Directions. Your goal in these problems is to explore and gain familiarity with these objects and topics. Don't let the computer have all the fun (i.e. don't ask AI to do the problems for you). The struggle and time spent thinking about these problems is more important than completing all of them or figuring them all out perfectly. Discuss the problems with each other and me! Math is more fun when we work on it together!

A note on terminology: **Open Problems** have been open for a long time. They are either hard or we haven't found the right approach yet. **Challenges** are doable and may lead to research ideas. **Further Study** invites you to dig deeper in a related paper.

Definition. An *alternating sign matrix (ASM)* is a square $\{0, 1, -1\}$ matrix such that (1) each row and each column sums to 1 and (2) the nonzero entries alternate in sign along each row/column.

Example: $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

1. Construct square $\{0, 1, -1\}$-matrices that satify:
   (a) Property (1) but not (2).
   (b) Property (2) but not (1).
2. Explain why permutation matrices are alternating sign matrices.
3. Determine some necessary conditions for the placement of -1 entries in an ASM.
4. What is the maximum number of -1 entries in an $n \times n$ ASM? Give a construction for a matrix attaining the maximum. How many $n \times n$ matrices attain the maximum?
5. The set of $n \times n$ ASMs is enumerated by $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. Use this formula to find the number of 5×5 ASMs.

   Further Study. This enumeration was proved by [Ku96, Ze96]; for a lovely exposition of the story behind the conjecture and proof of this formula, see [B99].
6. Let $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ and $n!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q$. Compute the q-analogue of the ASM numbers, $A_n(q) := \displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}$, for $n = 2, 3$.

   Open Problem. Find a (nice!) statistic on $n \times n$ ASMs that has generating function $A_n(q)$.

Further Study. $A_n(q)$ is a natural generating function for *descending plane partitions (DPPs)*, thus a solution to the above problem could be a step toward finding a nice bijection between ASMs and DPPs. The statistic is known in the permutation case; see, for instance, [S11b].

Further Study. Go to `FindStat.org`, click on 'All Statistics' in the banner across the top, then click on 'Alternating sign matrices'. Click on a statistic that sounds interesting to you. Read and play around with the information on the statistic page. Compute the statistic on our example ASM.

Challenge. Find a new interesting statistic (or map) on ASMs and add it to the database at `FindStat.org`.

7. Alternating sign matrices have nice bijections to several other sets of objects. Below, several sets of objects are defined and a map from ASMs to each of these objects is described (see also [P01]). In each case, do the following:
   (a) Finish transforming the given ASM A into the new object.
   (b) Explain why the map is well-defined, in the sense that it sends an ASM to the right kind of object.
   (c) Determine what a -1 corresponds to under the map.
   (d) Figure out the inverse map by drawing your own example of the new object and working backward to find the corresponding ASM.
   (e) Explain why the map is a bijection.

Definition. A *monotone triangle* of *order* n is an equilateral triangular array of numbers with bottom row $1\ 2\ \cdots\ n$ that satisfies the inequality $\leq$ along all southwest to northeast diagonals, the inequality $\leq$ along all northwest to southeast diagonals, and the inequality $<$ across all rows from left to right.

To map from an ASM A to a monotone triangle, first write a matrix M of the column partial sums of A. Then draw an array whose ith row consists of all the j such that $M_{ij} = 1$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} \qquad \begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & 2 & & 5 & & & \\ & & . & & . & & . & & \\ & . & & . & & . & & . & \\ . & & . & & . & & . & & . \end{array}$$

Definition. A *six-vertex configuration* with *domain wall boundary conditions* is a directed graph on an $n \times n$ grid of vertices with boundary edges pointing inward on the left and right and outward on the top and bottom such that each internal vertex has two edges pointing inward and two pointing outward.

