# Bonds, Bond Prices, Interest Rates, and the Risk and Term Structure of Interest Rates

ECON 40364: Monetary Theory & Policy

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Spring 2022

# Readings

- Text:
  - Mishkin Ch. 4, Mishkin Ch. 5 pg. 85-100, Mishkin Ch. 6
  - GLS Ch. 34
- Other:
  - Poole (2005): "Understanding the Term Structure of Interest Rates"
  - Bernanke (2016), "What Tools Does the Fed Have Left? Part 2: Targeting Longer-Term Interest Rates"

# Bonds

- Generically, a financial security entitles the holder to periodic cash flows
- We will generically refer to a "bond" as a debt instrument where a borrower promises to pay the holder of the bond (the lender) fixed and known payments according to some pre-specified schedule
- Hence, bonds are known as fixed income securities
  - Equity (stocks) offers unknown payout streams
- There are many different types of bonds. Differ according to:
  - Details of how bond is paid off
  - Time to maturity
  - Default risk
- The yield to maturity is a measure of the interest rate on the bond, although the interest rate is often not explicitly laid out. Will use terms interest rate and yield interchangeably
- Want to understand how interest rates are determined and how and why they vary across different characteristics of bonds

# Present Value

- Present discounted value (PDV): something received in the future is worth less to you than if you received the thing in the present
- Discount rate: rate at which you discount future cash flows from a security
- Suppose the discount rate is $i$ and is constant
- For a future cash flow ($CF$), how many dollars would be equivalent to you today:

$$PV_t = \frac{CF_{t+n}}{(1+i)(1+i)(1+i)\cdots\times(1+i)}$$

- Here $t$ is the "present," $t+n$ is the future ($n$ periods away), and $i$ is the discount rate
- The formula reduces to:

$$PV_t = \frac{CF_{t+n}}{(1+i)^n}$$

# Present Value: Example I

- Suppose you are promised \$10 in period $t+1$
- You could put \$1 in bank in period $t$ and earn $i = 0.05$ in interest between $t$ and $t+1$
- How many dollars would you need in the present to have \$10 in the future?

$$
\begin{aligned}
(1+i)PV_t &= CF_{t+1} \\
PV_t &= \frac{CF_{t+1}}{1+i} \\
&= \frac{10}{1.05} = 9.5238
\end{aligned}
$$

# Present Value: Example II

- Suppose you are promised \$10 in period $t + 3$
- You could put \$1 in bank in period $t$ and earn $i = 0.05$ across each set of adjacent periods between now and then
- If you put \$1 in bank in period $t$ and kept it there (re-investing any interest income), you would have $(1 + i)(1 + i)(1 + i)$ dollars in $t + 3$. Hence, the present value of \$10 three periods from now is:

$$\begin{aligned} PV_t(1+i)(1+i)(1+i) &= CF_{t+3} \\ PV_t &= \frac{CF_{t+3}}{(1+i)^3} \\ &= \frac{10}{1.05^3} = 8.6384 \end{aligned}$$

# Present Value and the Price of an Asset

- A financial asset is something which entitles the holder to periodic payments (cash flows)
- The classical theory of asset prices is that the price of an asset is equal to the present discounted value of all future cash flows
- The price of a bond (or any asset) is just the present discounted value of cash flows
- The yield is the discount rate you use to discount those future cash flows
- The yield is also equal to the expected or required return: not necessarily equal to realized return if security is sold prior to maturity
- Key issue in asset pricing: how do you determine which yield to use to discount cash flows and hence price an asset?
- Basic gist: the riskier an asset, the higher the required return you demand to hold it, and therefore the more you discount future cash flows

# Different Types of Bonds

- The following are different types of bonds/debt instruments depending on the nature of how they are paid back:
  1. Simple loan: you borrow $X$ dollars and agree to pay back $(1+i)X$ dollars at some specified date (interest plus principal) (e.g. commercial loan)
  2. Fixed payment loan: you borrow $X$ dollars and agree to pay back the same amount each period (e.g. month) for a specified period of time. "Full amortization" (e.g. fixed rate mortgage)
  3. Coupon bond: you borrow $X$ dollars and agree to pay back fixed coupon payments, $C$, each period (e.g. year) for a specified period of time (e.g. 10 years), at which time you pay off the "face value" of the bond (e.g. Treasury Bond)
  4. Discount bond: you borrow $X$ dollars and agree to pay back $Y$ dollars after a specified period of time with no payments in the intervening periods. Typically, $Y > X$, so the bond sells "at a discount" (e.g. Treasury Bill)
- Interest rate is not explicit for coupon or discount bonds, though the coupon rate is often called an interest rate, though this does not in general capture expected return on holding bond

# Yield to Maturity

- The yield to maturity (YTM) is the (fixed) interest rate that equates the PDV of cash flows with the price of the bond in the present
- This measures the return (expressed at an annualized rate) that would be earned on holding a bond if it is held until maturity (and there is no default, so more accurately this is an expected return)
- The YTM does not necessarily correspond to the realized return if the bond is not held until maturity (i.e. if you sell a bond before its maturity date)
- The YTM is another way of conveying the price of a bond, taking future cash flows as given

# YTM on a Simple Loan

- The YTM on a simple loan is just the contractual interest rate
- For a one period loan, the YTM is the same thing as the return
- Let $P$ be the price of the loan, $CF$ the payout after one year, and $i$ the interest rate. Then:

$$P = \frac{CF}{1+i}$$

- Or:

