# Webs and alternating sign matrices

## Lecture 1: Alternating sign matrices and their beautiful bijections

CMND Summer School

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# Alternating sign matrices

## Definition

**Alternating sign matrices** (ASM) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

# Examples of alternating sign matrices

- All seven of the $3 \times 3$ ASMs.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Two of the forty-two $4 \times 4$ ASMs.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

42 years of alternating sign matrices

https://sites.google.com/view/42-years-of-asms

# What does a large random ASM look like?

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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\end{pmatrix}$$

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\begin{pmatrix}
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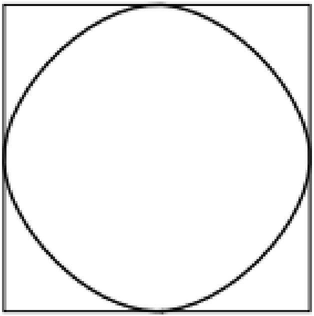
\end{pmatrix}
$$

# What does a large random ASM look like?

Colomo and Pronko (2010) / Aggarwal (2020) found the limit shape of the boundary between the *temperate* (a mixture of $\{0, 1, -1\}$) and *frozen* (all zeros) regions in a uniformly random ASM.



# Alternating sign matrix enumeration

**Theorem (Zeilberger 1996; Kuperberg 1996)**

*$n \times n$ alternating sign matrices are counted by:* $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$.

1, 2, 7, 42, 429, 7436, 218348, 10850216, ...

This was conjectured by Mills, Robbins, and Rumsey (1983) and proved in different ways by Zeilberger (1996), Kuperberg (1996), Fischer (2006), and Fischer-Kovonlinka (2020/2022).

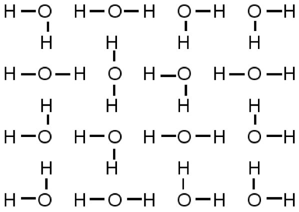
**Open Problem**

*Find a statistic on ASMs with generating function* $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!_q}{(n+j)!_q}$.

# Physics connection - Square ice

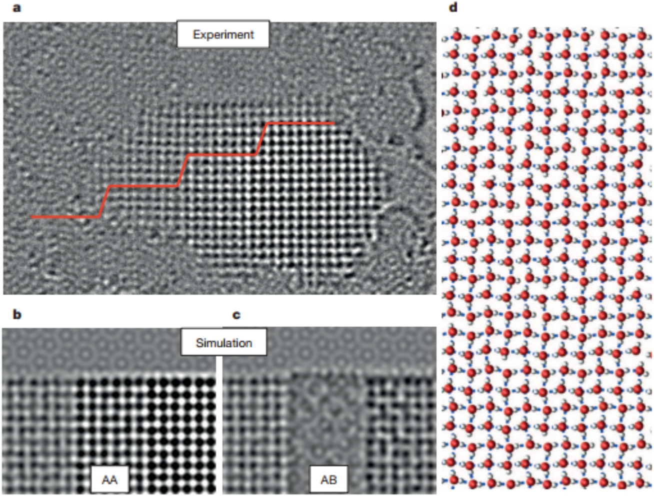
Kuperberg's enumeration proof relied on a beautiful bijection with configurations of the **square ice model of statistical physics.**

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



# Square ice in graphene nanocapillaries

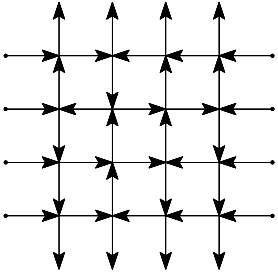
G. Algara-Siller, O. Lehtinen, F. C. Wang, R. R. Nair, U. Kaiser, H. A. Wu, A. K. Geim & I. V. Grigorieva

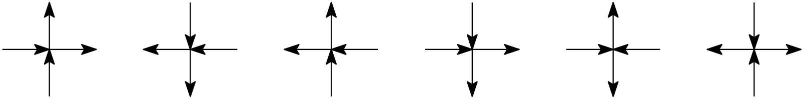


# Physics connection - Square ice

Kuperberg's enumeration proof relied on a beautiful bijection with configurations of the **six-vertex model with domain wall boundary conditions.**

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$





0 0 0 0 1 -1

# Known alternating sign matrix bijections

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

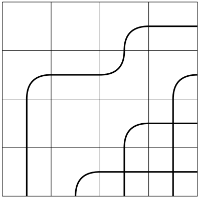
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

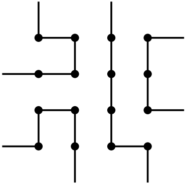
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Bumpless pipe dream



Fully-packed loop



Corner sum matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

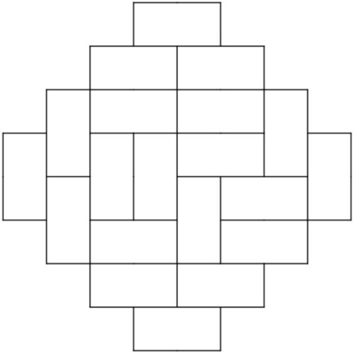
# Alternating sign matrix statistics

The **number of negative ones** $N(A)$ yields a nice *two-enumeration*:

**Theorem (Elkies, Kuperberg, Larsen, Propp 1992)**

*The two-enumeration* $\sum_{A \in ASM_n} 2^{(\text{number of } -1\text{'s in } A)}$ *equals* $2^{\binom{n+1}{2}}$.

*This is the number of domino tilings of the Aztec diamond.*



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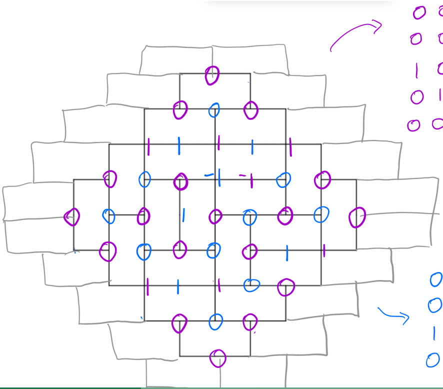

$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

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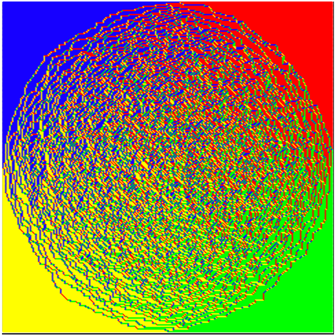

# Alternating sign matrices and domino tilings

**Theorem (Mills, Robbins, Rumsey 1992)**

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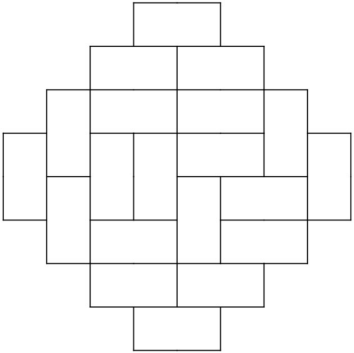

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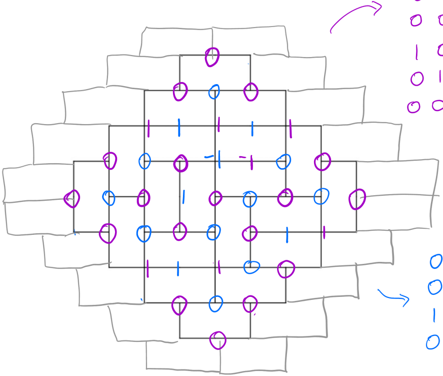


$$\begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}$$

# Alternating sign matrix statistics

The **positive inversion number** of an ASM $A$ is:

$$\nu(A) = \sum_{1 \le i < k < n} \sum_{1 \le \ell \le j \le n} A_{ij} A_{k\ell}.$$

This also equals $\mathrm{NW}(A)$, defined as:

- the number of  vertices in the corresponding six-vertex configuration, or
- the number of entries $A_{ij} = 0$ whose first nonzero entry to their right is a 1 and the first nonzero entry below is a 1.