To map from an ASM to a six-vertex configuration, make a directed graph on the grid with the 1 entries as sinks horizontally and sources vertically, the -1 entries as sources horizontally and sinks vertically, and the 0 entries as *transmitting vertices*, meaning the horizontal edge directions are both left or both right and the vertical edge directions are both up or both down.

$$\begin{array}{ccccccc}
 & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\cdot & 0 & 0 & 1 & 0 & 0 & \cdot \\
\cdot & 0 & 1 & -1 & 0 & 1 & \cdot \\
\cdot & 0 & 0 & 0 & 1 & 0 & \cdot \\
\cdot & 1 & -1 & 1 & 0 & 0 & \cdot \\
\cdot & 0 & 1 & 0 & 0 & 0 & \cdot \\
 & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array}$$

Definition. A *fully-packed loop (FPL)* is a graph on an $n \times n$ grid of vertices with $2n$ exterior edges at every other spot such that each interior vertex has degree 2.

To map from an ASM to an FPL, draw dots around the ASM as shown, in every other spot. Draw an edge from each of these boundary dots to the nearest matrix entry. Then draw paths between dots on the grid that go straight at each 1 and -1 and turn (to the left or right) at each 0. The direction turned at each 0 is determined by the condition that in the end, each interior vertex has degree 2.

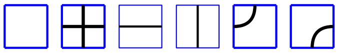
$$\begin{array}{ccccccc}
 & \cdot & & \cdot & & \cdot & \\
 & 0 & 0 & 1 & 0 & 0 & \\
\cdot & 0 & 1 & -1 & 0 & 1 & \cdot \\
 & 0 & 0 & 0 & 1 & 0 & \\
\cdot & 1 & -1 & 1 & 0 & 0 & \cdot \\
 & 0 & 1 & 0 & 0 & 0 & \\
 & \cdot & & \cdot & & \cdot &
\end{array}$$

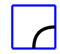
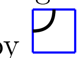
Definition. A *corner sum matrix* is an $(n+1) \times (n+1)$ matrix with first column and first row entries all 0, with last column and last row equal to $0, 1, \ldots, n$, whose rows are weakly increasing left to right, whose columns are weakly increasing top to bottom, and whose entries differ from each adjacent entry by 0 or 1.

Definition. A *height function* is an $(n+1) \times (n+1)$ matrix with first row and first column $0, 1, \cdots, n$, last row and last column $n, \cdots, 1, 0$, and interior entries differing by exactly one.

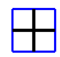
To map from an ASM $A^n_{i,j=1}$ to a corner sum matrix $C^n_{i,j=0}$, let C_{ij} equal the sum of all entries $A_{k\ell}$ such that $k \le i$ and $\ell \le j$. To map from a corner sum matrix to a height function $H^n_{i,j=0}$, let $H_{ij} = i + j - 2C_{ij}$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad H = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 3 & 4 \\ 2 & 3 & 2 & 3 & 4 & 3 \\ 3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 5 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

Definition ([LLS21])**.** A *bumpless pipe dream (BPD)* is a tiling of an $n \times n$ grid with tiles  such that n pipes traveling from the south border to the east border are formed.

To map from an ASM to a BPD, draw pipes starting from the bottom. Turn right at 1, turn left at -1, and go straight at 0. That is, replace each 1 by , each -1 by , and each 0 by whichever of the other four tiles is needed to form an appropriate set of pipes.

$$\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & \cdot \\ 0 & 1 & -1 & 0 & 1 & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot \\ 1 & -1 & 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{array}$$

8. Compute the permutation associated to the BPD of our example ASM A. Label the pipes $1, 2, \ldots, n$ across the bottom from left to right. Number the pipes along the right by tracking where each initial pipe goes. For the first  tile encountered involving pipes k and ℓ, consider the pipe labels as crossing.

9. Compute the *key* of the ASM, as in [We21] or [BSS25] (with change of convention from SW to NW) and check that this gives the same permutation.

   Further Study. Use BPDs to compute the *Schubert polynomial* of the permutation (see [LLS21] or [We21]).

