Name: **Version #1**

Instructor: Bullwinkle

Exam III
November 19, 2019

- The Honor Code *is* in effect for this examination. All work is to be your own.
- Please turn off all cellphones and electronic devices.
- Calculators are **not** allowed.
- The exam lasts for 1 hour and 15 minutes.
- Be sure that your name and your instructor's name are on the front page of your exam.
- Be sure that you have all 9 pages *of* the test.

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	(•)	(b)	(c)	(d)	(e)
2.	(•)	(b)	(c)	(d)	(e)
3⊡	(•)	(b)	(c)	(d)	(e)
4□	(•)	(b)	(c)	(d)	(e)
5.	(•)	(b)	(c)	(d)	(e)
6.	(•)	(b)	(c)	(d)	(e)
7.	(•)	(b)	(c)	(d)	(e)
8.	(•)	(b)	(c)	(d)	(e)
9.	(•)	(b)	(c)	(d)	(e)
10.	(•)	(b)	(c)	(d)	(e)

Please do NOT write in this box.

Multiple Choice __________

11. __________

12. __________

13. __________

Total __________

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3.	(a)	(b)	(c)	(d)	(e)
4	(a)	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	(d)	(e)
6.	(a)	(b)	(c)	(d)	(e)
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Please do NOT write in this box.

Multiple Choice	________
11.	________
12.	________
13.	________
Total	________

Multiple Choice

1.(6pts) Let $W = Span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$,

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}.$$

The decomposition of $\mathbf{y}$ as a sum of a vector in W and one in $W^{\perp}$ is:

(a) $\begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix} + \begin{bmatrix} 8/3 \\ 11/3 \\ 5 \\ 19/3 \end{bmatrix}$ (c) $\begin{bmatrix} 5 \\ 1/3 \\ 14/3 \\ 13/3 \end{bmatrix} + \begin{bmatrix} -2 \\ 11/3 \\ 1/3 \\ 5/3 \end{bmatrix}$

(d) $\begin{bmatrix} 1/3 \\ 2 \\ -5/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 8/3 \\ 2 \\ 20/3 \\ 14/3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 0 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 3 \\ 7 \end{bmatrix}$

Solution. The projections of $\mathbf{y}$ onto the three orthogonal vectors are $\begin{bmatrix} 1/3 \\ 1/3 \\ 0 \\ -1/3 \end{bmatrix}$ $\begin{bmatrix} 14/3 \\ 0 \\ 14/3 \\ 14/3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5/3 \\ -5/3 \\ 5/3 \end{bmatrix}$ which add to $\begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$. The difference between $\mathbf{y}$ and this vector is $\begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$.

2.(6pts) Find the orthogonal projection of

$$\mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

onto the subspace spanned by

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} 10/3 \\ 4/3 \\ -2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ (c) $\begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix}$ (d) $\begin{bmatrix} 4/3 \\ 0 \\ -4/3 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Solution. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for the subspace W spanned by these vectors. Therefore, the orthogonal projection of the vector $\mathbf{y}$ onto the span is

$$\widehat{\mathbf{y}} = \text{proj}_W \mathbf{y} = \frac{\mathbf{y} \bullet \mathbf{u}_1}{\mathbf{u}_1 \bullet \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \bullet \mathbf{u}_2}{\mathbf{u}_2 \bullet \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10/3 \\ 4/3 \\ -2/3 \end{bmatrix}.$$

3.(6pts) Which two matrices give the QR-factorization of the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \\ -1 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) $Q = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} R = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$

(b) $Q = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} R = \begin{bmatrix} 4 & 8 \\ 0 & 4 \end{bmatrix}$

(c) $Q = \begin{bmatrix} -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 1/2 \\ -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 1/2 \end{bmatrix} R = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$

(d) $Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ -1 & -1 \\ 1 & 0 \end{bmatrix} R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) $Q = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \\ -1/2 & -1 \\ 1/2 & 0 \end{bmatrix} R = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}$

Solution. Applying Gram-Schmidt to the columns of A to get orthonormal columns yields the 2 vectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Both of these vectors dot with themselves to give 4 so they both have length 2. Dividing through by the length gives us orthonormal vectors $\mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.

Thus, these are the columns of Q. Then to find R we compute $Q^T A$ which is $\begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$.

4.(6pts) Which of the following is the least squares solution $\hat{\mathbf{x}}$ to the equation

$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \\ 1 & -2 \\ 2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix} ?$$

(a) $\begin{bmatrix} 7/5 \\ 1/5 \end{bmatrix}$ (b) $\begin{bmatrix} 11/9 \\ 1/9 \end{bmatrix}$ (c) $\begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (e) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solution. $A^T A\mathbf{x} = A^T\mathbf{b}$ is $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 14 \\ 2 \end{bmatrix}$, from which $\hat{\mathbf{x}} = \begin{bmatrix} 7/5 \\ 1/5 \end{bmatrix}$.

4.

