Notes on Pavlo Pylyavskyy's Lecture

July 1, 2026

1 June 29

Based on the joint work with Pavel Galashin. Quivers with subadditive labelings: classification and integrability.

1.1 Zamolodchikov periodicity

Definition 1 (Mutations of a bipartite graph)**.** Let x_i label the vertices of a bipartite graph. A mutation on the graph is the following procedure:

- Label x_i at each vertex.
- Do $x_i \mapsto \frac{\prod_{\{i,j\} \text{ is and edge}} x_j+1}{x_i}$ alternatingly.

Example 1.

$$\circ \text{———} \bullet \text{———} \circ$$

1	1	1
2	1	2
2	5	2
3	5	3
3	2	3
1	2	1
1	1	1

Example 2.

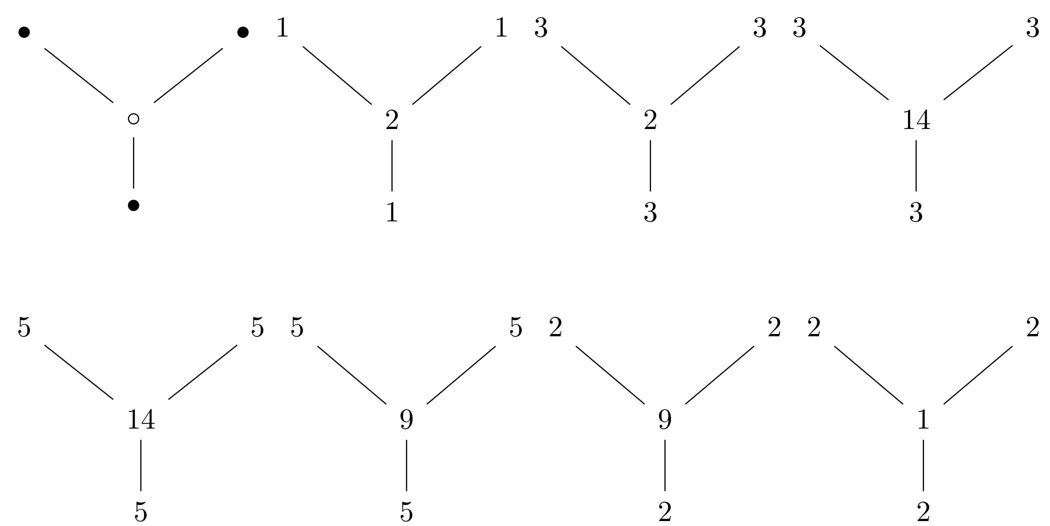
$$\begin{array}{ccc}
\circ & \text{———} & \bullet \\
\\
x & & y \\
\\
\frac{y+1}{x} & & \frac{x+y+1}{xy} \\
\\
\frac{x+1}{y} & & \frac{x+y+1}{xy} \\
\\
\frac{x+1}{y} & & x \\
\\
y & & x
\end{array}$$

Zamolodchikov 1991 noticed such periodicity.

1.2 Subadditive labelings

Question 1. Which bipartite graphs provide periodicity?

Example 3.



Example 4. This graph doesn't work. Also observe that we only get integers.

Definition 2 (Subadditive labelings for bipartite graphs)**.** A *subadditive labeling* for a bipartite graph is a function from its vertices to the positive real numbers such that such that for each vertex v we have [1]

$$2f(v) > \sum_{\{u,v\} \text{ is an edge of the graph}} f(u).$$

[1] Someone asked why number 2 comes up being important. Reasons given: Cartan matrix has 2 along the diagonal. The mutation relation has x_i^2 on the left hand side, etc.

$$
\begin{tikzcd}[arrows=dash]
 & \bullet \arrow[d] & \\
\bullet \arrow[r] & \circ \arrow[r] \arrow[d] & \bullet \\
 & \bullet &
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash]
 & 1 \arrow[d] & \\
1 \arrow[r] & 2 \arrow[r] \arrow[d] & 1 \\
 & 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash]
 & 3 \arrow[d] & \\
3 \arrow[r] & 2 \arrow[r] \arrow[d] & 3 \\
 & 3 &
\end{tikzcd}
\quad
\begin{tikzcd}[arrows=dash]
 & 3 \arrow[d] & \\
3 \arrow[r] & 41 \arrow[r] \arrow[d] & 3 \\
 & 3 &
\end{tikzcd}
$$

Example 5. Prove that Figure 4 does not have a subadditive labelings.

Theorem 1.1 ([GP20])**.** *A bipartite graph has the Zamolodchikov periodicity if and only if (the* if *direction is known previously) it has a subadditive labeling.*

Intuition. If the system is periodic, then it has a fixed point.

Answer: A, D, E Dynkin diagrams, which appear in the classification of semisimple Lie algebras, simple plane curve singularities, and the cluster algebras of finite type.