Definition. A *plane partition* is a stack of cubes in a corner. More formally, a plane partition t in an $a \times b \times c$ *bounding box* is a finite set of positive integer lattice points (i, j, k) with $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq k \leq c$ such that if $(i, j, k) \in t$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, $1 \leq k' \leq k$ then $(i', j', k') \in t$. (The lattice points can be thought of as the corner of a unit cube that is furthest from the origin.)

A plane partition t is *symmetric* if whenever $(i, j, k) \in t$ then $(j, i, k) \in t$ as well. t is *cyclically symmetric* if whenever $(i, j, k) \in t$ then $(j, k, i) \in t$ as well. t is *totally symmetric* if it is both symmetric and cyclically symmetric.

Let $t_{i,j} = k$ where k is the largest number such that $(i, j, k) \in t$ (and 0 if there is no such k). That is, $t_{i,j}$ is the number of blocks in the stack at (x, y)-coordinate (i, j). The *complement* t^C of t inside a given $a \times b \times c$ bounding box is defined as $t^C_{i,j} = c - t_{a-i+1,b-j+1}$ for all $1 \leq i \leq a, 1 \leq j \leq b$. That is, $t^C_{i,j}$ equals the number of empty cubes above $t_{a-i+1,b-j+1}$ in the bounding box. A plane partition t is *self-complementary* inside a given bounding box if $t = t^C$. A *totally symmetric self-complementary plane partition (TSSCPP)* is a plane partition which is both totally symmetric and self-complementary.

10. Using blocks, build an example of each of the following. Each example should include only the symmetry mentioned and no other. (Take a picture of each!)

(a) A stack of blocks that is not a plane partition.

(b) A plane partition.

(c) A symmetric plane partition.

(d) A cyclically symmetric plane partition.

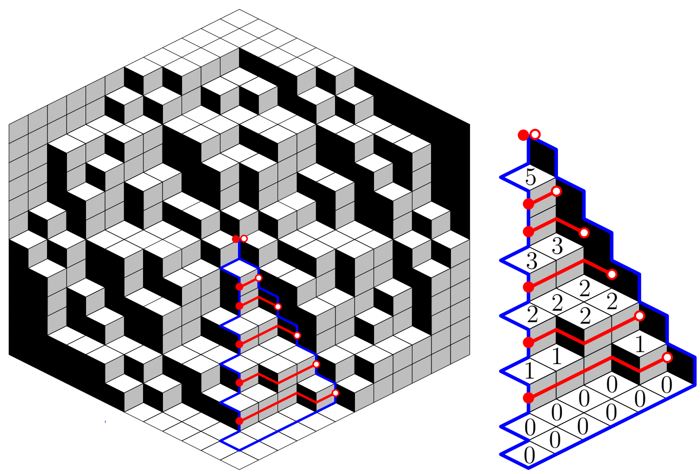
(e) A totally symmetric plane partition.

(f) A self-complementary plane partition.

(g) A totally symmetric self-complementary plane partition.

Open Problem. Find an explicit (nice!) bijection between $n \times n$ ASMs and TSSCPPs in a $2n \times 2n \times 2n$ box.

Here is a large example of a totally symmetric self-complementary plane partition. It has a lot of symmetry, so all you need to reconstruct the whole thing is its *fundamental domain*, in this case 1/12 of the hexagon, on the right. (The fundamental domain of symmetric plane partitions is 1/2 of the hexagon, cyclically symmetric is 1/3, and totally symmetric is 1/6.) Using the fundamental domain, you can organize the information in two different ways: as a *magog triangle* or boolean triangle.



Definition. A *magog triangle* of *order* n is an equilateral triangular array of numbers with bottom row $1\ 2\ \cdots\ n$ that satisfies the inequality $\leq$ along all southwest to northeast diagonals, the inequality $\geq +1$ along all northeast to southwest diagonals (i.e. if x is directly NW of y, then $x+1 \geq y$), the inequality $<$ across all rows from left to right, and an upper bound of n on all entries.