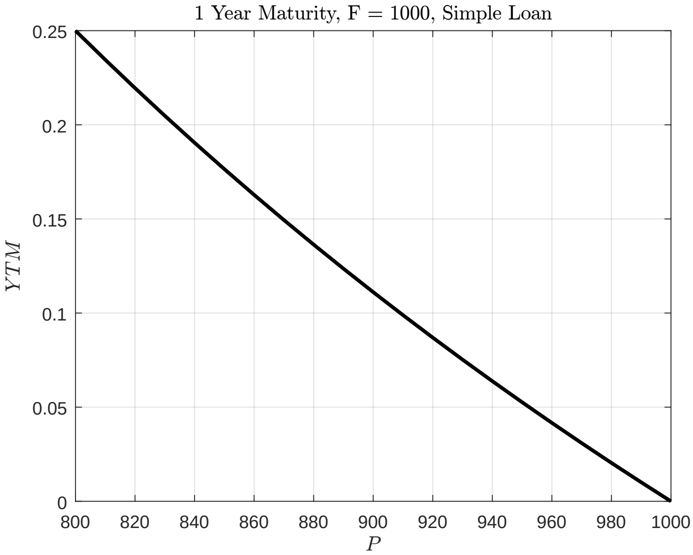
$$1+i = \frac{CF}{P}$$

- Or:

$$i = \frac{CF-P}{P}$$

- Where $\frac{CF-P}{P}$ is the return (or rate of return) on the loan

# Price and Yield on a Simple Loan



# YTM on a Discount Bond

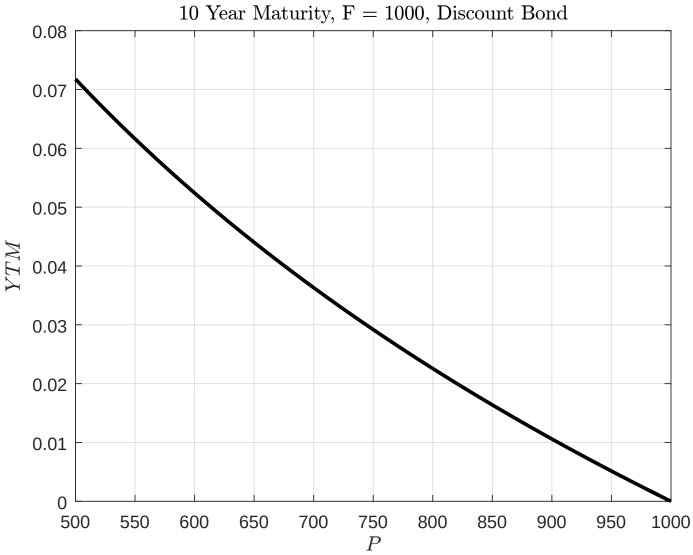
- The YTM on a discount bond is similar to a simple loan, just with different maturities
- In particular, for a face value of $F$, maturity of $n$, and price of $P$, the YTM satisfies:

$$P = \frac{F}{(1+i)^n}$$

- Or:

$$1 + i = \left(\frac{F}{P}\right)^{\frac{1}{n}}$$

# Price and Yield on a Discount Bond



# YTM on a Coupon Bond

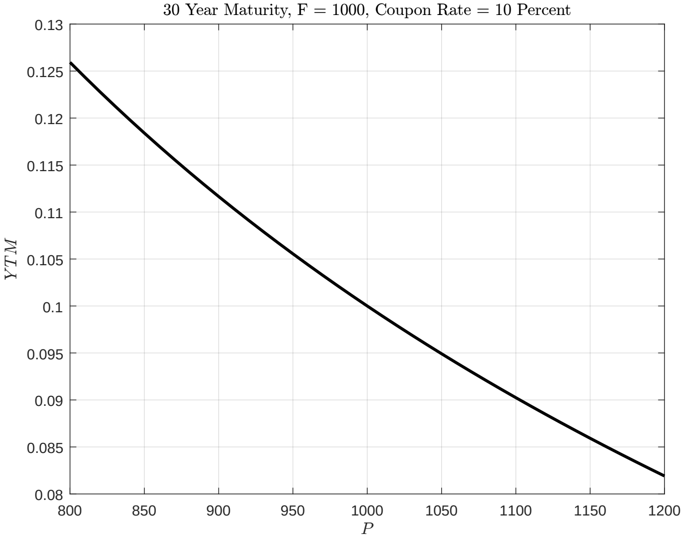
- Suppose a bond has a face value of \$100 and a maturity of three years
- It pays coupon payments of \$10 in years $t+1$, $t+2$, and $t+3$ (the coupon rate in this example is 10 percent)
- The face value is paid off after period $t+3$
- The period $t$ price of the bond is \$100
- The YTM solves:

$$100 = \frac{10}{1+i} + \frac{10}{(1+i)^2} + \frac{10}{(1+i)^3} + \frac{100}{(1+i)^3}$$

- Which works out to $i = 0.1$
- More generally, for an $n$ period maturity:

$$P = \sum_{j=1}^{n} \frac{C}{(1+i)^j} + \frac{FV}{(1+i)^n}$$

# Price and Yield on a Coupon Bond



# Coupon Rate vs. Yield

- In the example two slides ago, the coupon rate (10 percent) and yield (10 percent) are the same
- In general, this is not going to be true
- Will only be true if the bond "sells at par" – meaning the price of the bond equals the face value. If bond sells at a discount to face value, yield is greater than coupon rate and vice-versa
- Coupon rate is not the same thing as "the" interest rate!

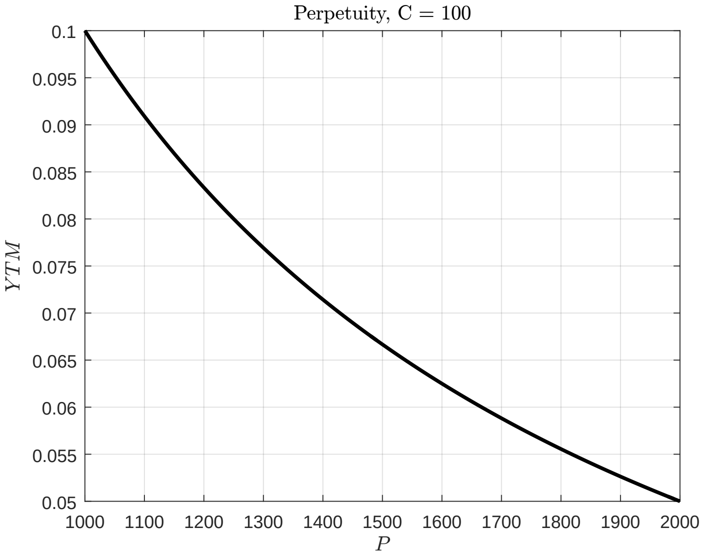
# Perpetuities

- Perpetuities (also called "consols") are like coupon bonds, except they have no maturity date
- Here, the relationship between price, yield, and coupon payments works out cleanly and is given by:

$$i = \frac{C}{P}$$

- For a coupon bond with a sufficiently long maturity, this is a reasonable approximation to the bond's YTM (because the PDV of the face value after many years is close to zero)
- This expression is also sometimes called the current yield as an approximation to the YTM on a coupon bond

