CMND 2026 Schubert Calculus

Lecture 2 Problem Set: Combinatorics in Schubert calculus

Lecturer: Colleen Robichaux

1. Prove that $s_{(k)} = h_k$ for $k \in \mathbb{Z}_{>0}$.

2. Say a tableaux T is lattice if, when reading the entries in T from top to bottom, right to left, we have always read weakly more i's than $i+1$'s for all $i > 0$. Let $d^{\nu}_{\lambda,\mu}$ denote the set of semistandard Young tableaux of shape ν/λ and weight μ that are lattice. Compute $d^{\nu}_{\lambda,\mu}$ for $\lambda = \mu = (2,1)$ and $\nu = (3,2,1)$ using the above rule.
Challenge: In fact, $d^{\nu}_{\lambda,\mu} = c^{\nu}_{\lambda,\mu}$. Are there any cases in which you can prove it?

3. Compute $c^{\nu}_{\lambda,\mu}$ for $\lambda = (4,1)$, $\mu = (3,2,1)$, and $\nu = (5,4,2)$.

4. Consider partitions λ, μ, and ν.
   (a) Give examples of triples λ, μ, ν for which $c^{\nu}_{\lambda,\mu} > 0$ and some for which $c^{\nu}_{\lambda,\mu} = 0$.
   (b) Describe conditions on λ, μ, and ν that must hold if we know $c^{\nu}_{\lambda,\mu} > 0$. Are they sufficient as well?

5. For a partition λ of length n, define the alternant $a_\lambda = \sum_{w\in S_n} \mathrm{sign}(w) x^{w(\lambda)}$, where S_n are the permutations of $[n]$ and $\mathrm{sign}(w)$ denotes the sign of the permutation w. Define $\delta_n := (n-1, n-2, \ldots, 1, 0)$. Then we may alternatively define Schur functions as
$$s_\lambda = \frac{a_{\lambda+\delta_n}}{a_{\delta_n}}.$$
Use this definition to prove the Pieri rule, that is, $s_\lambda s_{(a)} = \sum_{\nu\subseteq k\times(n-k)} c^{\nu}_{\lambda,(a)} s_\nu$, where
$$c^{\nu}_{\lambda,(a)} = \begin{cases} 1 & \text{if } \nu/\lambda \text{ is a horizontal strip of size } a \\ 0 & \text{otherwise.} \end{cases}$$

6. Let T be a semistandard Young tableau and consider an integer $k > 0$. We say an entry i in T is paired if $i = k$ and there is a box containing $k+1$ directly below it OR if $i = k+1$ and there is a box containing k directly above it. Let σ_k be the operator on tableaux defined by the following:
   - If $i \notin \{k, k+1\}$ OR i is paired, then σ_k leaves i the same.
   - Then if a row r has i unpaired k's and j unpaired $k+1$'s, replace those entries with j k's and i $k+1$'s. That is :

$$\sigma_3\left(\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{4}\boxed{5}\right) = \boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{4}\boxed{5}.$$

Now let's think about σ_k.

(a) Apply σ_3 to T below. What is the weight of $\sigma_3(T)$?

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ \hline 2 & 3 & 3 & 4 & 4 & 4 & 5 \\ \hline 4 & 4 & 5 & 5 & 5 & 6 \\ \cline{1-6} 5 & 6 & 7 \\ \cline{1-3} \end{array}.$$

(b) Prove that if $T \in \mathsf{SSYT}(\lambda, \mu)$ is semistandard, so is $\sigma_k(T)$.

(c) Suppose $T \in \mathsf{SSYT}(\lambda, \mu)$. Describe the weight of $\sigma_k(T)$.

(d) Use the above steps to Schur functions s_λ are symmetric using their tableaux-theoretic definition.

7. Let $T \in \mathsf{SSYT}(\lambda)$. Define $T|_{\geq j}$ denote the tableaux obtained by restricting T to its entries weakly greater than j. Similarly define $T|_{>j}$. We'll walk through the proof of Stembridge that

$$a_{\lambda+\delta_n} s_\mu = \sum_T a_{\lambda+\delta_n+\mathsf{wt}(T)},$$

where the sum is over $T \in \mathsf{SSYT}(\mu)$ such that $\lambda + \mathsf{wt}(T|_{\geq j})$ is a integer partition for all $j \geq 1$.

(a) Prove that

$$a_{\lambda+\delta_n} s_\mu = \sum_{T \in \mathsf{SSYT}(\mu)} a_{\lambda+\delta_n+\mathsf{wt}(T)}.$$

(b) Say $T \in \mathsf{SSYT}(\mu)$ is bad if $\lambda + \mathsf{wt}(T|_{\geq j})$ is a not integer partition for some $j \geq 1$. If T is bad, pick the largest j then smallest k that produces the "bad" property. Let T^* denote the tableaux formed by applying σ_k to $T|_{\geq j}$, and leaving the rest of the entries in T unchanged. Prove T^* is semistandard.

(c) Prove that if T is bad, then T^* is bad and $(T^*)^* = T$.

(d) Prove that if T is bad, then

$$a_{\lambda+\delta_n+\mathsf{wt}(T)} = -a_{\lambda+\delta_n+\mathsf{wt}(T^*)}.$$

(e) Use the above parts to finish the proof of the main statement.

8. Use the previous problem to prove another LR rule, i.e. a combinatorial description of $c^\nu_{\lambda,\mu}$. Is it one that we have already seen, or not?

9. Prove that the structure coefficients of Schur polynomials and the structure coefficients of Schubert classes in the Grassmannian are equal (i.e. $c^\nu_{\lambda,\mu} = d^\nu_{\lambda,\mu}$ in the lecture).