5.(6pts) Consider the initial value problem

$$\sin(2x) + \cos(3y)\frac{dy}{dx} = 0 \qquad y(\pi/2) = \pi/3$$

Which of the following implicitly defines the solution?

(a) $\frac{-\cos(2x)}{2} + \frac{\sin(3y)}{3} = \frac{1}{2}$

(b) $\frac{-\cos(2x)}{2} + \frac{\sin(3y)}{3} = \frac{-1}{2}$

(c) $-\cos(2x) + \sin(3y) = 1$

(d) $\sin(2x) + \cos(3y) = -1$

(e) $-2\cos(2x) + 3\sin(3y) = 2$

Solution. Separable: $\cos(3y)dy = -\sin(2x)dx$. Integrating we obtain: $\frac{\sin(3y)}{3} = \frac{\cos(2x)}{2} + C$. Initial condition $\frac{\sin(\pi)}{3} = \frac{\cos(\pi)}{2} + C$ gives $C = 1/2$, so the solution is $\frac{-\cos(2x)}{2} + \frac{\sin(3y)}{3} = \frac{1}{2}$.

6.(6pts) Find the solution to the initial value problem

$$t\frac{dy}{dt} + 3y = \frac{t}{1+t^4}, \qquad y(1) = 0.$$

(a) $y = \frac{1}{4t^3} \cdot \ln\left(\frac{1+t^4}{2}\right), t > 0$

(b) $y = \frac{1}{4t^3} \cdot \ln\left(\frac{1+t^4}{2}\right), t < 0$

(c) $y = \ln\left(\frac{1+t^4}{2t^3}\right), t > 0$

(d) $y = \ln\left(\frac{1+t^4}{2t^3}\right), t < 0$

(e) $y = \ln(1+t^4) - (\ln 2)t^{-3}, t > 0$

Solution. We rewrite the equation as $\frac{dy}{dt} + \frac{3}{t}y = \frac{1}{1+t^4}$. The solution will exist and be unique on $(0, \infty)$ since $\frac{3}{t}$ has a discontinuity at 0 and $1 > 0$. Integrating factor is $\mu = t^3$. Multiplying both sides with it gives $t^3\frac{dy}{dt} + 3t^2y = \frac{t^3}{1+t^4}$. Integrating and dividing by μ gives $y = \frac{1}{4t^3} \cdot [\ln(1+t^4) + C], t > 0$. Initial condition gives $0 = \frac{1}{4} \cdot [\ln(2) + C]$ so $C = -\ln 2$. We obtain $y = \frac{1}{4t^3} \cdot [\ln(1+t^4) - \ln 2] = \frac{1}{4t^3} \cdot \left[\ln\left(\frac{1+t^4}{2}\right)\right]$ $t > 0$

7.(6pts) Let $y(t)$ be the unique solution of the initial value problem

$$(t^2 - t)\frac{dy}{dt} + \cos(\pi t)y = \frac{t^2 - t}{t - 2} \qquad y(3/2) = 0$$

What is the largest interval where y is defined?

(a) $1 < t < 2$ (b) $t > 1$ (c) $0 < t < 2$

(d) $t < 1/2$ (e) $t > 0$

Solution. We rewrite the equation as: $\frac{dy}{dt} + \frac{\cos(\pi t)}{t^2 - t}y = \frac{1}{t-2}$. The coefficients have discontinuity at 0,1,2. Since $1 < 3/2 < 2$, we deduced based on Theorem 2.4.1 that $1 < t < 2$ is the largest interval for which the solution is defined.

8.(6pts) Which of the following functions can be used as an integrating factor for the equation $y' + ty = \cos t$?

(a) $e^{t^2/2}$ (b) t (c) $t^2/2$ (d) e^t (e) $e^{\cos t}$

Solution. An integrating factor can be found as $\mu = e^{\int t dt} = e^{t^2/2}$.

9.(6pts) A tank initially contains 100 liters of pure water. Then, at $t = 0$, a sugar solution with concentration of $4g/L$ starts being pumped into the tank at a rate of $5L$/min. The tank is kept well mixed, and the solution is being pumped out at the rate of $4L$/min. Which of the following is the initial value problem for the quantity of sugar $y(t)$, in grams, in the tank at time t?

(a) $\frac{dy}{dt} = 20 - \frac{4y}{100+t}$, $y(0) = 0$

(b) $\frac{dy}{dt} = 5y - 4(100 + t)$, $y(0) = 0$

(c) $\frac{dy}{dt} = 20 - 4y$, $y(0) = 0$

(d) $\frac{dy}{dt} = 4$, $y(0) = 100$

(e) $\frac{dy}{dt} = 20 - \frac{y}{(100+t)^2}$, $y(0) = 100$

Solution. $\frac{dy}{dt}$ = rate in − rate out. Each rate is equal to the (concentration) × (flow rate), so we obtain $\frac{dy}{dt} = 4 \cdot 5 - \frac{y}{100+(5-4)t} \cdot 4$ so $\frac{dy}{dt} = 20 - \frac{4y}{100+t}$ (g/min). The initial condition is $y(0) = 0$ since there is pure water in the tank at time $t = 0$ (no sugar).