# Price and Yield on a Perpetuity



# Observations

- Several observations are noteworthy from the previous slides:
  1. The bond price and yield are negatively related. This is true for all types of bonds. Bond prices and interest rates move in opposite directions
  2. For discount bonds, we would not expect price to be greater than face value – this would imply a negative yield
  3. For a coupon bond, when the bond is priced at face value, the yield to maturity equals the coupon rate
  4. For a coupon bond, when the bond is priced less than face value, the YTM is greater than the coupon rate (and vice-versa)

# Yields (Interest Rates) and Returns

- Returns and yields (interest rates) are in general not the same thing
- Rate of return: cash flow plus new security price, divided by current price
- Useful way to think about it (terminology here is related to equities): "dividend rate plus capital gain," where capital gain is the change in the security's price
- The return on a coupon bond held from $t$ to $t+1$ is:

$$R = \frac{C + P_{t+1} - P_t}{P_t}$$

- Or:

$$R = \underbrace{\frac{C}{P_t}}_{\text{Current Yield}} + \underbrace{\frac{P_{t+1} - P_t}{P_t}}_{\text{Capital Gain}}$$

- Return will differ from current yield (approximation to YTM) if bond prices fluctuates in unexpected ways

# Discount Bond

- Suppose that you hold a discount bond with face value \$1000, a maturity of 30 years, and a current yield to maturity of 10 percent
- The current price of this bond is $\frac{1000}{1.1^{30}} = 57.31$.
- Suppose that interest rates stay the same after a year. Then the bond has a price of $\frac{1000}{1.1^{29}} = 63.04$
- Since there is no coupon payment, your one year holding period return (holding period refers to length of time you hold the security) is just the capital gain:

$$R = \frac{P_{t+1} - P_t}{P_t} = \frac{63.04 - 57.31}{57.31} = 0.10$$

- If interest rates do not change, then the return and the yield to maturity are the same thing

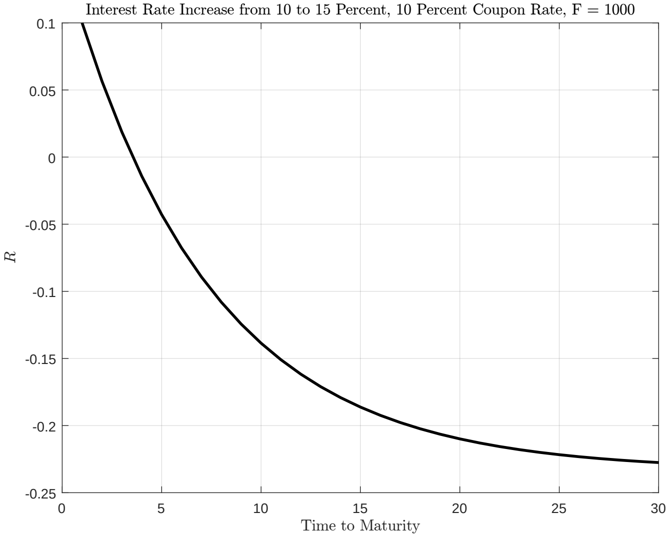
# Interest Rate Risk

- Continue with the same setup
- But now suppose that interest rates go up to 15 percent in period $t+1$ and are expected to remain there
- Then the price of the bond in period $t+1$ will be: $\frac{1000}{1.15^{29}} = 17.37$ – i.e. the bond price falls
- Your return is then:

$$R = \frac{P_{t+1} - P_t}{P_t} = \frac{17.37 - 57.31}{57.31} = -0.69$$

- On a discount bond, an increase in interest rates exposes you to capital loss
- True more generally for a coupon bond – while you know current yield at time of investment, capital gain is unknown
- What we call interest rate risk (close related concept is duration risk)

# Return and Time to Maturity, Coupon Bond



# Observations

- Return and initial YTM are equal if the holding period is the same as time to maturity (1 period). The capital gain is simply the face value (which is fixed) minus the initial price
- Increase in interest rates results in returns being less than initial yield
- Reverse is true
- Return is more affected by interest rate change the longer is the time to maturity
- If you hold the bond until maturity, your return is locked in at initial YTM
- Concept of return is relevant even if you do not sell the bond and realize the capital loss. There is an opportunity cost – if interest rates rise, had you not locked yourself in on a long maturity bond you could have purchased a bond in the future with a higher yield
- Longer maturity bonds are therefore riskier than short maturity bonds

# Determinants of Bond Prices (and Interest Rates)

- We've talked about specifics of bonds
- But what determines prices (and hence yields)?
- Two related approaches:
  - Conceptual (Mishkin): demand and supply
  - Micro-founded (GLS): explicit consumption-saving maximization problem
- For simplicity, think of all bonds as discount bonds (makes life easier)

