Fun with webs and tableaux — Jessica Striker

Directions. Your goal in these problems is to explore and gain familiarity with these objects and topics. Don't let the computer have all the fun (i.e. don't ask AI to do the problems for you). The struggle and time spent thinking about these problems is more important than completing all of them or figuring them all out perfectly. Discuss the problems with each other and me! Math is more fun when we work on it together!

Definition. Given a standard Young tableau (SYT) T with r rows, its *lattice word* $\mathcal{L}(T)$ is a word in the numbers $\{1,\ldots,r\}$ whose kth letter is the row of the entry k in T. $\mathcal{L}(T)$ has the *Yamanouchi* property, meaning in each prefix, the number of i letters is at most the number of $(i-1)$ letters.

1. Let $T =$

1	2	3	4
5	6	8	9
7	10	11	12

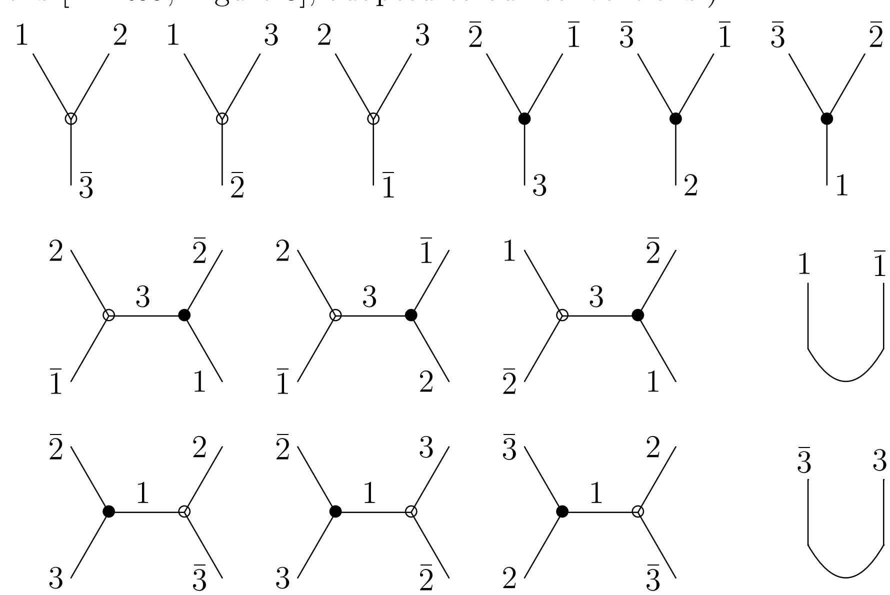
. Write its lattice word $\mathcal{L}(T)$.

Definition (Convention of [FLL19])**.** An SL_r*-tensor diagram* is a properly bicolored edge-weighted graph embedded in a disk with labeled boundary vertices and whose interior vertices have degree r. A tensor diagram is a *web* if it is planar.

Fact: SL_r-tensor diagrams span the space of SL_r-invariant polynomials.
Motivating question: Find nice bases.

Theorem ([Ku96b, KK99])**.** *Nonelliptic (no cycles of length less than* 6*)* SL_3*-webs index a rotation-invariant basis for* SL_3*. The growth rules and depth map are mutually inverse bijections between* $3 \times c$ *SYT and nonelliptic* SL_3*-webs with* $3c$ *black boundary vertices.*

2. Starting with $\mathcal{L}(T)$ from Problem 1, use the $r = 3$ growth rules, shown below, to grow the SL_3-web $\mathcal{W}(T)$ corresponding to T. (The $r = 3$ growth rules are due to [KK99]; this figure is [PPR09, Figure 3], adapted to our conventions.)



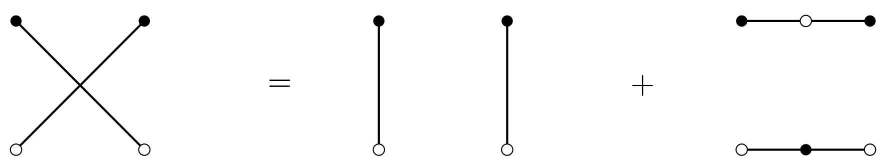
3. Apply promotion to T, then use the $r = 3$ growth rules to grow the new SL_3-web. The new web should be the rotation of $\mathcal{W}(T)$ [PPR09, Thm. 2.5].