Define the **weight** of an ASM to be $\mathrm{wt}(A) := \prod_{(i,j) \in \mathrm{NW}(A)} x_i$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \mathrm{wt}(A) = x_1^2 x_3.$$

# Patterns in alternating sign matrices

First way to generalize permutation patterns:

**Definition (Johansson-Linusson, 2007)**

An alternating sign matrix $A \in ASM(n)$ **classically avoids** a pattern $p \in S_m$ if there is no submatrix in $A$ with ones in the same positions as in $p$.

$\mathrm{Av}_n(\sigma)$ denotes the set of $n \times n$ ASMs that classically avoid $\sigma$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$ contains 132 and 231 but classically avoids 123.

Results of [Johansson-Linusson, 2007]:

- $\mathrm{Av}_n(132)$ is given by the large Schröder numbers: $1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots$
- $\mathrm{Av}_n(132, 123)$ is given by every second Fibonacci number.

# eXceptional patterns with eXponential growth

Let $X := \{1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412\}$.

**Theorem (Bouvel, Egge, Smith, S., Troyka 2026)**

*If a permutation $\sigma \in X$, then there exists $c$ such that $|\operatorname{Av}_n(\sigma)| \leq c^n$ for all $n$. Otherwise, $|\operatorname{Av}_n(\sigma)| \geq \lfloor n/3 \rfloor!$ for all $n$.*

This is in contrast to the case of pattern avoidance in permutations (or more generally, $0-1$ matrices), which has an exponential upper bound for each pattern avoidance class [Marcus and Tardos 2004].

# Key pattern avoidance in alternating sign matrices

Second way to generalize permutation patterns:

**Definition**

An alternating sign matrix $A \in ASM(n)$ **key-avoids** $\sigma \in S_m$ if: the permutation called the key associated to $A$ avoids $\sigma$.

$\mathrm{Av}_n^{\mathrm{key}}(\sigma)$ denotes the set of $n \times n$ ASMs that key-avoid $\sigma$.

Lascoux associated to each ASM a permutation, its *key*.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \to \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

# Computing the southwest key of an ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

# Computing the southwest key of an ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ \boxed{1} & \boxed{0} & \boxed{0} & \boxed{-1} & 1 \\ \boxed{0} & \boxed{0} & \boxed{1} & \boxed{0} & 0 \\ 0 & 0 & \boxed{0} & \boxed{1} & 0 \end{pmatrix}$$

# Computing the southwest key of an ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0} & 1 \\ \boxed{1} & \boxed{0} & \boxed{0} & \boxed{0} & 0 \\ 0 & 0 & \boxed{1} & \boxed{0} & 0 \end{pmatrix}$$

# Computing the southwest key of an ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & \boxed{-1} & 1 & 0 \\ 0 & \boxed{0} & \boxed{0} & 0 & 1 \\ 1 & \boxed{0} & \boxed{0} & 0 & 0 \\ 0 & \boxed{0} & \boxed{1} & 0 & 0 \end{pmatrix}$$

# Computing the southwest key of an ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{0} & \boxed{0} & 1 & 0 \\ 0 & \boxed{0} & \boxed{0} & 0 & 1 \\ 1 & \boxed{0} & \boxed{0} & 0 & 0 \\ 0 & \boxed{1} & \boxed{0} & 0 & 0 \end{pmatrix}$$

# Key-avoidance in alternating sign matrices

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 2 & & 4 & & 5 & & \\ & 1 & & 3 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

**Proposition (Ayyer, Cori, Gouyou-Beauchamps 2011)**

*A permutation avoids the pattern* 312 *if and only if its associated monotone triangle is gapless.*

**Theorem (Bouvel, Smith, S. 2025)**

312-*key-avoiding* $n \times n$ *ASMs are in bijection with gapless monotone triangles of order* $n$.

So key-avoidance generalizes the 312-avoidance of [ACGB 2011] to all patterns!

# 321 and 312 key-avoiding ASMs

**Theorem (Bouvel, Smith, S. 2025)**

*The class of $n \times n$ ASMs key-avoiding both 312 and 321 is in bijection with gapless monotone triangles of size $n$ with at most two distinct values in any column. Then,* $|\mathrm{Av}_n^{\mathrm{key}}(312, 321)| = C_n := \frac{1}{n+1}\binom{2n}{n}$.

$$\begin{array}{ccccc}
2 & & & & \\
1 & 3 & & & \\
1 & 3 & 4 & & \\
1 & 2 & 3 & 5 & \\
1 & 2 & 3 & 4 & 5
\end{array}$$

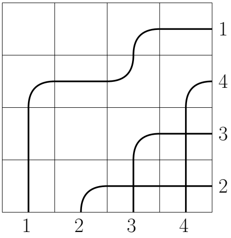
# Bumpless pipe dreams

**Definition (Lam, Lee, Shimozono 2021)**

A **bumpless pipe dream** (BPD) of size $n$ is a tiling of an $n \times n$ grid by tiles: , , , , , , such that $n$ pipes traveling from the south border to the east border are formed. A BPD is **reduced** if no two pipes cross twice.

Define the **weight** of a reduced bumpless pipe dream $D$ as $\mathrm{wt}(D) := \prod_{(i,j)\in \mathrm{blank}(D)} x_i$. Define the permutation $\pi$ by reading the order of the pipes down the right side.

$\pi = 1432$



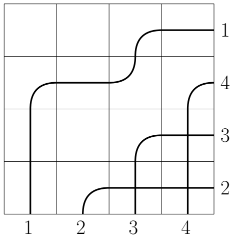

$\mathrm{wt}(D) = x_1^2 x_3$

# Bumpless pipe dreams

On the corresponding ASM $A$, the weight is $\mathrm{wt}(A) := \prod_{(i,j) \in \mathrm{NW}(A)} x_i$ where $\mathrm{NW}(A)$ is the set of positions of the entries $A_{ij} = 0$ whose first nonzero entries to the right and below equal 1.