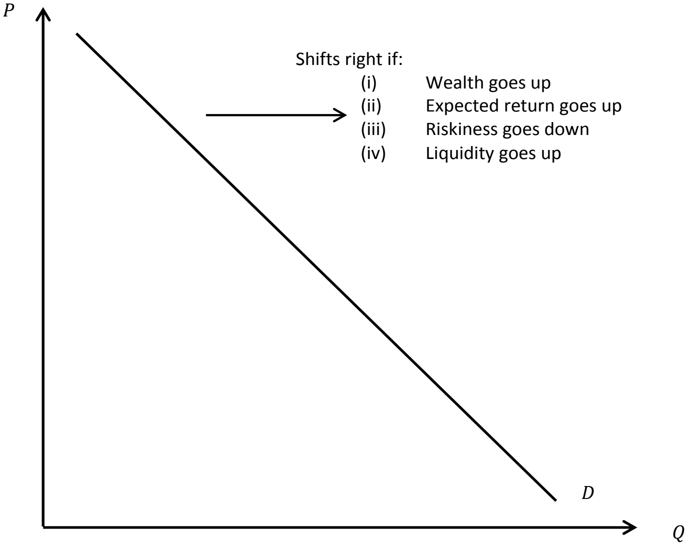
# Conceptual Approach: Portfolio Choice

- A bond (or any asset) is a way to transfer resources intertemporally
- Demand for a bond is based on the following factors:
  1. Wealth/income: assets are normal goods, so the more wealth, the more you want to hold at every price
  2. Expected returns: you hold assets to earn returns. The higher the expected return, the more of it you demand
  3. Risk: assume agents are risk averse. Holding expected return constant, you would prefer a less risky return. The more risky an asset is, the less of it you demand
  4. Liquidity: refers to the ease with which you can sell an asset and uncertainty over future price fluctuations. The more liquid it is, the more attractive it is to hold (it's easier to sell at a known price if you need to raise cash in a pinch)

# Demand for Bonds

- How does the demand for bonds vary with the price of bonds?
- As the price goes down, the yield goes up
- Therefore, holding everything else fixed, the expected return (yield) on holding a bond goes up as the price falls
- Therefore, demand slopes down
- Demand will shift (change in quantity demanded for a given price) with changes in other factors

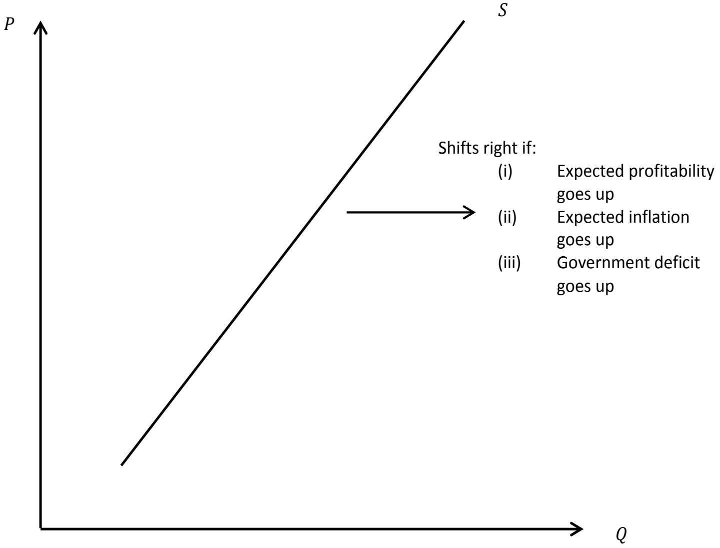
# Bond Demand


P
Shifts right if:
(i) Wealth goes up
(ii) Expected return goes up
(iii) Riskiness goes down
(iv) Liquidity goes up
D
Q

# Supply of Bonds

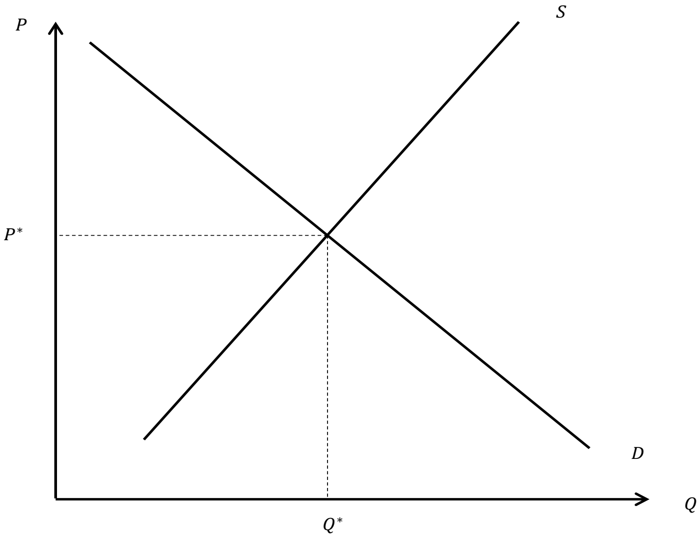
- Issuers of bonds are borrowers. You are issuing a bond to raise funds in the present to be paid back in the future
- Recall that bond prices move opposite interest rates
- As the bond price increases, the yield decreases
- Therefore, at a higher bond price, the cost of borrowing is lower
- So there will be more supply of bonds at a higher price – supply slopes up
- Changes in other factors, holding price fixed, will cause the supply curve to shift

# Bond Supply

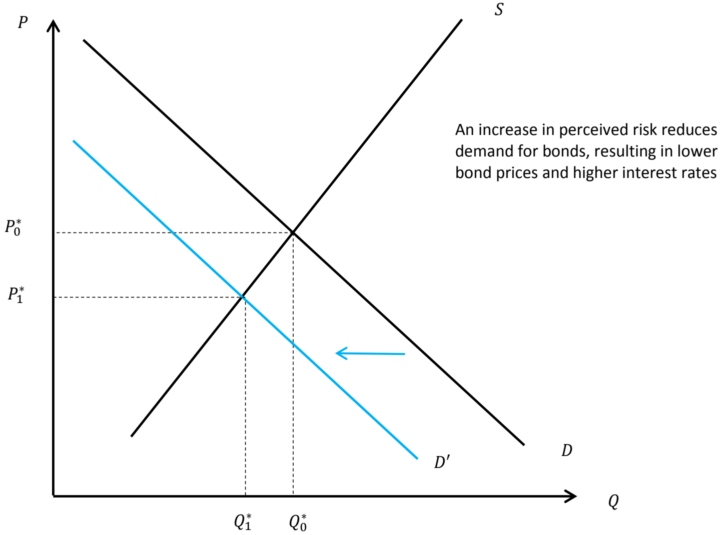


P
S
Q
Shifts right if:
(i) Expected profitability goes up
(ii) Expected inflation goes up
(iii) Government deficit goes up

