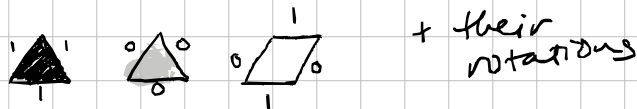
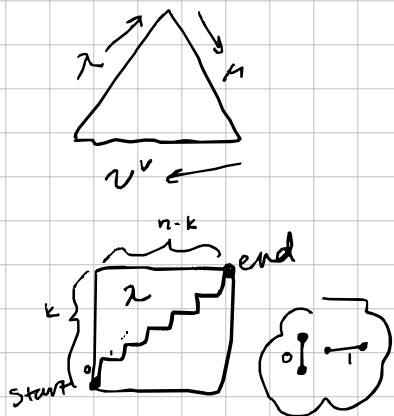


# Puzzles, Saturation, + Vanishing

Different rules for LR-coeffs exhibit different geometric properties

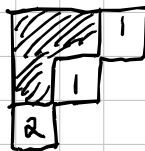
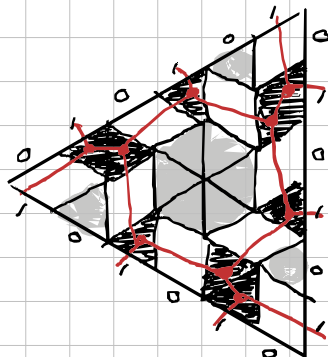
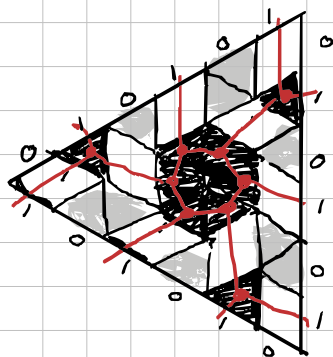
Another ex: Knutson-Tao introduce a new rule in terms of puzzles, i.e. tilings of an  $n \times n \times n$  triangular grid labelled by binary strings representing  $\pi, \mu, \nu$ .  
The tiling uses pieces



Ex) for  $\pi = (2, 1)$   $\mu = (2, 1)$   $\nu = (3, 2, 1)$  in  $Gr(3, 6)$

We saw

$C_{\pi, \mu}^{\nu} = 2$  using tableaux.  
These correspond to puzzles



Let's talk about another fundamental problem:  
the Hermitian eigenvalue problem

Recall: an  $n \times n$  matrix  $A$  <sup>over  $\mathbb{C}$</sup>  is Hermitian if it is self-adjoint, i.e. equal to its conjugate-transpose

ex)  $\begin{pmatrix} 3 & 1-i & -7i \\ 1+i & 0 & 3i \\ -7-i & -3 & i \end{pmatrix}$  is Hermitian

**Problem** Consider Hermitian matrices  $A, B, C$  s.t.  $A+B=C$ , where  $\alpha, \beta, \gamma$  are their respective eigenvalues. Then how must  $\gamma$  relate to  $\alpha, \beta$ ?

$\alpha, \beta, \gamma$  are the largest eig. of  $A, B, C$  respectively.

Observation 1: we know  $\text{tr}(A+B) = \text{tr}(C)$   
"trace eq"  $\sum_{i=1}^n \alpha_i + \beta_i = \text{tr}(A) + \text{tr}(B) = \sum_{i=1}^n \gamma_i$

But there must be more!

**Thm** (Spectral Thm) Unitary matrix  $U$  s.t.  $UAU^* = \alpha \text{In} = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$ , i.e.  $U^{-1} = U^*$   
 $A$  can be diag in some ON basis.

Using this ON basis, we can prove

$$\begin{cases} \gamma_1 = \alpha_1 + \beta_1 \\ \gamma_n = \alpha_n + \beta_n \end{cases}$$

Exercise: prove it!

We can generalize these ideas into a "minmax principle" by Weyl (1912)

From this, similar as above, we can obtain  
 $\delta_{i+j-1} \leq \alpha_i + \beta_j$ ,  $i+j-1 \leq n$

More and more ineq discovered. But which fully characterize the problem?

Horn (Horn) (1962)

$$\sum_{k \in K} \delta_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \quad (*)$$

where  $I, J, K \subseteq [n]$  s.t.  $|I| = |J| = |K| = r \in \mathbb{Z}_{\geq 0}$   
 s.t.  $(I, J, K) \in T_r^n$  some set.

Say  $(I, J, K)$  is admissible if  $(*)$  holds.

Horn conjectured that the following set  $T_r^n$  below is admissible + these characterize the eigenvalue problem.

(see  $I = (i_1 < i_2 < \dots < i_r)$ )

$T_r^n$  is described inductively:

For  $r=1$ :  $(I, J, K) \in T_1^n$  if  $K_1 = i_1 + j_1 + 1$ .

For  $r > 1$ :  $(I, J, K) \in T_r^n$  if

$$\sum_{s=1}^r K_s + \binom{r+1}{2} = \sum_{s=1}^r i_s + \sum_{s=1}^r j_s$$

and  $\forall 1 \leq p \leq r-1$  +  $(I', J', K') \in T_p^n$ ,

$$\sum_{k' \in K'} K_{k'} + \binom{p+1}{2} = \sum_{i' \in I'} i_{i'} + \sum_{j' \in J'} j_{j'}$$

Don't

By the structure of  $\text{Tr}^n$ , if true, this shows that the eigenvalues obey a natural natural convexity property + are characterized by it.