The permutation of the BPD is the NW-key of the ASM.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



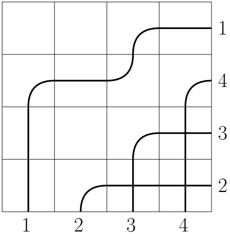

$$\mathrm{wt}(A) = \mathrm{wt}(D) = x_1^2 x_3$$

# Bumpless pipe dreams

**Theorem (Lam, Lee, Shimozono 2021)**

*Reduced BPDs with permutation $\pi$ give a combinatorial expansion of the Schubert polynomial:* $\mathfrak{S}_\pi = \sum_{D \in \mathrm{BPD}^{red}(\pi)} \mathrm{wt}(D)$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$




$$\mathfrak{S}_{1432} = \cdots + x_1^2 x_3 + \cdots$$

**Theorem (Lascoux 2002 / Weigandt 2021)**

*Reduced alternating sign matrices with NW-key $\pi$ give an expansion of the Schubert polynomial:* $\mathfrak{S}_\pi = \sum_{A \in \mathrm{ASM}^{red}(\pi)} \mathrm{wt}(A)$.

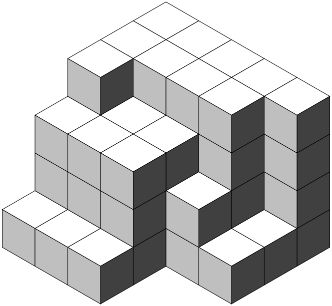
# Plane partitions

## Definition

A **plane partition** $\pi$ is a set of positive integer lattice points $(i, j, k)$ such that if $(i, j, k) \in \pi$, $i' \leq i$, $j' \leq j$, $k' \leq k$ then $(i', j', k') \in \pi$.

## Theorem (MacMahon 1896)

*The number of plane partitions inside $a \times b \times c$ is* $\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

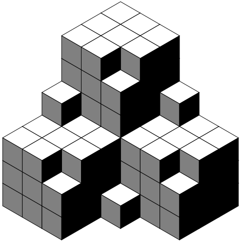


# Symmetry classes of plane partitions

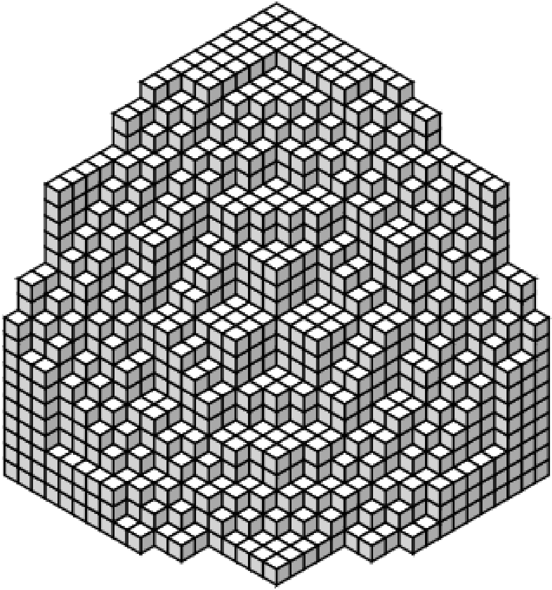
## Definition

- $\pi$ is **symmetric** if $(i, j, k) \in \pi$ implies $(j, i, k) \in \pi$.
- $\pi$ is **cyclically symmetric** if $(i, j, k) \in \pi$ implies $(j, k, i) \in \pi$.
- $\pi$ is **totally symmetric** if both of the above hold.
- $\pi$ is **self-complementary** if it is equal to its complement.
- $\pi$ is **transpose-complementary** if its transpose is equal to its complement.

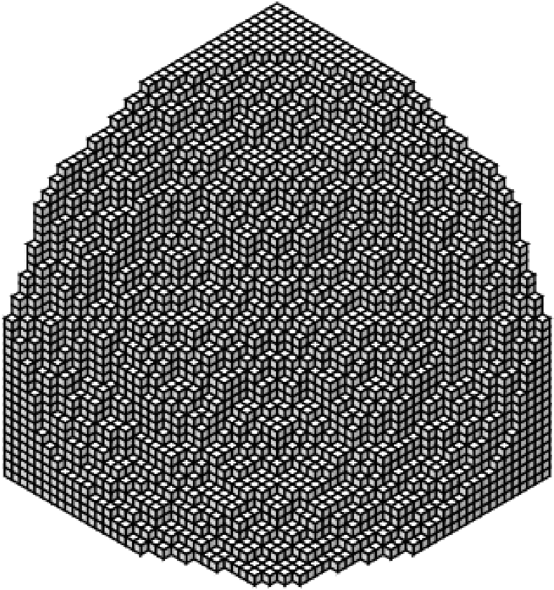
**Totally symmetric self-complementary plane partitions**



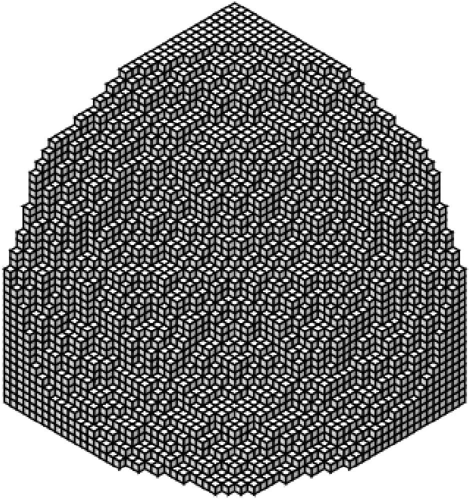
# What does a large random TSSCPP look like?



# What does a large random TSSCPP look like?



# What does a large random TSSCPP look like?

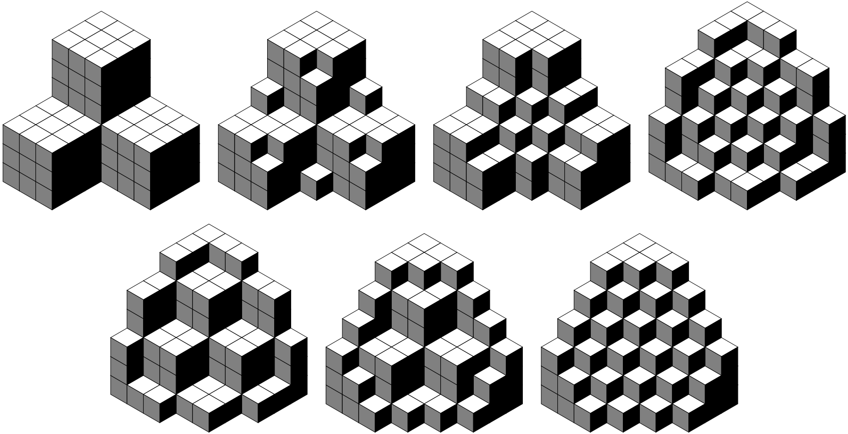


Ayyer and Chhita have a conjectured limit shape (2021)

# Totally symmetric self-complementary plane partitions

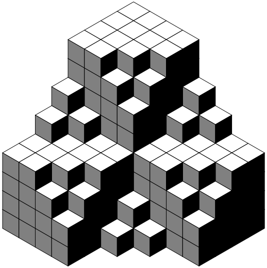
**Theorem (G. Andrews 1994)**

*TSSCPPs inside a $2n \times 2n \times 2n$ box are counted by:* $\displaystyle\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$.