Definition. In a web, the *depth* of a face is the number of edges that must be crossed to get from the face to the *base face* (the face between 1 and n). The SL_3-*depth map*, a boundary vertex gets a label 1 if crossing the incident edge (in the direction of increasing vertex label) increases the depth, 2 if the depth stays the same, and 3 if the depth decreases.

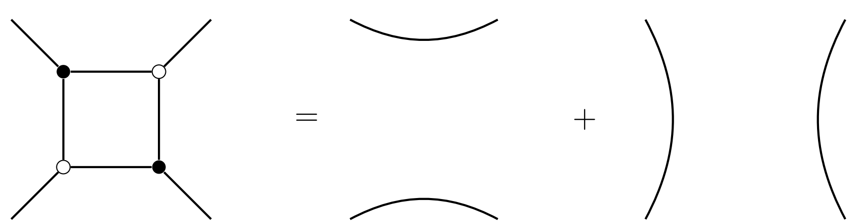
4. Draw a nonelliptic SL_3-web (or use the web you grew in Problem 2 or 3). Use the depth map to find the lattice word. Use the lattice word to determine the SYT that corresponds to your web.

5. The above story easily adapts to SL_2. Determine the growth rules and a depth map that give mutually inverse bijections between $2 \times c$ SYT and SL_2-webs. (There is no need for an extra condition such as nonelliptic here; all SL_2-webs are basis webs.)

6. Draw an SL_2-tensor diagram that has one or more crossings in it and use the skein relation below to express it as a linear combination of basis webs.



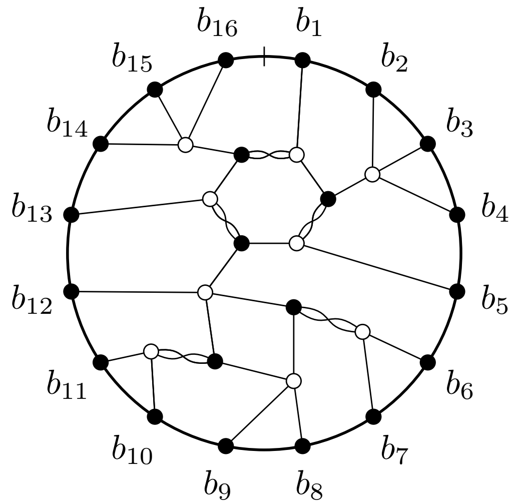
7. Draw an elliptic SL_3-web that has one or more 4-cycles in it and use the skein relation below to express it as a linear combination of basis webs.



Definition ([GPPSS26])**.** An *r-hourglass plabic graph (HPG) W* is an SL_r web in which edges with weight $k \geq 2$ are replaced by k edges twisted like an hourglass. trip_i on W is a map on boundary vertices defined by taking the ith left at each $\circ$ and the ith right at each $\bullet$. The trip_i *permutation* is defined as $\mathsf{trip}_i(W)(b) = a$ if the trip_i starting at b ends at a.

8. Let W be the 4-hourglass plabic graph given below. Find its trip permutations trip_1, trip_2, and trip_3.

$$\mathsf{trip}_1(W) = 5\ 3 \ldots$$
$$\mathsf{trip}_2(W) = 10\ 4 \ldots$$
$$\mathsf{trip}_3(W) = 13\ 12 \ldots$$



9. Apply *square* and *benzene* moves from [GPPSS26, Fig. 2] to W to create new webs in the move equivalence class. (Trace the web in pencil, then erase as needed to do the moves.)

Definition. A 4-hourglass plabic graph is *fully reduced* if no web in its move equivalence class has a square face with an hourglass edge on it. It is *top* if for all benzene faces, in each hourglass edge, the white vertex precedes the black vertex in counterclockwise order.