Theorem 1.2 ([Vin71])**.** *These are the only graphs with subadditive labelings.*

Lemma 1. *A graph with subadditive labelings cannot have a vertex of degree 4.*

Lemma 2. *A connected graph with subadditive labelings can have at most one vertex of degree 3.*

Proof. Take two degree 3 vertices that are closest to each other in terms of walking distance. Label the vertices by We have

$$
\begin{aligned}
2a &> c_1 \\
2b &> c_2 \\
2c_1 &> a + b + c_2 \\
\implies c_1 &> c_2 \\
\implies c_2 &> c_3 \\
&\vdots
\end{aligned}
$$

We obtain a reverse inequality had we started from the other side, a contradiction. □

Exercise 1. If we have all degrees ≤ 2, we can have a path, but not a circle. Suppose we had a circle and a subadditive labeling by $x_1, \ldots, x_n$. Then we have

$$
\begin{aligned}
2x_1 &> x_2 + x_n \\
2x_2 &> x_1 + x_3 \\
&\vdots \\
2x_n &> x_1 + x_{n-1}
\end{aligned}
$$

If we sum them all together, we have that $2\sum x_i > 2\sum x_i$, which is a contradiction.

If we have one vertex of degree 3, we cannot have each branch of length ≥ 2. We label the vertices by

$$
\begin{tikzcd}[arrows=dash]
 & & x_2 \arrow[d] & & \\
 & & x_1 \arrow[d] & & \\
y_2 \arrow[r] & y_1 \arrow[r] & a \arrow[r] & z_1 \arrow[r] & z_2
\end{tikzcd}
$$

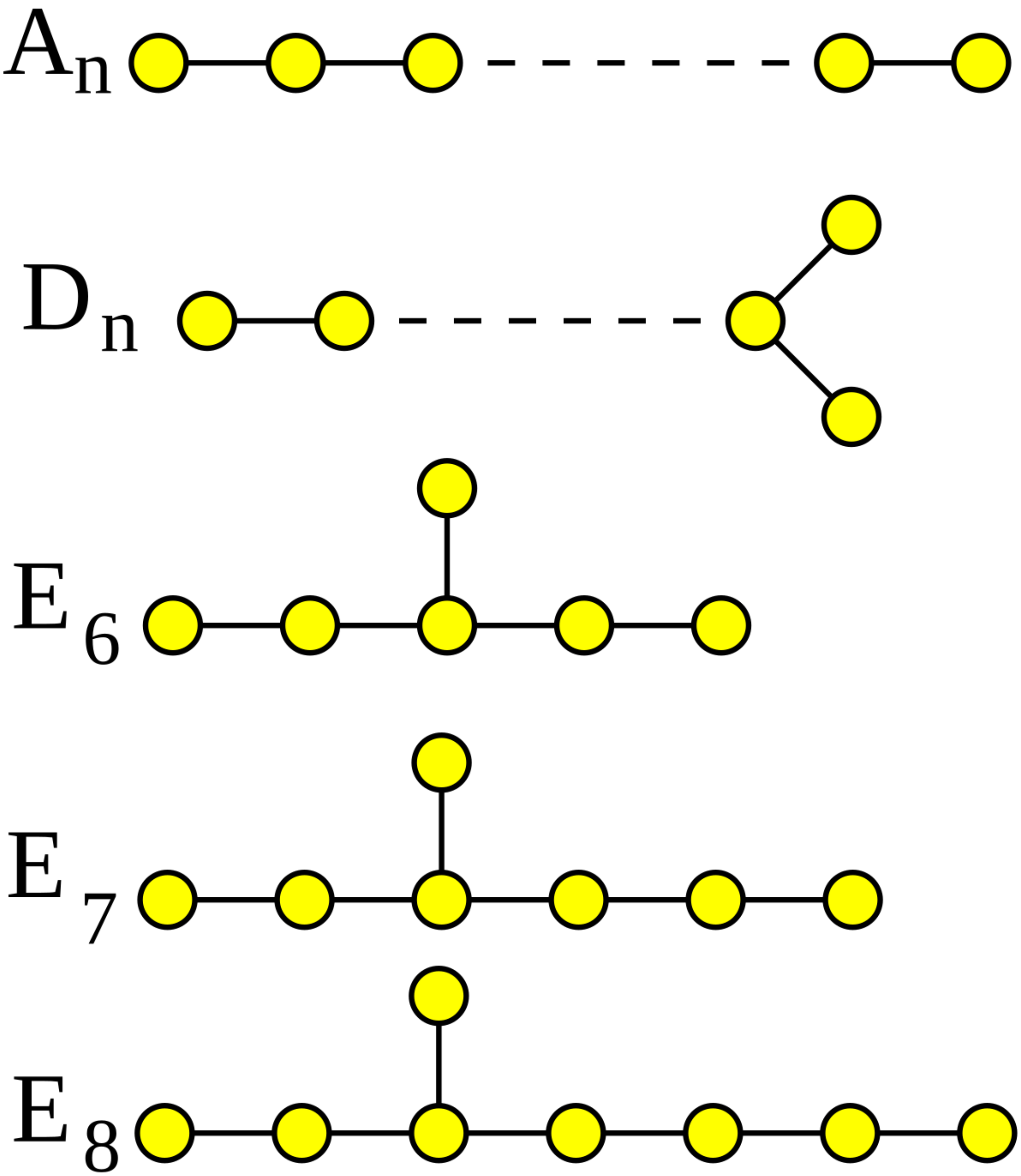


Figure 1: Dynkin diagrams of types A,D, and E.