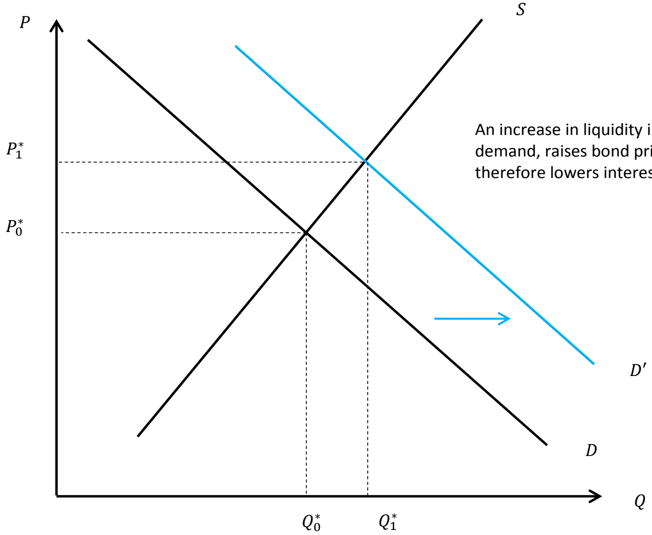
# Bond Market Equilibrium



# An Increase in Risk

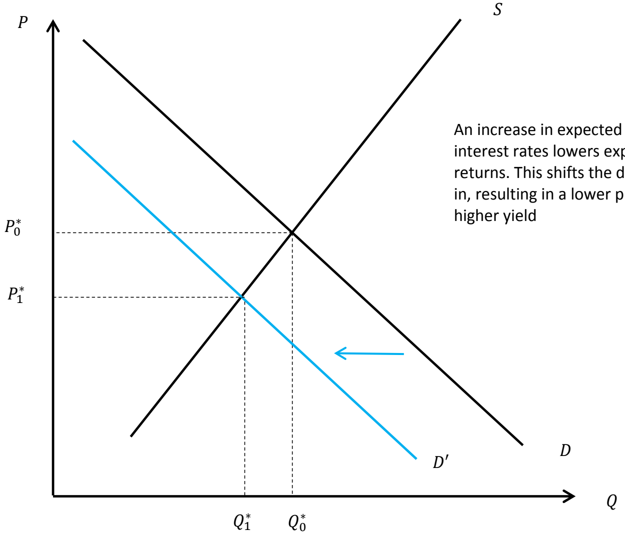


# An Increase in Liquidity



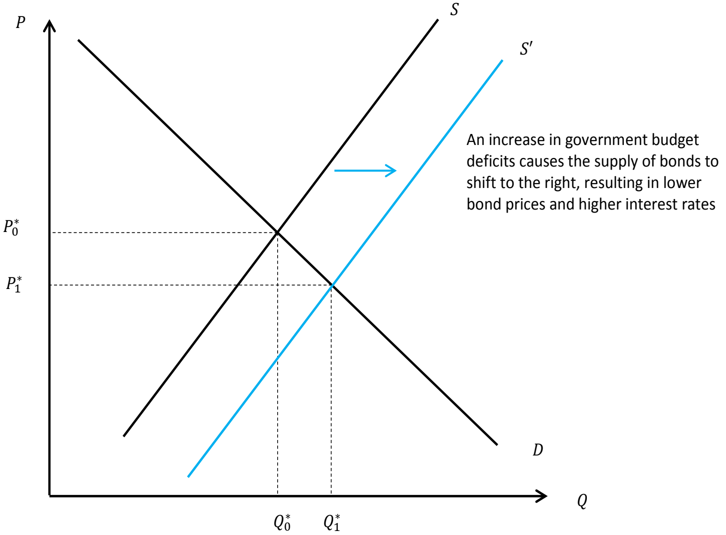

An increase in liquidity increases demand, raises bond prices, and therefore lowers interest rates

# An Increase in Expected Future Interest Rates




An increase in expected future interest rates lowers expected bond returns. This shifts the demand curve in, resulting in a lower price and higher yield

# An Increase in Government Budget Deficit



# Micro-Founded Approach

- Household lives (at least) two periods: $t$ and $t+1$
- Can purchase/sell one period discount bonds at $P_t^B$
- Assume no uncertainty

$$U = u(C_t) + \beta u(C_{t+1})$$

- Budget constraints:

$$C_t + P_t^B B_t = Y_t$$

$$C_{t+1} = Y_{t+1} + B_t$$

- You hold/issue $B_t$ to smooth consumption relative to income

# Euler Equation and Stochastic Discount Factor (SDF)

- Optimization yields an Euler equation:

$$P_t^B = \frac{\beta u'(C_{t+1})}{u'(C_t)}$$

- Right hand side is known as stochastic discount factor, $m_{t,t+1}$
- General asset pricing condition:

$$P_t^A = \mathbb{E}\left[\sum_{j=1}^{\infty} m_{t,t+j} D_{t+j}^A\right]$$

- Where $\mathbb{E}$ is expectations operator (allows for uncertainty over future), $A$ indexes an asset, $D_{t+j}^A$ is the cash flow generated by the asset in period $t+j$, and $m_{t,t+j} = \frac{\beta^j u'(C_{t+j})}{u'(C_t)}$ is the stochastic discount factor, and the summation allows security to pay cash flows in multiple periods (so more general than a discount bond)
- If we abstract from supply (assume bonds in zero net supply), then $C_t = Y_t$ and this determines equilibrium bond prices (and hence yields)

