CMND 2026 Schubert Calculus

Lecture 3 Problem Set: Saturation and Vanishing

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1. Prove that $\mathcal{P}_{\lambda,\mu}^{\nu} \cap \mathbb{Z}^{n^2} \neq \emptyset$ if and only if $c_{\lambda,\mu}^{\nu} > 0$.

2. Compute $c_{\lambda,\mu}^{\nu}$ for $\lambda = (2,1)$, $\mu = (2,1,1)$, and $\nu = (3,2,2)$ using both the puzzle rule and the lattice rule.
   Challenge: Can you find a bijection between the puzzles and the lattice tableaux that works in general?

3. Prove that $c_{\lambda,\mu}^{\nu} > 0$ implies that $c_{N\lambda,N\mu}^{N\nu} > 0$ for each $N > 0$.

4. Consider $\mathcal{P}_{\lambda,\mu}^{\nu}$.
   (a) What does a rational vertex of $\mathcal{P}_{\lambda,\mu}^{\nu}$ represent with respect to the system of equations defining $\mathcal{P}_{\lambda,\mu}^{\nu}$?
   (b) Prove that $\mathcal{P}_{\lambda,\mu}^{\nu} = \emptyset$ if and only if $\mathcal{P}_{\lambda,\mu}^{\nu}$ contains a rational vertex.
   (c) Prove that $N\mathcal{P}_{\lambda,\mu}^{\nu} = \mathcal{P}_{N\lambda,N\mu}^{N\nu}$.

5. Say a tableaux T is row lattice if, when reading the entries in T from right to left starting in the first row and working down to the last row, we have always read weakly more i's than $i+1$'s for all $i > 0$. Say a tableaux T is column lattice if, when reading the entries in T from top to bottom starting in the rightmost column and working leftwards to the first column, we have always read weakly more i's than $i+1$'s for all $i > 0$. Prove that T is row lattice if and only if T is column lattice.

6. Let T be a semistandard Young tableau and consider an integer $k > 0$. We say an entry i in T is paired if $i = k$ and there is a box containing $k+1$ directly below it OR if $i = k+1$ and there is a box containing k directly above it. Let σ_k be the operator on tableaux defined by the following:

   - If $i \notin \{k, k+1\}$ OR i is paired, then σ_k leaves i the same.
   - Then if a row r has i unpaired k's and j unpaired $k+1$'s, replace those entries with j k's and i $k+1$'s. That is :

   $$\sigma_3\left(\boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{3}\boxed{4}\boxed{5}\right) = \boxed{1}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{4}\boxed{5}.$$

   Now let's think about σ_k.

(a) Apply σ_3 to T below. What is the weight of $\sigma_3(T)$?

$$T = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ \hline 2 & 3 & 3 & 4 & 4 & 5 & 5 \\ \hline 4 & 4 & 5 & 5 & 5 & 6 \\ \cline{1-6} 5 & 6 & 7 \\ \cline{1-3} \end{array}.$$

(b) Prove that if $T \in \mathsf{SSYT}(\lambda, \mu)$ is semistandard, so is $\sigma_k(T)$.

(c) Suppose $T \in \mathsf{SSYT}(\lambda, \mu)$. Describe the weight of $\sigma_k(T)$.

(d) Use the above steps to Schur functions s_λ are symmetric using their tableaux-theoretic definition.

7. Let $T \in \mathsf{SSYT}(\lambda)$. Define $T|_{\geq j}$ denote the tableau obtained by restricting T to its entries weakly greater than j. Similarly define $T|_{>j}$. We'll walk through the proof of Stembridge that

$$a_{\lambda+\delta_n} s_\mu(x_1, x_2, \ldots, x_n) = \sum_T a_{\lambda+\delta_n+\text{weight}(T)}, \tag{1}$$

where the sum is over $T \in \mathsf{SSYT}(\mu)$ such that $\lambda + \mathsf{wt}(T|_{\geq j})$ is a integer partition for all $j \geq 1$. Warning: In this proof, we prove the result for Schur *polynomials* rather than Schur functions. We define the Schur polynomial as

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_T x^{\text{weight}(T)},$$

where we sum over $T \in SSYT(\lambda)$ whose entries are in the set $[k] = \{1, \ldots, k\}$.

(a) Prove that

$$a_{\lambda+\delta_n} s_\mu(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathsf{SSYT}(\mu)} a_{\lambda+\delta_n+\text{weight}(T)}.$$

(b) Say $T \in \mathsf{SSYT}(\mu)$ is bad if $\lambda + \text{weight}(T|_{\geq j})$ is not an integer partition for some $j \geq 1$. If T is bad, pick the largest j then smallest k that produces the "bad" property. Let T^* denote the tableaux formed by applying σ_k to $T|_{\geq j}$, and leaving the rest of the entries in T unchanged. Prove T^* is semistandard.

(c) Prove that if T is bad, then T^* is bad and $(T^*)^* = T$.

(d) Prove that if T is bad, then

$$a_{\lambda+\delta_n+\text{weight}(T)} = -a_{\lambda+\delta_n+\text{weight}(T^*)}.$$

(e) Use the above parts to prove Equation (1).

(f) Use Equation (1) to give a combinatorial rule for $c^\nu_{\lambda,\mu}$:

$$s_\lambda(x_1, x_2, \ldots, x_n) s_\mu(x_1, x_2, \ldots, x_n) = \sum_\nu c^\nu_{\lambda,\mu} s_\nu(x_1, x_2, \ldots, x_n).$$