker coordinates of V.

    (a) Prove ϕ is an embedding.

    (b) Take the ring $\mathbb{C}[P_J]$. Here J ranges over the tuples $(i_1, \ldots, i_k)$ such that $1 \leq i_1 < \ldots < i_k \leq n$. Consider the ideal $I(n,k)$ of homogeneous polynomials in $\mathbb{C}[P_J]$ that are identically zero on $\mathsf{Gr}(k,n)$.

        i. Prove $I(n,k)$ contains no linear polynomials.

        ii. Find the generators of $I(n,k)$.