# A missing bijection

**Open problem**: Find a beautiful bijection between $n \times n$ ASM and TSSCPP in a $2n \times 2n \times 2n$ box.



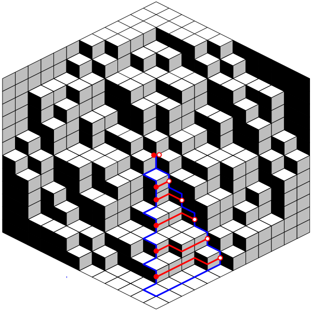
?

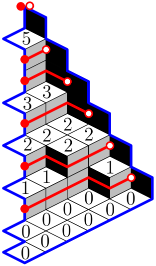
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Sub-bijections known in the following cases:

- Permutation [S. 2018]
- Two-diagonal [Biane and Cheballah 2012, Bettinelli 2019]
- Gapless monotone triangle [ACGB 2011, S. 2011]
- **1432-key-avoiding** [Huang and S. 2024]

# Known TSSCPP bijections

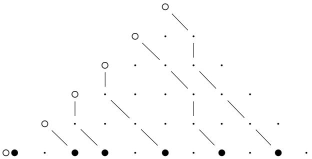





Magog triangle

Non-intersecting lattice paths

6
4 5
3 4 5
2 3 5 6
1 2 3 4 6
1 2 3 4 5 6



# New TSSCPP object: Magog matrices

Magog triangle

$$\begin{array}{ccccccccccc} & & & & & 6 & & & & & \\ & & & & 4 & & 5 & & & & \\ & & & 3 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 5 & & 6 & & \\ & 1 & & 2 & & 3 & & 4 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Magog matrix

$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

# Comparing ASM and magog matrices (Holmlund and S. 2024)

## ASM vs magog matrices

Properties of both:

- entries $\in \{0, 1, -1\}$
- rows and columns sum to one
- row and column partial sums $\geq 0$
- column partial sums $\leq 1$

Property of ASMs:

- row partial sums $\leq 1$

Property of Magog matrices:

- For all $0 \leq i \leq n-2, 0 \leq j \leq n-2$,
$$\sum_{i'=1}^{i} a_{i',j+1} + \sum_{j'=1}^{j+1} a_{i+1,j'} - \sum_{i'=1}^{i} a_{i'j} \geq 0$$

Magog but not ASM

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ASM but not Magog

$$\begin{pmatrix} \color{red}{0} & \color{green}{1} & 0 & 0 \\ \color{red}{1} & \color{green}{-1} & 1 & 0 \\ \color{green}{0} & \color{green}{0} & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

ASM and Magog

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# New TSSCPP object: Magog matrices

Magog triangle

$$\begin{array}{ccccccccccc} & & & & & 5 & & & & & \\ & & & & 4 & & 5 & & & & \\ & & & 2 & & 4 & & 5 & & & \\ & & 2 & & 3 & & 4 & & 5 & & \\ & 2 & & 3 & & 4 & & 5 & & 6 & \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

$\longleftrightarrow$

Magog matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Theorem (Holmlund and S. 2024)**

*The magog matrices of order $n$ with no negative ones are the 132-avoiding permutation matrices.*

# The many faces of alternating sign matrices

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

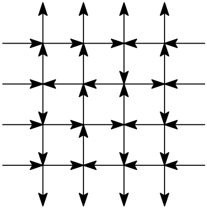
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

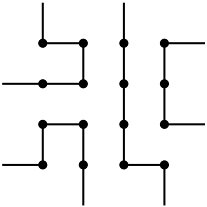
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Six-vertex model



Fully-packed loop



Corner sum matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

# The many faces of magog matrices (Bansal and S. 2026+)

Magog matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

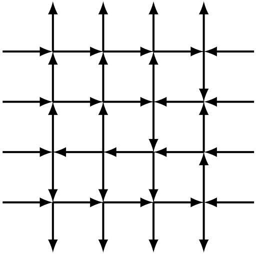
Magog triangle

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 3 & & 4 & & \\ & 1 & & 2 & & 3 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

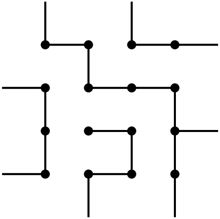
Magog height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 4 & 3 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

Magog vertex model



Magog fully-packed loop



Magog corner sum

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix}$$

# ASM vs. TSSCPP polytope

### Definition (S. / Behrend-Knight)

The $n$th *alternating sign matrix polytope* ASM($n$) is the convex hull in $\mathbb{R}^{n^2}$ of the $n \times n$ ASMs.

### Definition (Holmlund and S.)

The $n$th *TSSCPP matrix polytope* TSSCPP($n$) is the convex hull in $\mathbb{R}^{n^2}$ of all $n \times n$ magog matrices.

### Theorem (S. 2009 / *+BK 2007)

ASM($n$) *satisfies:*

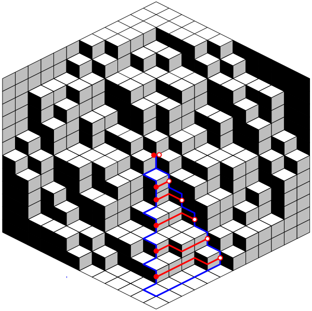
- * *Dimension* $(n-1)^2$
- * *Vertices are precisely the* $n \times n$ *ASMs*
- * *Inequality description*
- $4[(n-2)^2+1]$ *facets*
- *Nice face lattice description*
- *Projects to permutohedron*

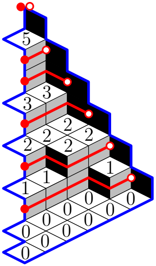
### Theorem (Holmlund and S. 2024)

TSSCPP($n$) *satisfies:*

- *Dimension at most* $(n-1)^2$
- *Vertices are precisely the set of* $n \times n$ *magog matrices*
- *Partial inequality description*

# Known TSSCPP bijections

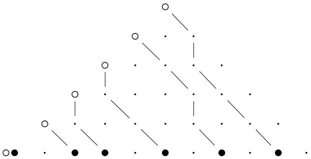




Magog triangle

Non-intersecting lattice paths

6
4 5
3 4 5
2 3 5 6
1 2 3 4 6
1 2 3 4 5 6



# TSSCPP boolean triangles

## Definition (S. 2018)

A *TSSCPP boolean triangle* of order $n$ is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}$.

Not a TSSCPP boolean triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 1 & & 0 \end{array}$$

# TSSCPP boolean triangles

## Definition (S. 2018)

A *TSSCPP boolean triangle* of order $n$ is a triangular integer array $\{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n-i \leq j \leq n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \geq \sum_{i=j}^{i'} b_{i,n-j}$.

A TSSCPP boolean triangle

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & & 1 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

# Boolean triangle polytope

**Definition (Holmlund and S. 2024)**

The $n$th *boolean triangle polytope* $\mathrm{BTP}(n)$ is the convex hull in $\mathbb{R}^{n^2}$ of all TSSCPP boolean triangles of order $n$.