11. Compare this definition to the definition of a monotone triangle. Explain the similarities and differences.

12. Use the following instructions to map the given TSSCPP to a magog triangle. First finish writing the fundamental domain box heights as a triangle (below, left), then add i to the ith southwest to northeast diagonal (below, center).

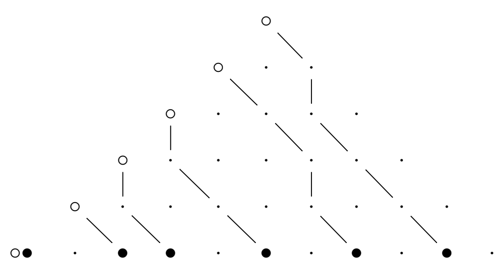
$$
\begin{array}{ccccccccccc}
 & & & & & 5 & & & & & \\
 & & & & 3 & & 3 & & & & \\
 & & & 2 & & 2 & & \cdot & & & \\
 & & 1 & & 1 & & \cdot & & \cdot & & \\
 & 0 & & 0 & & \cdot & & \cdot & & \cdot & \\
0 & & 0 & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 6 & & & & & \\
 & & & & 4 & & \cdot & & & & \\
 & & & 3 & & \cdot & & \cdot & & & \\
 & & 2 & & \cdot & & \cdot & & \cdot & & \\
 & 1 & & \cdot & & \cdot & & \cdot & & \cdot & \\
1 & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot
\end{array}
\qquad
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

13. Now use the map that sends a monotone triangle to an ASM to transform the magog triangle into a $\{0, 1, -1\}$-matrix (above, right). How does this differ from an alternating sign matrix?

    Further Study. Look up the definition of *magog matrices* in [HoS24] and learn about some enumerative properties of these matrices.

Definition ([St18])**.** A *TSSCPP boolean triangle* of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}$.

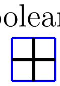
14. Use the following instructions to map the example TSSCPP to a Boolean triangle. First finish drawing the nonintersecting lattice paths that trace out each layer (below, left). These are the paths that trace out each layer of the fundamental domain on the prior page, rotated counterclockwise. Then, change diagonal steps to 0 and vertical steps to 1 (below, right).



$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & \cdot & & 1 & & & \\
 & & \cdot & & \cdot & & 0 & & \\
 & \cdot & & \cdot & & \cdot & & 0 & \\
\cdot & & \cdot & & \cdot & & \cdot & & 0
\end{array}
$$

15. Check that the resulting triangle satisfies the defining inequality.

    Further Study. TSSCPP Boolean triangles with each row weakly decreasing (i.e. all 1 entries are left-justified) are in bijection with permutation matrices [St18]. Draw such a TSSCPP Boolean trangle and work through the bijection from [St18] to transform it to a permutation (or figure it out yourself!).

16. Flip the TSSCPP boolean triangle upside-down and change the shape to be left-justified. Replace each 1 by a  and each 0 by a . This produces a *pipe dream*.

Further Study. The paper [HuS24] gives a partial solution to the ASM-TSSCPP bijection problem, in the case that the permutation of the corresponding bumpless pipe dream *avoids* the pattern 1432. Work through the steps of this bijection to find out what ASM corresponds to the above TSSCPP.

17. Write out all the monotone triangles of order 3.
    (a) Define a natural partial order on them. Is it a lattice? If so, is it distributive?
    (b) Then look at the subposet consisting only of the permutation monotone triangles. Is this a known permutation poset?

18. Write out all the magog triangles of order 3.
    (a) Define a natural partial order on them. Is it a lattice? If so, is it distributive?
    (b) Then look at the subposet consisting only of the permutation monotone triangles. Is this a known permutation poset?

Further Study. See [St18, Sec. 4] for a discussion of these monotone and magog triangle posets, their permutation subposets, and their subposets of join irreducibles. See also [St11].

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