$$\begin{tikzcd}
a \arrow[dr, dash] & & & \\
& c_1 \arrow[r, dash] & c_2 \arrow[r, dash] & \cdots \\
b \arrow[ur, dash] & & &
\end{tikzcd}$$

Let $S_1 = x_1 + y_1 + z_1, S_2 = x_2 + y_2 + z_2$. Then we have inequalities

$$\begin{aligned}
&2a > S_1 \\
&2S_1 > 3a + S_2 \\
&2S_2 > S_1 \\
\implies &4S_1 > 6a + 2S_1 > 6a + S_1 \\
\implies &3S_1 > 6a \\
\implies &S_1 > 2a \lightning .
\end{aligned}$$

If there is one branch of length $= 1$, then the other two branches cannot be both ≥ 3. If one branch is $= 1$, one branch is $= 2$, the other branch cannot be ≥ 5. To show this, we prove a lemma extending Lemma 2. For

$$c_n \text{ ——— } c_{n-1} \text{ ——— } \cdots \text{ ——— } c_1$$

$$\begin{aligned}
2c_1 &> c_2 \\
2c_2 &> c_1 + c_3 > \frac{c_2}{2} + c_3 \\
\frac{3}{2}c_2 &> c_3 \\
\frac{n}{n-1}c_{n-1} &> c_n
\end{aligned}$$

These inequalities telescope, so $(n+1)c_1 > nc_2$.

Now for

$$\begin{tikzcd}
& & & a_1 \arrow[d, dash] & & & \\
& & & \vdots \arrow[d, dash] & & & \\
& & & a_p \arrow[d, dash] & & & \\
b_1 \arrow[r, dash] & \cdots \arrow[r, dash] & b_q \arrow[r, dash] & k \arrow[r, dash] & c_1 \arrow[r, dash] & \cdots \arrow[r, dash] & c_r
\end{tikzcd}$$

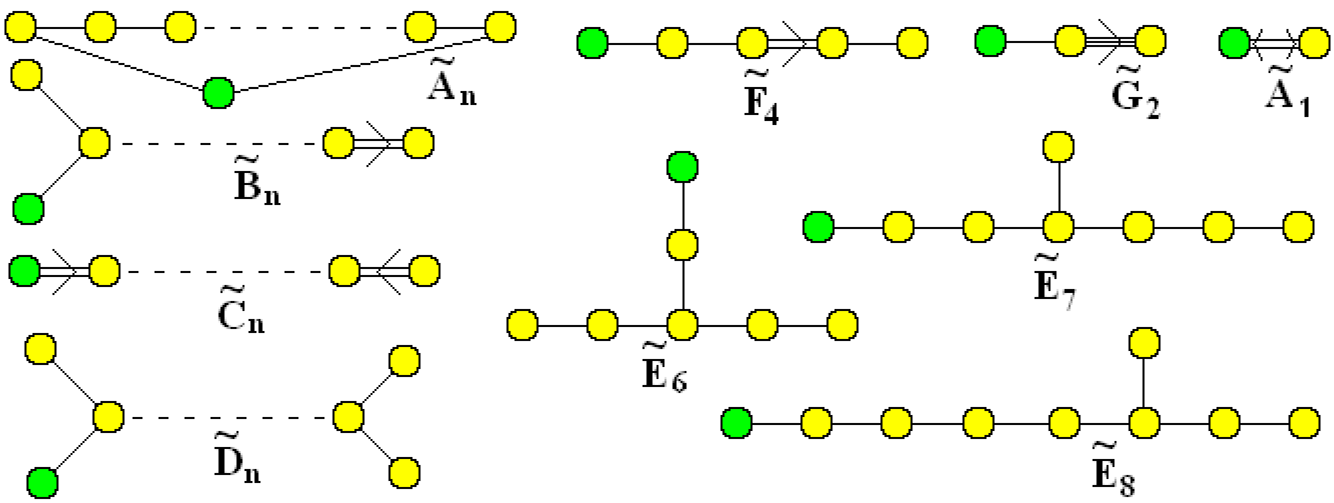


Figure 2: Affine Dynkin diagrams.

We have

$$
\begin{aligned}
2a_p &> a_{p-1} + k > \frac{p-1}{p} a_p + k && (1)\\
2b_q &> b_{q-1} + k > \frac{q-1}{q} b_q + k && (2)\\
2c_r &> c_{r-1} + k > \frac{r-1}{r} c_r + k && (3)\\
2k &> a_p + b_q + c_r && (4)\\
& && (5)
\end{aligned}
$$

Now we add them all up to obtain

$$\frac{1}{p} a_p + \frac{1}{q} b_q + \frac{1}{r} c_r > k.$$

Definition 3. A labeling is *weakly subadditive* if for each vertex we have $2f(v) \geq \sum_{\{u,v\} \text{ is an edge of the graph}} f(u)$.