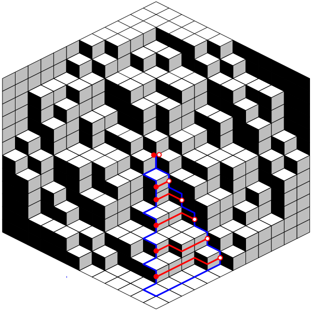
**Theorem (Holmlund and S. 2024)**

$\mathrm{BTP}(n)$ *satisfies the following:*

- *Dimension* $\binom{n}{2}$.
- *Vertices are precisely the* TSSCPP *boolean triangles of order* $n$
- *Inequality description*
- *Has* $\dfrac{(n-1)(3n-2)}{2}$ *facets*

# TSSCPP boolean triangles

Define the **weight** of a TSSCPP as $\mathrm{wt}(T) := \prod_{(i,j)\in \mathrm{one}(T)} x_{n-i}$, where $\mathrm{one}(T)$ denotes the set of indices of the entries of its boolean triangle that equal 1.



$$\begin{array}{ccccccccc} &&&& 0 &&&& \\ &&& 0 && 1 &&& \\ && 1 && 0 && 0 && \\ & 1 && 0 && 1 && 0 & \\ 0 && 0 && 0 && 0 && 0 \end{array}$$

$$\mathrm{wt}(T) = x_2^2 x_3 x_4$$

# Identification of permutation TSSCPP

**Definition**

Let *permutation TSSCPP* of order $n$ be all TSSCPP inside a $2n \times 2n \times 2n$ box whose corresponding boolean triangles have weakly decreasing rows.

Not a permutation TSSCPP

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

# Identification of permutation TSSCPP

## Definition

Let *permutation TSSCPP* of order $n$ be all TSSCPP inside a $2n \times 2n \times 2n$ box whose corresponding boolean triangles have weakly decreasing rows.

A permutation TSSCPP

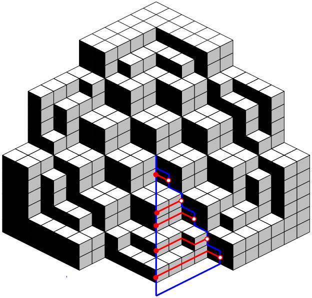
$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 0 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 1 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

# Permutation case bijection

**Theorem (S. 2018)**

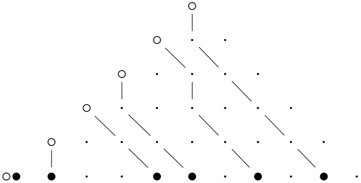
*There is a natural, weight-preserving bijection between $n \times n$ permutation matrices and permutation TSSCPP in a $2n \times 2n \times 2n$ box.*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow$$



# Permutation case bijection

Non-intersecting lattice paths



$\leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc}
&&&&1&&&&\\
&&&\mathbf{0}&&\mathbf{0}&&&\\
&&1&&1&&\mathbf{0}&&\\
&\mathbf{0}&&\mathbf{0}&&\mathbf{0}&&\mathbf{0}&\\
1&&\mathbf{0}&&\mathbf{0}&&\mathbf{0}&&\mathbf{0}
\end{array}$$

$\leftrightarrow$

Monotone triangle

$$\begin{array}{ccccccccccc}
&&&&&4&&&&&\\
&&&&\mathbf{4}&&\mathbf{6}&&&&\\
&&&3&&4&&\mathbf{6}&&&\\
&&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}&&\\
&1&&\mathbf{3}&&\mathbf{4}&&\mathbf{5}&&\mathbf{6}&\\
1&&2&&3&&4&&5&&6
\end{array}$$

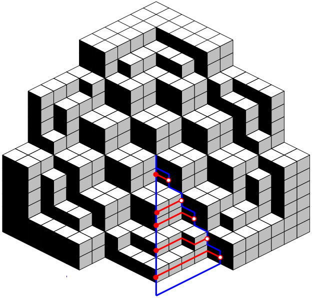
$\leftrightarrow$

Permutation matrix

$$\begin{pmatrix}
0&0&0&1&0&0\\
0&0&0&0&0&1\\
0&0&1&0&0&0\\
0&0&0&0&1&0\\
1&0&0&0&0&0\\
0&1&0&0&0&0
\end{pmatrix}$$

# Permutation case bijection

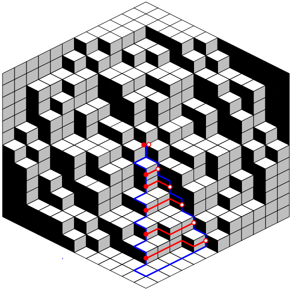
Can we extend this bijection beyond permutations?



$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Permutation case bijection and beyond

Yes! In the 1432-key-avoiding case.



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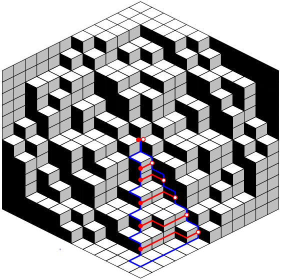
# A pipe dream perspective

Bumpless pipe dreams (reduced and nonreduced) are alternating sign matrices.

New perspective: TSSCPP boolean triangles are a subset of (reduced and nonreduced) pipe dreams!



$$\leftrightarrow \quad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array} \quad \leftrightarrow \quad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & & \\ 0 & 1 & & & \\ 0 & & & & \end{array}$$

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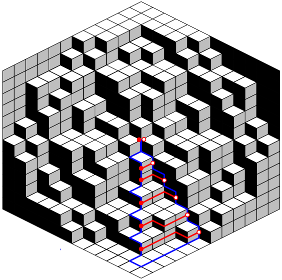
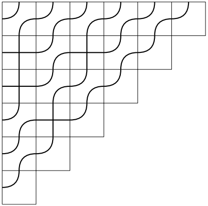


$\leftrightarrow$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array}$$

$\leftrightarrow$

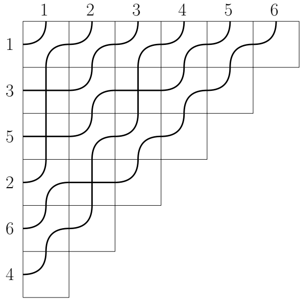


# Pipe dreams

**Definition (Bergeron, Billey 1993 / Fomin, Kirillov 1996)**

A **pipe dream** (PD) of size $n$ is a tiling of an $n \times n$ grid by tiles ⊞ and ⌟⌜, such that the positions on or below the main anti-diagonal only consist of elbow-tiles. A PD is **reduced** if no two pipes cross twice.

Define the **weight** of a pipe dream $D$ as $\mathrm{wt}(D) := \prod_{(i,j)\in \mathrm{cross}(D)} x_i$.
Define the permutation $\pi$ by reading the order of the pipes down the left side.




$\pi = 135264$

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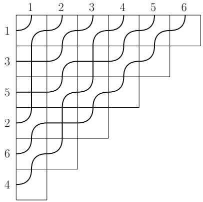


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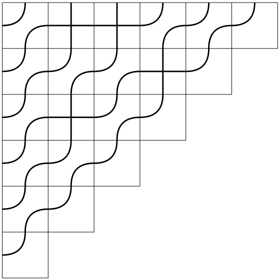
## Theorem (Bergeron, Billey 1993)

*Reduced pipe dreams with permutation $\pi$ give a combinatorial expansion of the Schubert polynomial:* $\mathfrak{S}_\pi = \sum_{D \in \mathrm{PD}^{red}(\pi)} \mathrm{wt}(D)$.