# Prices and Yields

- If operating in endowment economy with fixed bond supply, Euler equation determines bond prices (and hence yields)
- Yield on one period discount bond:

$$1 + i_{B,t} = \frac{1}{P_t^B}$$

- Useful to compare assets according to yields instead of prices. Why? Suppose you have a discount bond that pays out 2 in $t+1$ (label this $D$) but is otherwise identical to $B$
- Price of asset $D$ will be double price of $B$. But yields are identical:

$$1 + i_{B,t} = 1 + i_{D,t} = \frac{1}{P_t^B} = \frac{2}{P_t^D}$$

- Comparing yields puts assets with different levels of cash flow on "equal footing"

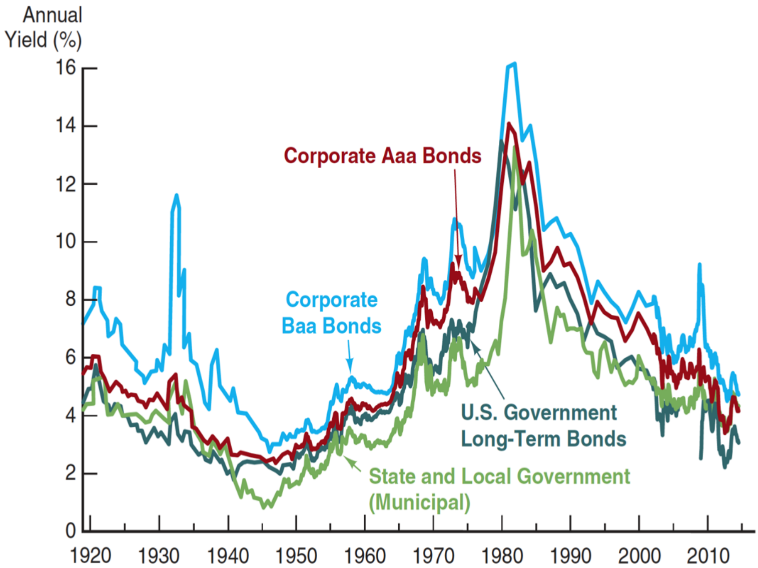
# Different Bond Characteristics

- Bonds with the same cash flow details (e.g. discount bonds) nevertheless often have very different yields
- Why is this?
- Aside from details about cash flows, bonds differ principally on two dimensions:
  1. Default risk
  2. Time to maturity
- The first, and potentially the second, generate uncertainty
- People don't like uncertainty: more uncertainty $\Rightarrow$ lower bond price (higher yield)

# Default Risk

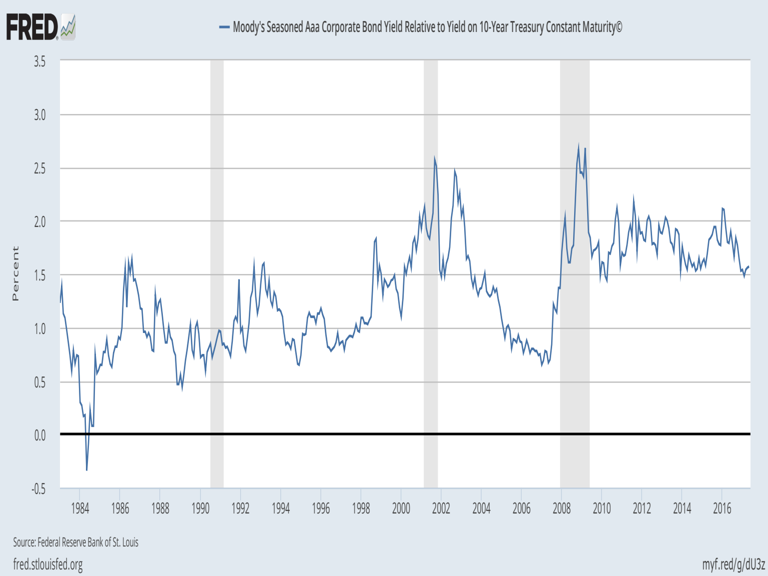
- Default occurs when the borrower decides to not make good on a promise to pay back all or some of his/her outstanding debts
- We think of federal government bonds as being (essentially) default-free: since government can always "print" money, should not explicitly default (though monetization of debt is effectively form of default)
- Corporate (and state and local government) bonds do have some default risk
- Rating agencies: Aaa is highest rated, then Bs, then Cs measure credit risk of lenders
- Risk premium: difference (i.e. spread) between yield on relatively more risky debt (e.g. Aaa corporate debt) and less risk debt (e.g. government debt), assuming same time to maturity

# Yields on Different Bonds



# Observations

- We see more or less exactly the pattern we would expect from our demand/supply analysis – riskier bonds have higher yields
- You demand a higher yield (expected return) to be compensated for taking on more risk
- Obvious exception: municipal bonds
- Why? Interest income on these bonds is exempt from federal taxes, which makes their expected returns higher, and therefore drives up price (and drives down yield)
- Important: interesting time variation in spreads (see next couple of slides)


FRED
Moody's Seasoned Aaa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity©
Percent
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-0.5
1984
1986
1988
1990
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Source: Federal Reserve Bank of St. Louis
fred.stlouisfed.org
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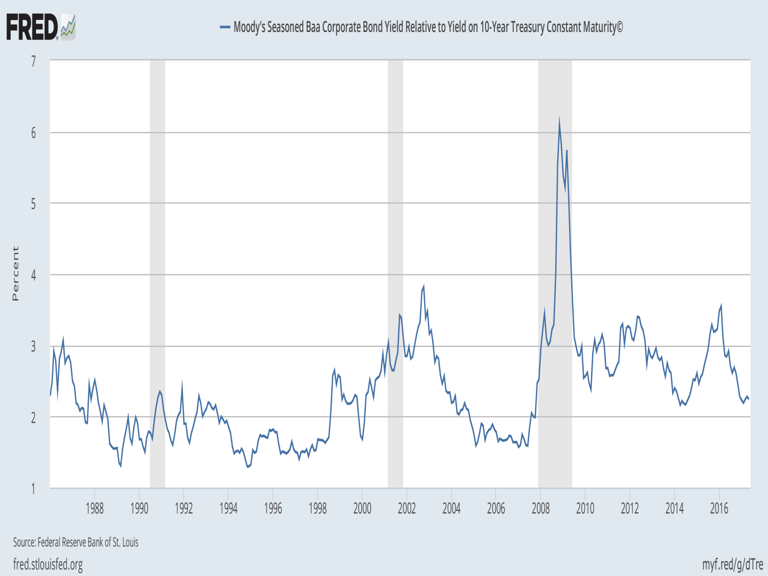