Example 6.

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & & | & & \\
1 & \text{——} & 2 & \text{——} & 1 \\
 & & | & & \\
 & & 1 & &
\end{array}
$$

However, a star with the center vertex of degree 5 does not have a weakly subadditive labeling.

Exercise 2. Classify the graphs with weakly subadditive labelings.

The answer is the affine Dynkin diagrams. The proofs are very similar from the previous exercise.

Conjecture 1 ([Zam91]). *For ADE graphs, our process (T-system)*[2] *is periodic.*

It is proved by [FS95] in type A, and by [FZ02] in types ADE [3].
Next time, we will define quivers and mutations, and use directed graphs.

2 June 30

2.1 Cluster mutation

Definition 4. A quiver is a directed graph. A *mutation* at vertex z follows these steps:

- For each pair of edges $x \to z, z \to y$, replace them by $x \to y$
- reverse the direction of all edges adjacent to z.
- cancel out 2-cycles.

A mutation is an involution.

Example 7.

$$
\begin{tikzcd}
1 \arrow[r] & 4 \arrow[d] & & 1 \arrow[r] \arrow[d] & 4 \arrow[d] & & 1 \arrow[d] & 4 \arrow[l] & \\
2 \arrow[u] \arrow[d] & 5 \arrow[l] & \overset{\mu_2}{\to} & 2 \arrow[r] & 5 \arrow[ul] \arrow[dl] & \overset{\mu_4}{\to} & 2 \arrow[r] & 5 \arrow[u] \arrow[dl] & \overset{\mu_6}{\to} \text{ the same quiver up to reversal of all arrows.} \\
3 \arrow[r] & 6 \arrow[u] & & 3 \arrow[r] \arrow[u] & 6 \arrow[u] & & 3 \arrow[r] \arrow[u] & 6 \arrow[u] &
\end{tikzcd}
$$

Definition 5 (Mutation of variables). Assign x_i to the i-th vertex, the mutation of variables follow these steps:

$$\mu_i(x_i) = \frac{\prod_{i \to j} x_j + \prod_{k \to i} x_k}{x_i}$$

and $\mu_i(x_j) = x_j$ if $j \neq i$.

Example 8. Using the same quiver in Example 7, we have

$$x_2' = \frac{x_1 x_3 + x_5}{x_2}, x_4' = \frac{x_1 + x_5}{x_4}, x_6' = \frac{x_3 + x_5}{x_6}.$$

If we have a bipartite quiver, and all arrows go from $\circ$ to $\bullet$, then mutations are the same as Definition 1.

Example 9.

$$
\begin{tikzcd}
a\circ \arrow[dr] & & \\
& c\bullet & d\circ \arrow[l] \\
b\circ \arrow[ur] & &
\end{tikzcd}
$$

Then $c' = \frac{abd+1}{c}$.

[2] Why is called T-system? Maybe it comes from [Kun89]. What about Y-systems? Maybe it's because Fomin–Zelevinsky uses the Y-variables.

[3] There exists non-simply laced version of this, for type B and C. We will stick with simply laced.

Definition 6. If a bipartite quiver in which all arrows go from $\circ$ to $\bullet$ comes back to itself (as an undirected graph because all the arrows will be reversed) after mutations at all $\bullet$ or $\circ$, then we say that the quiver is *recurrent*. Such a dynamical system is called a T-*system*.

Example 10.

$$\begin{tikzcd}
\circ \arrow[r] & \bullet \arrow[d, shift left] \arrow[d, shift right] \\
\bullet \arrow[u] & \circ \arrow[l]
\end{tikzcd}$$

This quiver is not recurrent.

Question 2. Are there any other recurrent quivers periodic?

Example 11. We use the same quiver as in Example 7.

$$\begin{tikzcd}
1 \arrow[r] & 1 \arrow[d] \\
1 \arrow[u] \arrow[d] & 1 \arrow[l] \\
1 \arrow[r] & 1 \arrow[u]
\end{tikzcd}
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] \\
2 \arrow[r] & 1 \arrow[u] \arrow[d] \\
1 \arrow[u] & 2 \arrow[l]
\end{tikzcd}
\begin{tikzcd}
4 \arrow[r] & 2 \arrow[d] \\
2 \arrow[u] \arrow[d] & 6 \arrow[l] \\
4 \arrow[r] & 2 \arrow[u]
\end{tikzcd}
\begin{tikzcd}
4 \arrow[d] & 5 \arrow[l] \\
11 \arrow[r] & 6 \arrow[u] \arrow[d] \\
4 \arrow[u] & 5 \arrow[l]
\end{tikzcd}
\begin{tikzcd}
4 \arrow[r] & 5 \arrow[d] \\
11 \arrow[u] \arrow[d] & 6 \arrow[l] \\
4 \arrow[r] & 5 \arrow[u]
\end{tikzcd}
\begin{tikzcd}
4 \arrow[d] & 2 \arrow[l] \\
2 \arrow[r] & 6 \arrow[u] \arrow[d] \\
4 \arrow[u] & 2 \arrow[l]
\end{tikzcd}$$