Theorem ([GPPSS26, Thms A and B; standard case])**.** *The collection of web invariants of contracted, fully reduced, top 4-hourglass plabic graphs give a rotation-invariant web basis for SL_4. The growth rules $\mathcal{G}$ and separation labeling are mutually inverse bijections between $4 \times c$ SYT and these web equivalence classes with $4c$ black boundary vertices. Moreover, for all $1 \leq i \leq 4$, $\mathsf{prom}_i(T) = \mathsf{trip}_i(\mathcal{G}(T))$.*

Definition ([GPPSS26, Def. 4.18])**.** Given an edge e in an HPG, let F denote the face "below" e when the white vertex is on the right. The *separation label* $\mathsf{sep}(e)$ of a non-hourglass edge e equals 1+ the number of trips passing through e that separate F from the base face. (The hourglass edge labels for $r = 4$ are the complement of their incident edge labels.)

10. Compute the separation labeling of W. The labels of the boundary edges form the lattice word of the corresponding tableau. (For W from Problem 8, this should equal the first tableau in Problem 11.)

    When computing this by hand, you do not need to compute the separation label for every edge by the trip definition. In small examples, it is often easier to infer some edge labels using the Yamanouchi property of the boundary word and the fact that the separation labeling is a proper edge coloring in the numbers $\{1, 2, 3, 4\}$ [GPPSS26, Prop. 4.19].

Definition ([GPPSS24])**.** The ith *promotion permutation* of T is defined as $\mathsf{prom}_i(T)(b) = a$ if and only if a is the unique value that moves from row $i+1$ to row i in the bth application of gromotion to T.

11. Compute the missing steps of the gromotion orbit shown below. Then finish computing the promotion permutations prom_1, prom_2, and prom_3. Check that they match the trip permutations from Problem 8. Note that $\mathsf{prom}_3 = \mathsf{prom}_1^{-1}$ and prom_2 is in involution [GPPSS24, Thm. 6.7]. See more examples in [PPS25+, Sec. 2].

$$\begin{array}{|c|c|c|c|}\hline 1&2&6&8\\\hline 3&5&7&14\\\hline 4&9&10&15\\\hline 11&12&13&16\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2&5&6&8\\\hline 3&7&10&14\\\hline 4&9&13&15\\\hline 11&12&16&1\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 3&5&6&8\\\hline 4&7&10&14\\\hline 9&12&13&15\\\hline 11&16&1&2\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 4&5&6&8\\\hline 7&10&13&14\\\hline 9&12&15&2\\\hline 11&16&1&3\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 5&6&8&14\\\hline 7&10&13&2\\\hline 9&12&15&3\\\hline 11&16&1&4\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 6&8&13&14\\\hline 7&10&15&2\\\hline 9&12&1&3\\\hline 11&16&4&5\\\hline\end{array} \rightarrow$$

$$\begin{array}{|c|c|c|c|}\hline 7&8&13&14\\\hline 9&10&15&2\\\hline 11&12&1&3\\\hline 16&4&5&6\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 8&10&13&14\\\hline 9&12&15&2\\\hline 11&1&3&6\\\hline 16&4&5&7\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 9&10&13&14\\\hline 11&12&15&2\\\hline 16&1&3&6\\\hline 4&5&7&8\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 10&12&13&14\\\hline 11&15&2&6\\\hline 16&1&3&8\\\hline 4&5&7&9\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 11&12&13&14\\\hline 15&1&2&6\\\hline 16&3&7&8\\\hline 4&5&9&10\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 12&13&14&6\\\hline 15&1&2&8\\\hline 16&3&7&10\\\hline 4&5&9&11\\\hline\end{array} \rightarrow$$

$$\begin{array}{|c|c|c|c|}\hline 13&14&2&6\\\hline 15&1&7&8\\\hline 16&3&9&10\\\hline 4&5&11&12\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 14&&&\\\hline &&&\\\hline &&&\\\hline &&&\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 15&&&\\\hline &&&\\\hline &&&\\\hline &&&\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 16&&&\\\hline &&&\\\hline &&&\\\hline &&&\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1&2&6&8\\\hline 3&5&7&14\\\hline 4&9&10&15\\\hline 11&12&13&16\\\hline\end{array}.$$

$$\begin{aligned}\mathsf{prom}_1(T) &= 5\ 3\ldots\\ \mathsf{prom}_2(T) &= 10\ 4\ldots\\ \mathsf{prom}_3(T) &= 13\ 12\ldots\end{aligned}$$