FRED
Moody's Seasoned Baa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity©
Percent
7
6
5
4
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2
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1988
1990
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Source: Federal Reserve Bank of St. Louis
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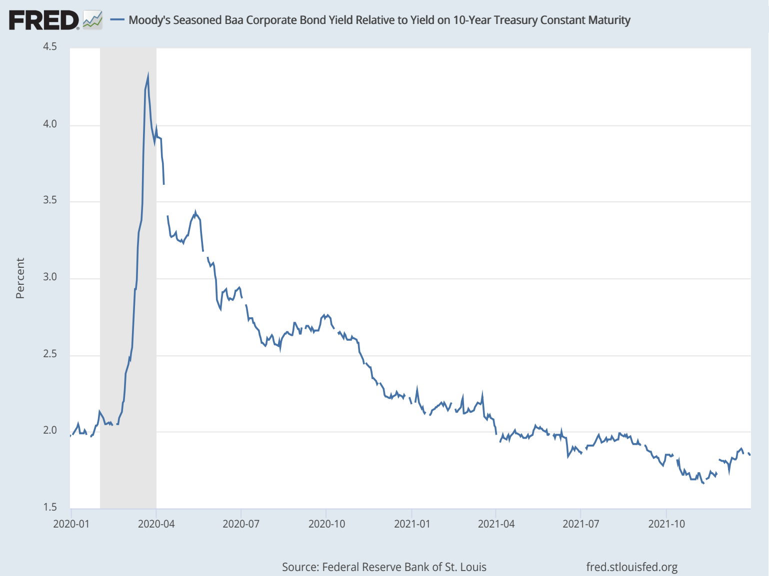
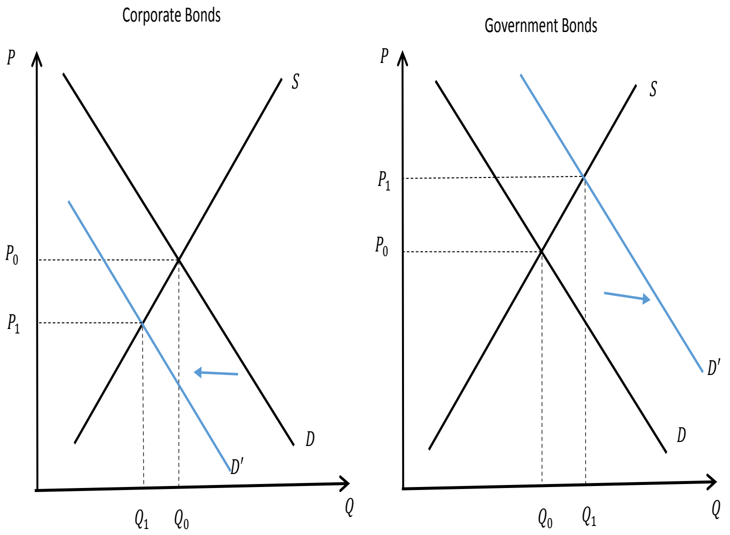

FRED
Moody's Seasoned Baa Corporate Bond Yield Relative to Yield on 10-Year Treasury Constant Maturity
4.5
4.0
3.5
3.0
2.5
2.0
1.5
Percent
2020-01
2020-04
2020-07
2020-10
2021-01
2021-04
2021-07
2021-10
Source: Federal Reserve Bank of St. Louis
fred.stlouisfed.org

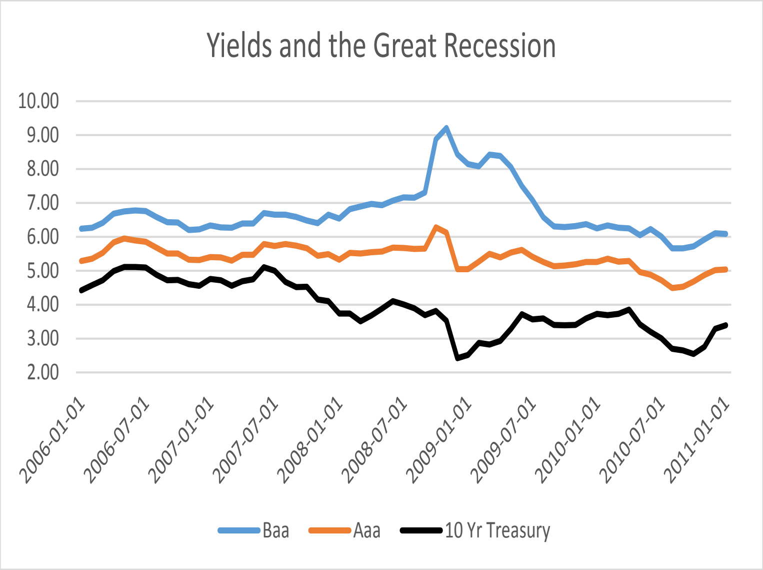
# Countercyclical Default Risk

- It stands to reason that default risk ought to be high when economy is in recession
- When default risk is high, we might expect a "flight to safety": reduces demand for risky bonds and increases demand for riskless bonds, which moves credit spread up
- This is consistent with what we see in the previous slides: credit spreads tend to rise during recessions