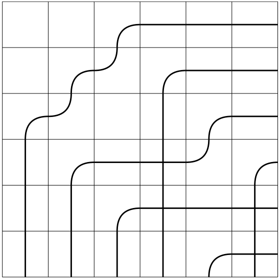
# A pipe dream bijection

**Theorem (Gao-Huang 2023)**

*Let $\pi \in S_n$. There is an explicit weight-preserving bijection $\varphi$ between reduced pipe dreams of $\pi$ and reduced bumpless pipe dreams of $\pi$.*



$\Leftrightarrow$



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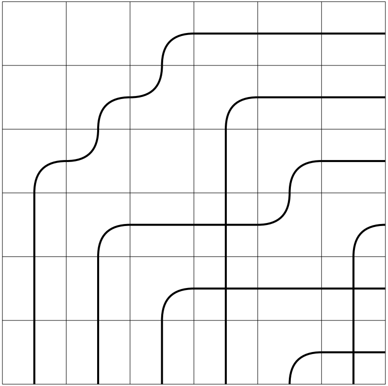

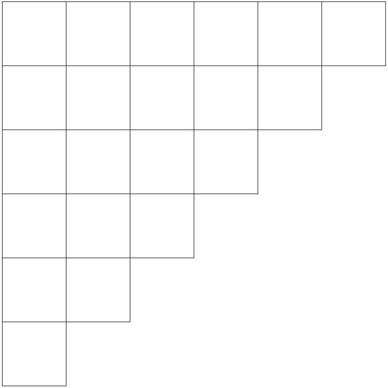

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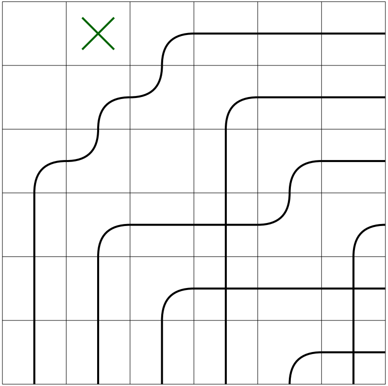

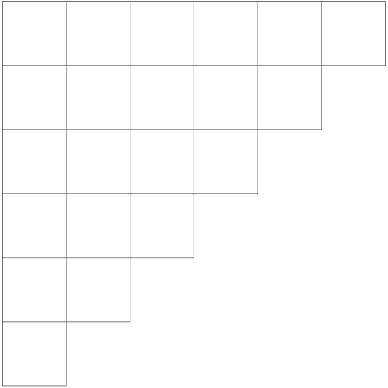

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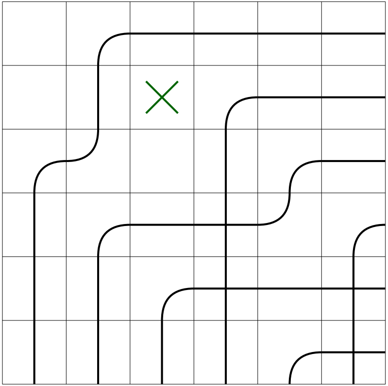

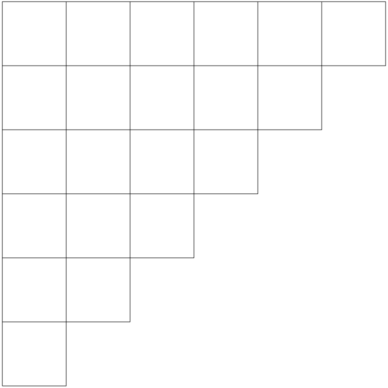

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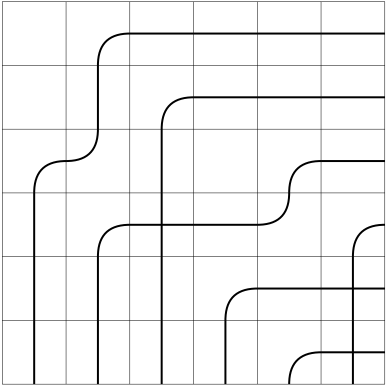

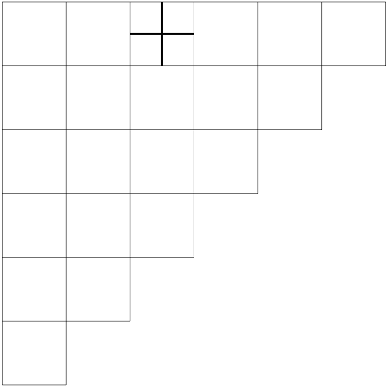

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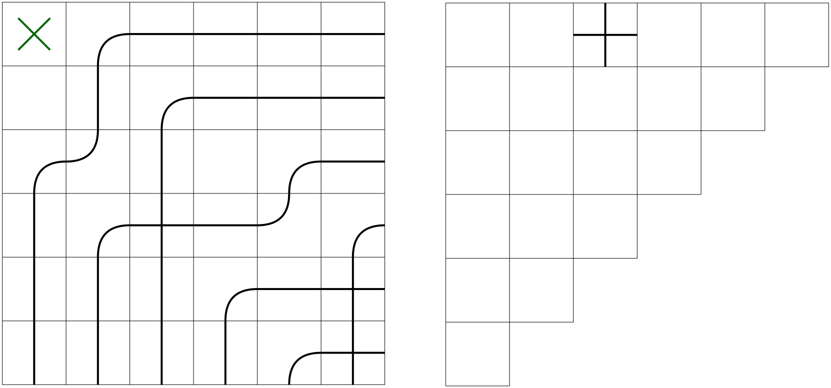

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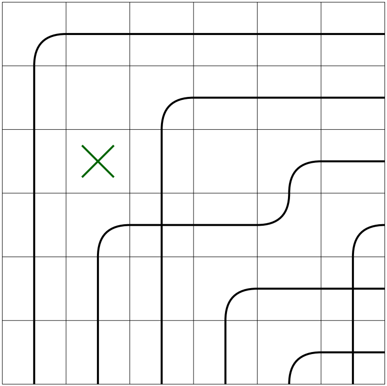

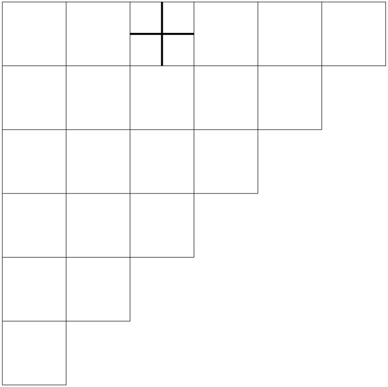

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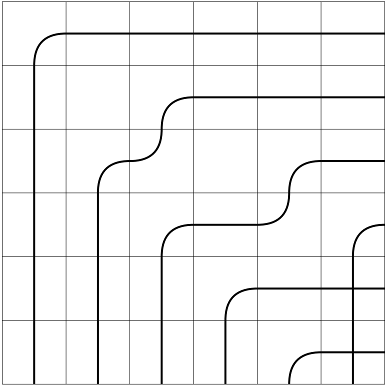