Using min max principle + orthogonal projections, results of Weilandt '80 + Hersch-Zwahlen '62 give that if  $u, v, w$  are flags determined by the ON bases and we suppose

$$L \in X_{\tilde{I}}(u.) \cap X_{\tilde{J}}(v.) \cap X_k(w.) \neq \emptyset \quad (*)$$

$$\Rightarrow \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \geq \sum_{k \in K} \gamma_k, \quad \text{so as noted by Thompson '88 Helmske-Rosenthal '95}$$

(\*) holds for  $(\tilde{I}, \tilde{J}, k) \Rightarrow (I, J, k)$  admissible

Work of many show that the vanishing of  $c_{n,1}^u$  control the Horn problem.

Flyachko<sup>198</sup> produced another set of inequalities + showed that they imply Horn's + characterize the problem **IF** the following holds:

Theorem (Knutson-Tao '99)

Conjecture (Saturation)

Connects deeply to representation theory of unitary group of "quantum vs classical"

$$c_{n,1}^u \neq 0 \Leftrightarrow \text{for all } N \geq 1, c_{N,2,N+1}^{N,u} \neq 0$$

Ex  $u = (3, 2)$   
 $3-u = (9, 6)$

But we have a combinatorial rule for this! How hard can it be?

( $\Rightarrow$ ) not too bad

( $\Leftarrow$ ) Hard! (How to know we can uniformly "shrink" a tableau?)

What to do?

Perhaps using another POV, the answer will be more clear?

Idea: Knutson-Tao use a "dual" object of hives for their puzzles which exhibit this scaling naturally.

Why else might we care about saturation?

Q: How efficiently can we tell if  $c_{\lambda\mu}^{\nu} > 0$ ?

easy to check given some evidence

(can give me an lattice SSYT that I can check is counted by  $c_{\lambda\mu}^{\nu}$ )  
 $\Rightarrow$  the problem is in NP

how easy to tell from scratch?

(can we decide in polynomial time?)

$\hookrightarrow$  Yes (DeLoera-McAllister,<sup>12</sup>  
Nagayama-Mulmuley-Schoni

Idea:

Encode set of lattice tableaux + determine if this set is empty

Let  $r_{iik}(T) = \#$  k's in row i of  $T \in \text{SSYT}(\nu/\lambda, \mu)$   
Take  $n = \max(\ell(\nu), \ell(\mu))$ .

Ex)  $T =$

|   |   |   |   |   |
|---|---|---|---|---|
|   |   |   | 1 | 1 |
|   |   | 1 | 2 |   |
| 1 | 2 |   |   |   |
| 3 |   |   |   |   |

$(r_{ik}) =$

|   |   |   |   |
|---|---|---|---|
| 2 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 |

We'll need:

- non-negative:  $r_{ik} \geq 0 \quad \forall i, k \in [n]$
- shape:  $\pi_i + \sum_k r_{ik} = v_i \quad \forall i \in [n]$
- content:  $\sum_i r_{ik} = \mu_k \quad \forall k \in [n]$
- semistd:
  - $r_{ik} = 0$  if  $i < k$
  - $\pi_{i+1} + \sum_{k' \leq k} r_{i+1, k'} \leq \pi_i + \sum_{k' < k} r_{i, k'}, i, k \in [n]$
- lattice:  $\sum_{i' < i} r_{i'k} \geq r_{i, k+1} + \sum_{i' < i} r_{i', k+1} \quad \forall i, k \in [n]$

Define  $P_{\pi, \mu}^v = \{(r_{ik}) \in \mathbb{R}^n \mid r_{ik} \text{ satisfy } (*) \forall i, k \in [n]\}$

Claim:  $P_{\pi, \mu}^v \cap \mathbb{Z}^{n^2} \neq \emptyset \Leftrightarrow c_{\pi, \mu}^v = 0$   
 PF: Exercise

Then to show  $c_{\pi, \mu}^v \neq 0$  it suffices to find an integer point in  $P_{\pi, \mu}^v$

How hard could it be?

In general, this is called Integer Linear Programming which is hard?

Can we get around this?

Are there special properties of this system?

We see all coeffs + constants are integers

$\therefore P_{\pi, \mu}^v \neq \emptyset \stackrel{\text{even}}{\Leftrightarrow} P_{\pi, \mu}^v$  contains a rational vertex

$\{N\alpha \mid \alpha \in P_{\pi, \mu}^v\}$  dilation

exercise:  $N \cdot P_{\pi, \mu}^v = P_{N\pi, N\mu}^{Nv} \Leftrightarrow N \cdot P_{\pi, \mu}^v \cap \mathbb{Z}^{n^2} \neq \emptyset$  for some  $N \in \mathbb{Z}_{>0}$

$\Leftrightarrow P_{N\pi, N\mu}^{Nv} \cap \mathbb{Z}^{n^2} \neq \emptyset$  for some  $N \in \mathbb{Z}_{>0}$

$\Leftrightarrow c_{N\pi, N\mu}^{Nv} \neq 0$  for some  $N \in \mathbb{Z}_{>0}$

saturation  $\Rightarrow c_{\pi, \mu}^v \neq 0$

$\therefore$  We just have to check if  $P_{\pi, \mu}^v \neq \emptyset$  (find if there is ANY solution to this system of eqns)

This is a Linear Programming Problem

$\Rightarrow$  can use ellipsoid method to decide if  $c_{\pi, \mu}^v = 0$ .