Example 12.

$$\begin{tikzcd}
x \arrow[r] & y \arrow[d] \\
w \arrow[u] & z \arrow[l]
\end{tikzcd}
\begin{tikzcd}
x' \arrow[d] & y \arrow[l] \arrow[d] \\
w \arrow[ur] & z \arrow[l]
\end{tikzcd}
\begin{tikzcd}
x' \arrow[d] & y \arrow[l] \\
w \arrow[r] & z' \arrow[u]
\end{tikzcd}
\begin{tikzcd}
x' \arrow[r] \arrow[d] & y' \arrow[d] \\
w \arrow[r] & z' \arrow[ul]
\end{tikzcd}
\begin{tikzcd}
x' \arrow[r] & y' \arrow[d] \\
w' \arrow[u] & z' \arrow[l]
\end{tikzcd}$$

The variables get to

$$x' = \frac{y+w}{x}, \tag{6}$$

$$z' = \frac{y+w}{z}, \tag{7}$$

$$y' = \frac{(y+w)(x+z)}{xyz}, \tag{8}$$

$$w' = \frac{(y+w)(x+z)}{xyw}, \tag{9}$$

$$x'' = \frac{(y+w)(x+z)}{yzw}, \tag{10}$$

$$z'' = \frac{(y+w)(x+z)}{xyw}, \tag{11}$$

$$\tag{12}$$

At this point, you can tell that it is recurrent[4]. We have $(\mu_\bullet \mu_\circ)^3 = 1$.

We can come up with T-systems with a recurrent quiver that are not periodic.

[4]You can also observe the Laurent phenomenon here, i.e., we only have monomials in the denominator.

Example 13.

$$\begin{array}{ccc} \bullet & \rightrightarrows & \circ \\ \uparrow & & \downarrow \\ \circ & \leftleftarrows & \bullet \end{array} \quad \begin{array}{ccc} 1 & \rightrightarrows & 1 \\ \uparrow & & \downarrow \\ 1 & \leftleftarrows & 1 \end{array} \quad \begin{array}{ccc} 2 & \rightrightarrows & 1 \\ \uparrow & & \downarrow \\ 1 & \leftleftarrows & 2 \end{array} \quad \begin{array}{ccc} 2 & \rightrightarrows & 6 \\ \uparrow & & \downarrow \\ 6 & \leftleftarrows & 2 \end{array} \quad \begin{array}{ccc} 21 & \rightrightarrows & 6 \\ \uparrow & & \downarrow \\ 6 & \leftleftarrows & 21 \end{array} \quad \cdots$$

$$1 \rightrightarrows 1 \quad 2 \rightrightarrows 1 \quad 2 \rightrightarrows 5 \quad 13 \rightrightarrows 5$$

2.2 Tensor product construction

Definition 7 (Tensor product of bipartite graphs)**.** Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two bipartite graphs[5]. Them $G \otimes H$ is the graph with vertices $V = V_G \times V_H$, and edges E are

- For each $\{u, u'\} \in E_G$, and each $v \in V_H$, we have $\{(u, v), (u', v)\} \in E$.
- For each $u \in V_G$, and each $\{v, v'\} \in E_H$, we have $\{(u, v), (u, v')\} \in E$.

The direction is

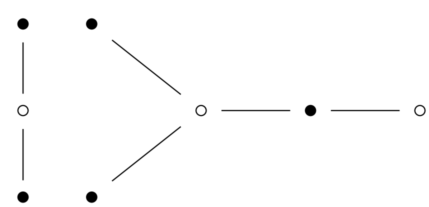
$$\begin{array}{ccc} \bullet\bullet & \longleftarrow & \bullet\circ \\ \downarrow & & \uparrow \\ \circ\bullet & \longrightarrow & \circ\circ \end{array}$$

Exercise 3. For any two bipartite graphs with directions $\circ$ to $\bullet$, the tensor product produces a recurrent quiver.

Conjecture 2 ([Zam91])**.** *If G and H are Dynkin diagrams of type A,D, or E, the tensor product $G \otimes H$ is a periodic T-system.*

For $A \otimes A$, [Vol07] showed it. For general types, [Kel13] showed it.

Example 14.



Their tensor product is a fan-like structure (c.f. Figure 3).

Question 3. Which ones are periodic?

Answer. The ones with a subadditive labeling.