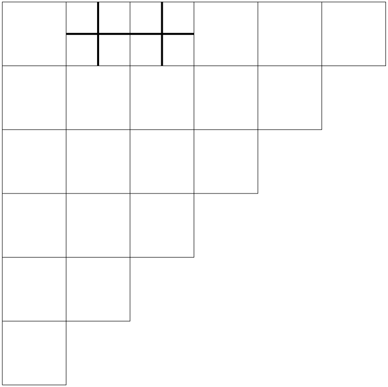

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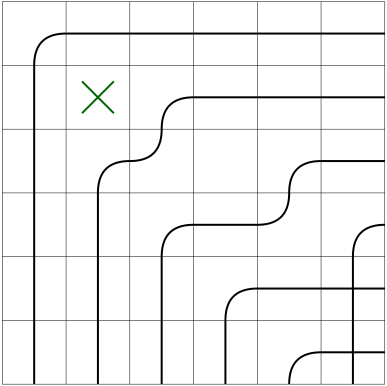

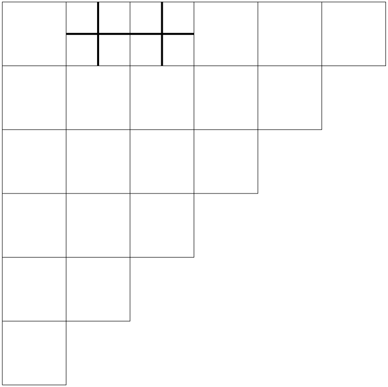

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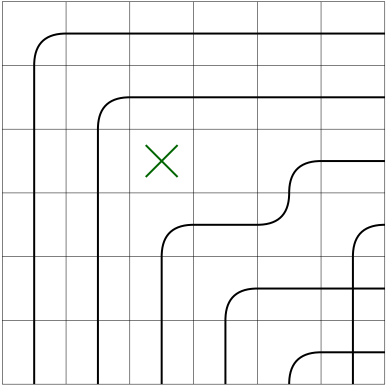

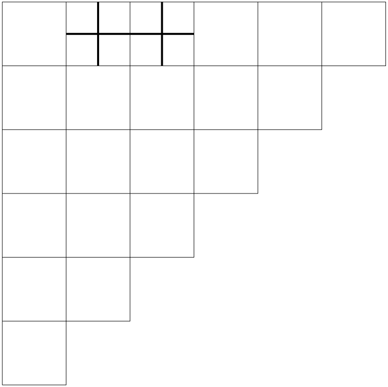

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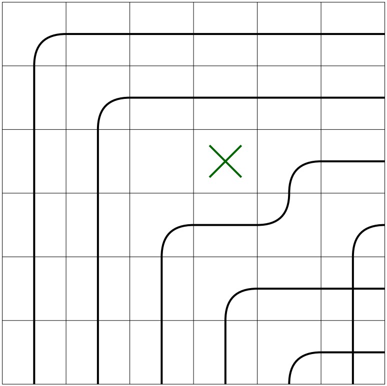

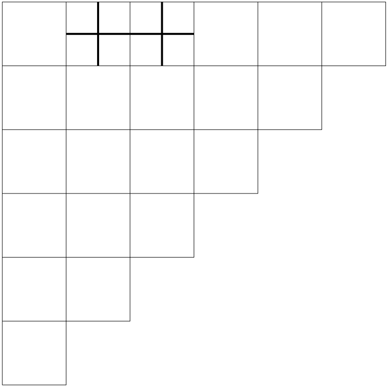

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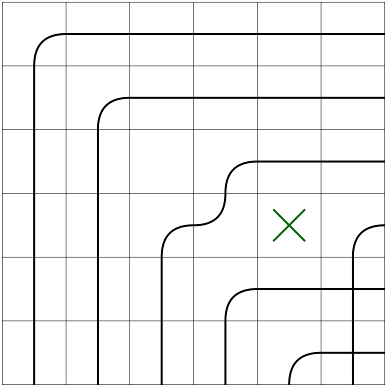

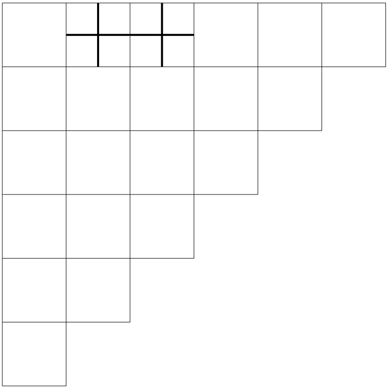

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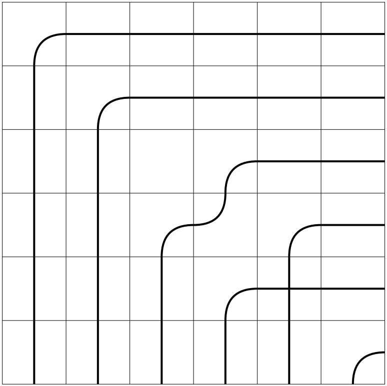

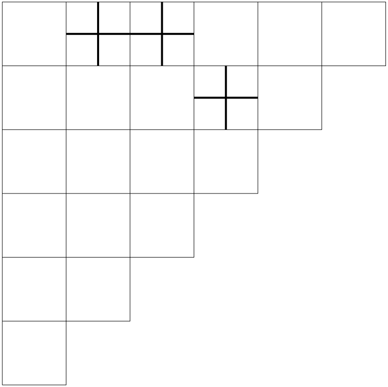

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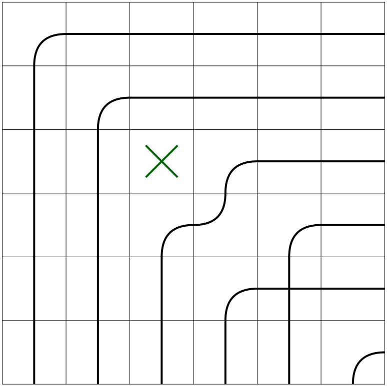

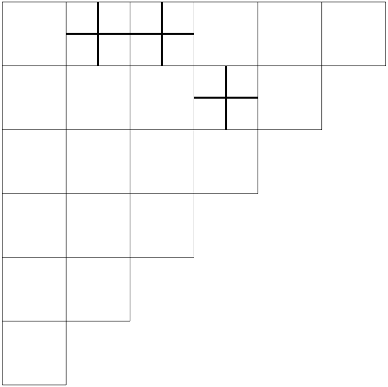

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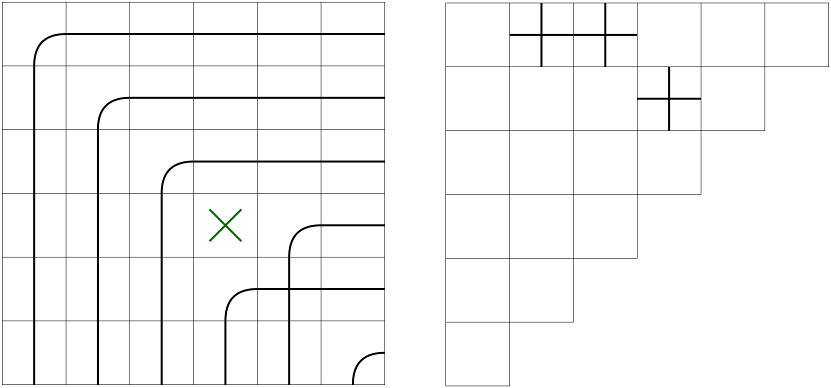