10.(6pts) A college graduate borrows \$8,000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously, the constant annual rate k at which the borrower needs to make payments continuously to pay off the loan in 3 years is:

(a) $\frac{800e^{0.3}}{1-e^{0.3}}$

(b) $800e^{-0.3}(1 - e^{0.3})$

(c) $\frac{8{,}000}{3}(1 - e^{0.3})$

(d) $\frac{800e^{-0.3}}{1-e^{0.3}}$

(e) $\frac{8{,}000e^{0.1}}{3(1-e^{0.1})}$

Solution. Let $S(t)$ be the value of the loan at time t. It satisfies $\frac{dS}{dt} = \frac{S}{10} + k$, with $k < 0$ and $S(0) = \$8,000$. Rewriting: $\frac{dS}{dt} - \frac{S}{10} = k$, with integrating factor $\mu = e^{-t/10}$. The solution is $S(t) = e^{t/10}\left[C - 10ke^{-t/10}\right] = Ce^{t/10} - 10k$.

Applying the initial condition gives: $8,000 = C - 10k$ so $C = 8,000 + 10k$. We substitute: $S(t) = (8,000 + 10k)e^{t/10} - 10k = 8,000e^{t/10} + 10k(e^{t/10} - 1)$. After 3 years, the loan is paid, so $S(t) = 0$: $0 = 8,000e^{3/10} + 10k(e^{3/10} - 1)$, which gives $k = \frac{800e^{0.3}}{1-e^{0.3}}$

Partial credit

11.(14pts) Let $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -4 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -3 \end{bmatrix}.$$

(a) Apply the Gram-Schmidt process to find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for W.

(b) Find the QR decomposition of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & -4 & 3 \\ -1 & 4 & 1 \\ -1 & -2 & -3 \end{bmatrix}$.

(c) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Compute $||Q\mathbf{x}||$.

Solution. a) and b) $Q = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & -1/2 \end{bmatrix}$, $R = Q^T A$.

c) $||Q\mathbf{x}|| = ||\mathbf{x}|| = \sqrt{6}$.

12.(14pts) A tank initially contains 50 liters of water and 20 grams of salt. Water containing a salt concentration of 2 g/L enters the tank at the rate of 5 L/min, and the well-stirred mixture leaves the tank *at the same rate.*

(a) Find an expression for the amount $Q(t)$ of salt in the tank (in grams) as a function of time t (in minutes).

(b) How long does it take for the amount of salt to reach 60 grams?

(c) Compute the limit of $Q(t)$ as t approaches infinity.

9.

Solution. a) $\frac{dQ}{dt}$ = rate in − rate out, $Q(0) = 20$. Each rate is equal to the (concentration) × (flow rate), so we obtain $\frac{dQ}{dt} = 2 \cdot 5 - \frac{Q}{50} \cdot 5$ so $\frac{dQ}{dt} = 10 - \frac{Q}{10}$ $(g/min), Q(0) = 20$.

$$\frac{dQ}{dt} + \frac{Q}{10} = 10$$

has $\mu = e^{t/10}$

$$Q(t) = e^{-t/10}\left[100e^{t/10} + C\right] = 100 + Ce^{-t/10}.$$

Applying the initial condition, we obtain $C = -80$ so

$$Q(t) = 100 - 80e^{-t/10}.$$

b) We plug $Q(t) = 60$ in the equation to get $60 = 100 - 80e^{-T/10}$ and so $1/2 = e^{-T/10}$ from which $T = 10 \ln 2$.

c) $t \to \infty$ gives $Q(t) = 100$.

13.(12pts) Find the function $y(t)$ which solves the initial value problem

$$t\frac{dy}{dt} + 4y = \frac{e^{-t}}{t^2}$$

(a) Find the general solution.

(b) Find the solution with $y(1) = 0$.

(c) What is the largest interval in which the solution in part (b) is defined?

11.

Solution. (a) Rewrite:

$$\frac{dy}{dt} + \frac{4}{t}y = \frac{e^{-t}}{t^3}$$

Multiplying both side with $\mu(t) = t^4$ gives

$$y = t^{-4}\left[\int te^{-t}dt\right] = t^{-4}\left[-e^{-t} - te^{-t} + C\right]$$

$$y = -t^{-4}(t+1)e^{-t} + Ct^{-4}$$

(b) $0 = -2e^{-1} + C$ so $C = 2e^{-1}$. The solution is $y = -t^{-4}(t+1)e^{-t} + 2t^{-4}e^{-1}$.

(c) Applying Theorem 2.4.1: Since $\frac{4}{t}$ and $\frac{e^{-t}}{t^3}$ are discontinuous at $t = 0$ and $1 > 0$ we obtain $t > 0$ the largest interval.

It can also been seen by looking at the solution directly, which is discontinuous at $t = 0$ so it only exists in the interval about $t_0 = 1$, which is $(0, \infty)$.