[5] allow multiple edges

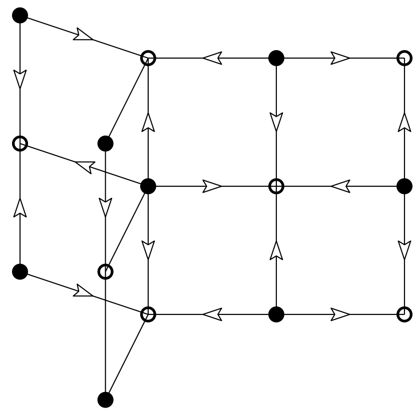
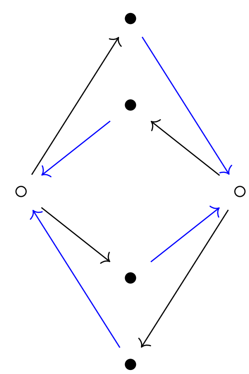


Figure 3: Tensor product of A_3 and D_5.

Example 15.



Definition 8 (Subadditive labelings for quivers)**.** A *subadditive labeling* of a quiver if a function from the vertices of that quiver to the positive real numbers such that for each vertex v we have

$$2f(v) > \sum_{u \to v} f(u),$$

and

$$2f(v) > \sum_{v \to u} f(u).$$

Theorem 2.1 ([GP20])**.** *A T-system is periodic if and only if its corresponding quivers has a subadditive labeling.*

Remark 1. Observe that the edges are either $\bullet \to \circ$ or $\circ \to \bullet$, so we have a bicoloring of the edges of the graph. By Theorem 1.1, each colored component is itself a Dynkin diagram of type A,D, or E (c.f. Figure 4).

Remark 2. Recurrence implies that for any pair of vertices u, v, the number of paths of the form $u {\color{blue}\to} w \to v$ is equal to the number of paths $u \to w {\color{blue}\to} v$.

Exercise 4. Let Q be a quiver, and let $\Gamma(Q)$ and $\Delta(Q)$ be bipartite graphs such that

- $\Gamma(Q)$ contains an undirected edge $\{u, v\}$ if $u\circ \to v\bullet$ in Q;

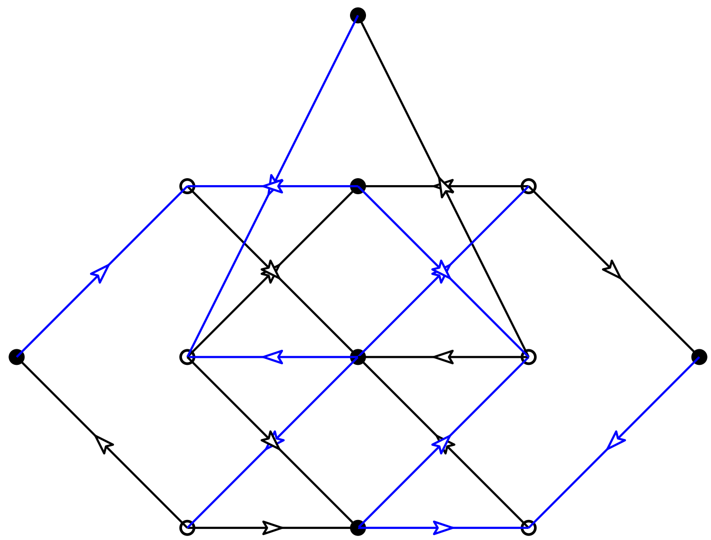


Figure 4: Bicoloring of edges.

- $\Delta(Q)$ contains an undirected edge $\{u, v\}$ if $u\bullet \to v\circ$ in Q.

Let A_Γ and A_Δ be their corresponding adjacency matrices. Prove that Q is recurrent if and only if $A_\Gamma A_\Delta = A_\Delta A_\Gamma$. To see that, consider each path of the form $i\circ \to j\bullet \to k\circ$, μ_j creates the edge $i \to k$. These accounts for exactly the 1s in $A_\Delta A_\Gamma$. However, since $\mu_\bullet$ (mutates at every black vertex) preserves the undirected graph, there must exists an ℓ such that μ_ℓ eliminates the edge $k \to i$, i.e., we have $k\circ \to \ell\bullet \to i\circ$. These accounts for exactly the 1s in $A_\Gamma A_\Delta$.

Question 4. Which pairs of Cartan matrices commute?

Answer. [Ste10].

Question 5. Which direction is harder? Commuting Cartans $\implies$ Periodicity seems harder. You can manually check the other direction.

Definition 9 (Tropical mutation)**.** A tropical mutation of x_i is the function

$$\mu_i(x_i) = \max\left(\sum_{i\to j} x_j,\ \sum_{j\to i} x_j\right) - x_i.$$

Exercise 5. Consider the $A_n \otimes A_m$ tropical system with the initial condition that is equal to 1 at one vertex and 0 elsewhere. Prove that this is periodic. If we mutate at the neighbors of 1, then the 1 will propagate until it hits the borders. The border will propagate 0s. At some point we have to mutate an isolated 1 to obtain -1, and then another propagation begins.

Definition 10 (Twist)**.** Take a bipartite graph G and its copy G'. The *twist* of G is defined on $G \cup G'$ with additional edges:

- For any pair $i \to j$ in G, create $j' \to i$;
- For any pair $i' \to j'$ in G', create $j \to i'$.