12. Find the *antiexcedences* of each prom_i (values $a = \mathsf{prom}_i(T)(b)$ such that $a < b$). Compare them to the entries of T in rows 1 thru i. What do you notice? (Compare to [GPPSS24, Thm. 6.12].)

13. Use the growth rules from [GPPSS26, Fig. 24] to grow a symmetrized six-vertex configuration from the lattice word of a $4 \times c$ SYT; you may use the tableau from the prior problem. Then convert to an hourglass plabic graph by replacing sinks with black vertices, sources with white vertices, transmitting vertices with hourglasses (correctly oriented), and unorienting all edges. You should recover the web from Problem 8.

14. Apply promotion to the tableau T, then use the growth rules to grow the new symmetrized six-vertex configuration. Apply square and/or benzene moves to find a web in the move equivalence class that is a rotation of the prior web.

15. Draw a non-fully-reduced SL_4 HPG that has a square with an hourglass edge. Use skein relations of [GPPSS26, Fig. 43] to express it as a linear combination of basis webs.

16. Use the growth rules to find a web for the tableau $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline 10 & 11 & 12 \\ \hline \end{array}$. Leave the web in symmetrized six-vertex form. What does this move equivalence class correspond to? (See [GPPSS26, Prop. 8.2].)

17. Use the growth rules to find the web for the tableau $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 7 \\ \hline 3 & 5 & 11 & 14 \\ \hline 4 & 8 & 12 & 15 \\ \hline 9 & 10 & 13 & 16 \\ \hline \end{array}$. Convert to plabic form. What does this move equivalence class correspond to? (See [GPPSS26, Prop. 8.3]; note this proposition is stated for *oscillating tableaux*. For this problem, we instead use the standardized tableau.)

Further Study. Use the algorithm in [GPPSS25] to construct an hourglass plabic graph from an $r \times 2$ SYT of your choosing. Verify that $\mathsf{trip}_i = \mathsf{prom}_i$ for all $1 \le i \le r-1$ [GPPSS25, Thm. 1.2].

Further Study. The web results on this worksheet are also true at the level of *oscillating* (and *fluctuating*) tableaux. Oscillating tableaux have lattice words with both positive and negative (barred) numbers in them. The negative numbers correspond to white boundary vertices. The Yamanouchi condition for lattice words of oscillating tableaux is that the number of i letters minus the number of $\bar{i}$ letters is at most the number of $(i-1)$ letters minus the number of $\overline{i-1}$ letters. Repeat some of the examples of this worksheet for oscillating tableaux. This level of generality is often useful and was essential for several proofs in [GPPSS26].

Challenge. What tableau properties are visible on the corresponding web? Can you characterize webs whose tableaux have these properties? Likewise, what notable features of webs can be seen on the corresponding tableaux?

Further Study. See [Cu26+] and [ST26+] for an example of the above; these papers study the case where the web for an $r \times 2$ SYT is a forest.

Open Problem. Find rotation-invariant web bases for SL_r with $r > 4$.

(GPPSS are working on this for $r = 5$.) A solution to this problem should also solve:

Open Problem. Find objects (webs or web equivalence classes!) in bijection with $r \times c$ SYT such that tableau promotion corresponds to rotation, for $r > 4$ or $c > 2$.

Challenge (Hard or impossible, but still worth considering)**.** Does the appearance of alternating sign matrices and/or plane partitions as webs uncover any new algebra or representation theory related to those objects? Might it help shed light on the ASM-TSSCPP bijection problem? (See [AS25+] for a characterization of the tableaux that correspond to the webs for symmetry classes of plane partitions.)

Challenge. What are some other notable web equivalence classes?

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