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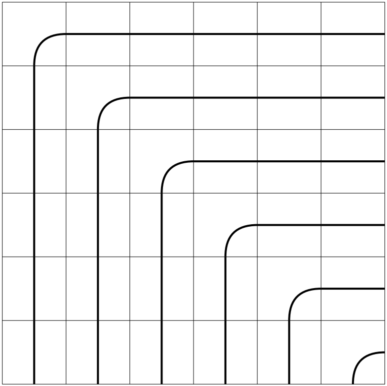

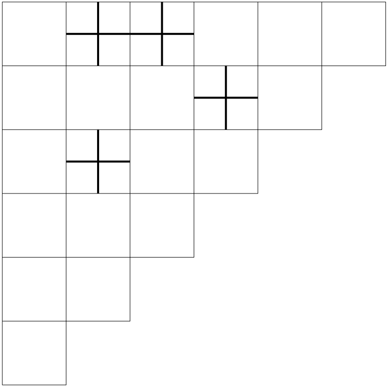

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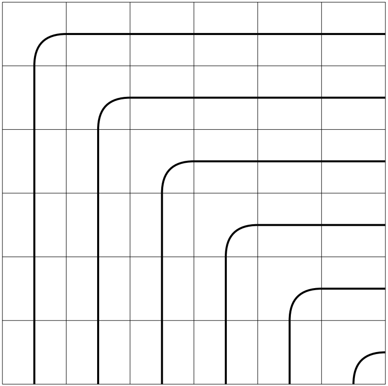
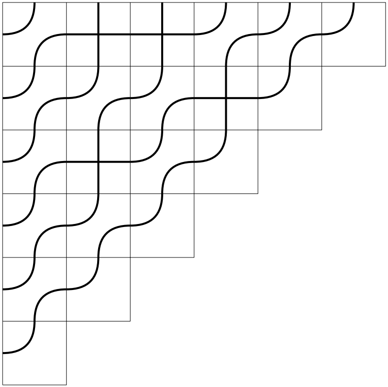
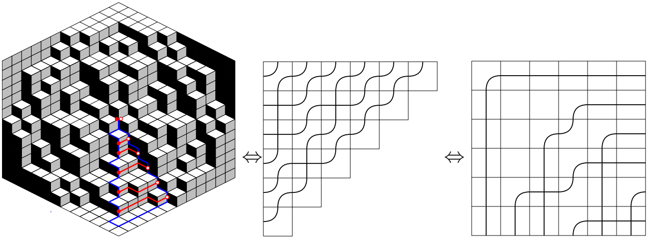




# A 1432-avoiding ASM-TSSCPP bijection

**Theorem (Huang-S. 2024)**

*Let $\pi \in S_n$. There is an explicit weight-preserving injection $\varphi$ from $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$. If $\pi$ avoids 1432, then $\varphi$ is a bijection.*



$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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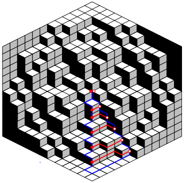
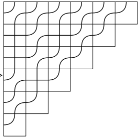
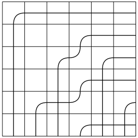
 $\Leftrightarrow$  $\Leftrightarrow$  $\Leftrightarrow$ $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

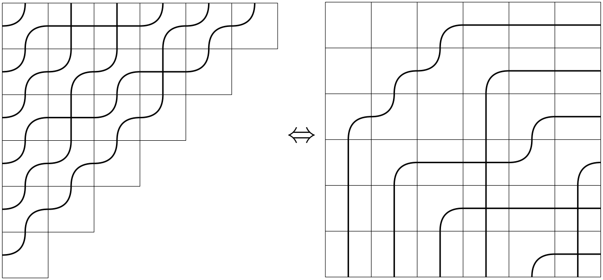
- Characterize TSSCPP by a pseudo-Yamanouchi property on the bounded compatible sequence associated to a pipe dream.
- Simple slides preserve this property.
- By a theorem of Gao, if $\pi$ avoids 1432, any two reduced pipe dreams are connected by simple slides.
- The pipe dream with left-justified crosses is pseudo-Yamanouchi.
- Thus $\varphi$ maps $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$ bijectively.

# Comparison of subset bijections

| $n$ | Perm | **1432 & 2143-av** | Gap-less | Two diag | **1432-av** | $\varphi$ | ASM |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 6 | 7 | 6 | 7 | 7 | 7 | 7 |
| 4 | 24 | 33 | 26 | 35 | 36 | 40 | 42 |
| 5 | 120 | 185 | 162 | 219 | 246 | 362 | 429 |
| 6 | 720 | 1175 | 1450 | 1594 | 2135 | 5125 | 7436 |
| 7 | 5040 | 8261 | 18626 | 12935 | 23067 | 112941 | 218348 |

# A pipe dream perspective

Is there hope to extend this bijection beyond the reduced case?

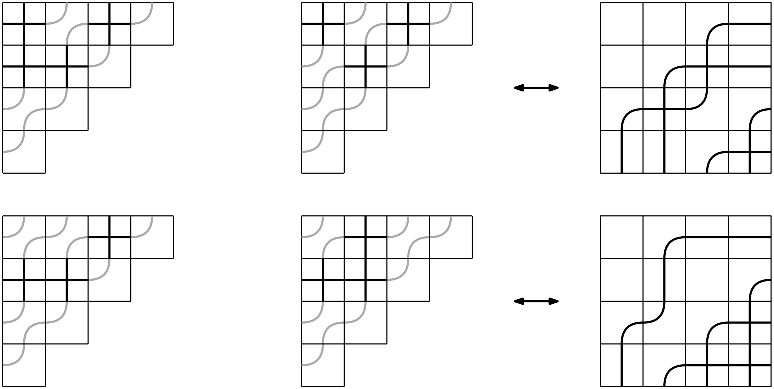


A new tool that may help:

**Theorem (Huang-Shimozono-Yu 2024+)**

*Let $\pi \in S_n$. There is an explicit weight-preserving bijection between all pipe dreams of $\pi$ and marked bumpless pipe dreams of $\pi$.*

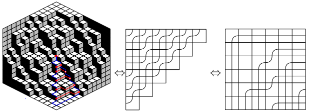
# Challenges



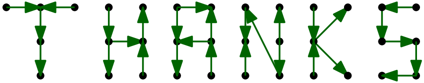
TSSCPP PDs

Images of BPDs under known bijections,
Not TSSCPP

Remaining BPDs for $n = 4$



$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$





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