Exercise 6. Prove that T-systems for ADE twists are periodic. Quiver mutation is automated on Keller's website. We define a bijection between simple reflections in the corresponding Coxeter group and sequences of mutations, i.e., $s_i \mapsto \mu_i \mu_{i'}$. We verify on a small example that the simple reflections satisfy the braid relations and are involutions. Then we realize that periodicity requires that the Coxeter element $(s_1 s_3 s_5 \cdots s_2 s_4 s_6 \cdots)$ has finite order, which is automatic because we are dealing with finite groups.

3 July 1

3.1 Volkov's argument

Volkov proved that $A \otimes A$ is periodic.

The *Grassmannian* is the variety parametrizing k-dimensional subspaces of n-dimensional space. A point in $G(2,4)$ can be written as a $k \times n$ matrix up to SL_k. The $k \times k$-minors of such a matrix form the *Plücker coordinates* which satisfy the *Plücker relations*.

Example 16. Consider $G(2,4)$.

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}$$

We can check that

$$\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}.$$

Set up the initial cluster using the zig-zag triangulation of the n-gon, shown in Figure 5.

Example 17. Let $n = 8, k = 4$. The mutable part will look exactly like $A \otimes A$. There are exactly 8 frozen variables given by consecutive indices. The PLücker coordinates are given by $\Delta_{[i,i+t]\cup[j,j+t']}$ where $t + t' = k - 2$. From here, mutations of all vertices on a diagonal flips that diagonal.

$$\Delta_{1348}\Delta_{2378} = \Delta_{1238}\Delta_{3478} + \Delta_{2348}\Delta_{1378}.$$

which means that Δ_{2378} mutates to Δ_{1348}. If you mutate randomly, you can arrive at non-Plücker variables. Now $\mu_\bullet$ is a rotation of this zig-zag, and rotation is periodic, so we obtain periodicity for free. See [Sco06, Pos06].

Lemma 3. *Initial cluster variables that we have chosen are algebraically independent.*

Otherwise, we could have periodicity for a special subset of cluster variables, which does not hold in general.

3.2 Weakly subadditivity

Question 6. How do we generalize the notion of periodicity to weakly subadditive labelings? (not periodic in general).

Example 18.

$$\bullet \Longrightarrow \circ$$

$$x \qquad\qquad y$$

$$\frac{1+y^2}{x} \qquad\qquad y$$

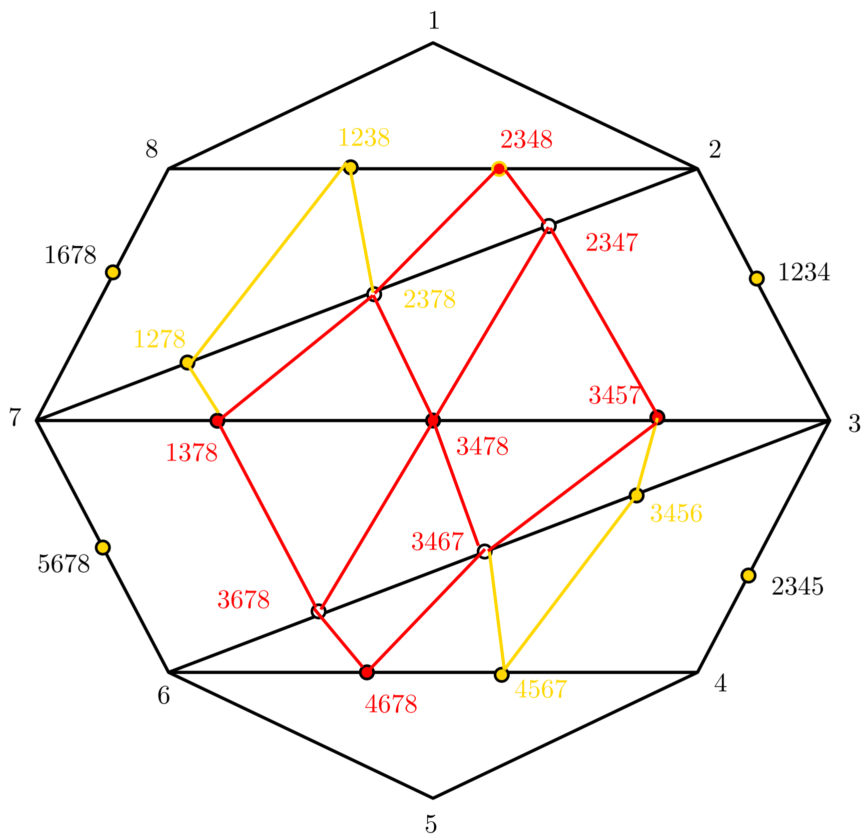


Figure 5: The zig-zag triangulation for $G(4,8)$. The mutable part is colored in red.

If we start with 1,1, then we obtain every *other* Fibonacci numbers. The x-vars satisfy

$$x_i - 7x_{i-1} + x_{i-2} = 0.$$

If we consider both of them, then

$$z_i - 3z_{i-1} + z_{i-2} = 0$$

Proof. Let $z_1 = x, z_2 = y, z_3 = \frac{1+y^2}{x}, \ldots$. We claim that $f(x, y) = \frac{x^2+y^2+1}{xy} =: A$ is a *conserved quantity*, i.e., $f(z_1, z_2) = f(z_2, z_3) = \cdots$. Compute

$$f(y, \frac{1+y^2}{x}) = \frac{y^2 + \left(\frac{1+y^2}{x}\right)^2 + 1}{y \cdot \frac{1+y^2}{x}} \tag{13}$$

$$= \frac{x^2y^2 + y^4 + 2y^2 + 1 + x^2}{xy(1+y^2)} \tag{14}$$

$$= \frac{x^2 + y^2 + 1}{xy}. \tag{15}$$

We can check that

$$z_i - Az_{i-1} + z_{i-2} = 0$$

always holds. $\square$

The linear relation tells us that, roughly, $z_i \approx K\lambda^i$, and $z_i z_{i-2} \approx z_{i-1}^2$

Example 19.

$$\bullet \Rrightarrow \circ$$

We have $w_i w_{i-2} \approx w_{i+1}^3$, but then we cannot have $w_i \approx K\lambda^i$, so there cannot be a linear relation among them.

Definition 11. Call a bipartite T-system *Zamolodchikov integrable* if each vertex gives a linearizable sequence.

Theorem 3.1 ([GP20]). *A Zamolodchikov integrable T-system has to be affine $\boxtimes$ finite.*

Conjecture 3. *Take $G \otimes H$ where G is A, D, E and H is $\tilde{A}, \tilde{D}, \tilde{E}$. The T-system of $G \otimes H$ is Zamolodchikov integrable.*

Known in $A \otimes A$.

Question 7. Give explicit formulas for the conserved quantities in these systems.

Question 8. What happens for affine $\boxtimes$ affine? See [GP19].

Exercise 7. Assume $\{x_i\}$ and $\{y_i\}$ satisfy known linear recurrences. What linear recurrence is satisfied by $\{x_{2i}\}, \{x_i + y_i\}, \{x_i y_i\}$. Do the case of

$$x_{i+2} - Ax_{i+1} + x_i = 0 \tag{16}$$

$$y_{i+2} - By_{i+1} + y_i = 0 \tag{17}$$

Let t_1, t_2 be the roots of $t^2 - At + 1 = 0$. Let λ_1, λ_2 be the roots of $t^2 - Bt + 1 = 0$. Then $x_i = c_1 t_1^i + c_2 t_2^i$ and $y_i = d_1 \lambda_1^i + d_2 \lambda_2^i$ for some constants c_1, c_2, d_1, d_2.

Then $x_{2i} + c_1 t_1^{2i} + c_2 t_2^{2i}$, so the linear recurrence comes from a polynomial with roots t_1^2, t_2^2. We have $t_1 + t_2 = A, t_1 t_2 = 1$, so $t_1^2 + t_2^2 = A^2 - 2$. Therefore, $t^2 - (A^2-2)t + 1 = 0$ has roots t_1^2, t_2^2, and $x_{2(i+2)} - (A^2-2)x_{2(i+1)} + x_{2i} = 0$.

Similarly, the sequence $\{x_i + y_i\}$ has a linear recurrence given by a quartic polynomial with roots $t_1, t_2, \lambda_1, \lambda_2$, which is $(t^2 - At + 1)(t^2 - Bt + 1) = 0$; the sequence $\{x_i y_i\}$ has a linear recurrence given by a quartic polynomial with roots $t_i \lambda_j$ for $i, j \in \{1, 2\}$.

Exercise 8.

$$\begin{tikzcd}
 & c & \\
a \arrow[ur] \arrow[dr] & & b \arrow[ul] \arrow[dl] \\
 & d &
\end{tikzcd}$$

What is the conserved quantity for this quiver? Show that this T-system is linearizable.

$$\begin{tikzcd}
 & 1 & & & 1 & & & 5 & \\
1 \arrow[ur] \arrow[dr] & & 1 \arrow[ul] \arrow[dl] & 2 \arrow[ur] \arrow[dr] & & 2 \arrow[ul] \arrow[dl] & 2 \arrow[ur] \arrow[dr] & & 2 \arrow[ul] \arrow[dl] \\
 & 1 & & & 1 & & & 5 &
\end{tikzcd}$$

We realize that we see Fibonacci numbers, so we guess out that the conserved quantity is $H = \frac{(ab+cd+1)^2}{abcd}$. We can prove it for mutations at a and b.

$$\begin{tikzcd}
 & c \arrow[dl] \arrow[dr] & \\
\frac{1+cd}{a} & & \frac{1+cd}{b} \\
 & d \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Check

$$H = \frac{\left(\frac{(1+cd)^2}{ab} + cd + 1\right)^2}{\frac{(1+cd)^2}{ab} cd}.$$

We have the linear recurrence

$$a_{n+2} - (H-2)b_{n+1} + a_n = 0.$$

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