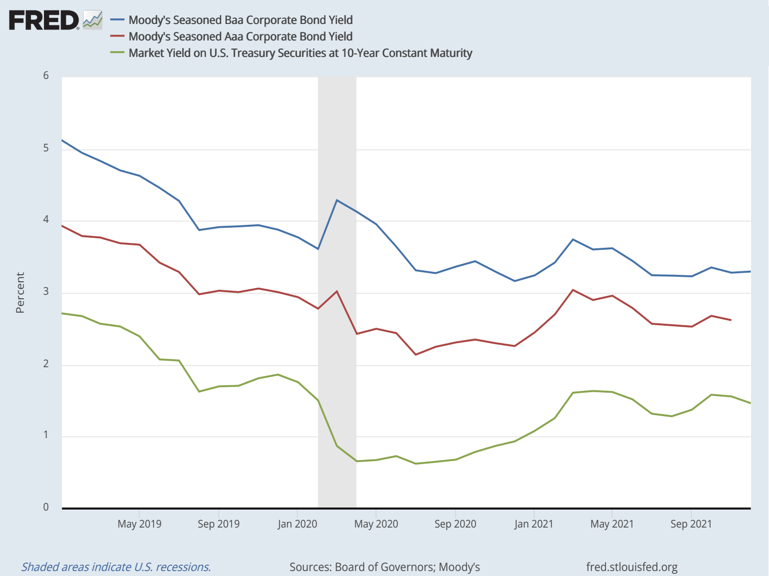
# Flight to Safety



# Yields During Great Recession



# Yields During COVID-19



# A Micro-Founded Approach

- Household lives for two periods. Receives endowment of income. Future endowment is uncertain
- Can save in period $t$ either via discount bonds $B_{1,t}$ or $B_{2,t}$. $B_{1,t}$ is risk-free. $B_{2,t}$ might default (completely, so pays back no principal)
- Current endowment is known. Future endowment is uncertain. Four possible states of the world in $t+1$:

$$\begin{aligned}
\text{State 1}: &\quad Y_{t+1} = Y_{t+1}^{h}, \ \text{risky bond pays} \\
\text{State 2}: &\quad Y_{t+1} = Y_{t+1}^{h}, \ \text{risky bond defaults} \\
\text{State 3}: &\quad Y_{t+1} = Y_{t+1}^{l}, \ \text{risky bond pays} \\
\text{State 4}: &\quad Y_{t+1} = Y_{t+1}^{l}, \ \text{risky bond defaults}
\end{aligned}$$

- Assume probabilities are $p_1$, $p_2$, $p_3$, and $p_4$, with $p_1 + p_2 + p_3 + p_4 = 1$.

# Expected Utility and Budget Constraints

- Household wishes to maximize expected utility:

$$
\begin{aligned}
U &= u(C_t) + \beta\,\mathbb{E}[u(C_{t+1})] \\
&= u(C_t) + \beta\,[p_1 u(C_{t+1}(1)) + p_2 u(C_{t+1}(2)) + p_3 u(C_{t+1}(3)) + p_4 u(C_{t+1}(4))]
\end{aligned}
$$

- Budget constraints must hold in each state of the world:

$$
\begin{aligned}
C_t + P^B_{1,t} B_{1,t} + P^B_{2,t} B_{2,t} &= Y_t \\
C_{t+1}(1) &= Y^h_{t+1} + B_{1,t} + B_{2,t} \\
C_{t+1}(2) &= Y^h_{t+1} + B_{1,t} \\
C_{t+1}(3) &= Y^l_{t+1} + B_{1,t} + B_{2,t} \\
C_{t+1}(4) &= Y^l_{t+1} + B_{1,t}
\end{aligned}
$$

# Facts About Expectations Operators

- Just probability-weighted sum of outcomes:

$$\mathbb{E}[X_{t+1}] = p_1 X_{t+1}(1) + p_2 X_{t+1}(2) + \cdots + p_n X_{t+1}(n)$$

- Useful facts:

$$\mathbb{E}[f(X_{t+1})] \neq f\left(\mathbb{E}[X_{t+1}]\right) \text{ unless } f(\cdot) \text{ is linear}$$

$$\mathbb{E}[aX_{t+1})] = a\,\mathbb{E}[X_{t+1}]$$

$$\mathbb{E}[X_{t+1} + Y_{t+1}] = \mathbb{E}[X_{t+1}] + \mathbb{E}[Y_{t+1}]$$

$$\frac{\partial\,\mathbb{E}[f(X_{t+1})]}{\partial X_{t+1}} = \mathbb{E}[f'(X_{t+1})]$$

$$\mathbb{E}[X_{t+1} Y_{t+1}] = \mathbb{E}[X_{t+1}]\,\mathbb{E}[Y_{t+1}] + cov(X_{t+1}, Y_{t+1})$$

$$cov(X_{t+1}, Y_{t+1}) = \mathbb{E}\left[\left(X_{t+1} - \mathbb{E}(X_{t+1})\right)\left(Y_{t+1} - \mathbb{E}(Y_{t+1})\right)\right]$$

# Optimization in an Endowment Economy

- Risk-free bond:

$$P_{1,t} = \mathbb{E}\left[\frac{\beta u'(Y_{t+1})}{u'(Y_t)}\right]$$

- Risky bond:

$$P_{2,t} = \mathbb{E}\left[\frac{\beta u'(Y_{t+1})}{u'(Y_t)} D_{2,t+1}\right]$$

- Where $D_{2,t+1}$ is the payout on the risky bond (either 1 or 0). Yields:

$$1 + i_{1,t} = \frac{1}{P^B_{1,t}} = \frac{u'(Y_t)}{\mathbb{E}[\beta u'(Y_{t+1})]}$$

$$1 + i_{2,t} = \frac{\mathbb{E}[D_{2,t+1}]}{P^B_{2,t}} = \frac{\mathbb{E}[D_{2,t+1}]u'(Y_t)}{\mathbb{E}[\beta u'(Y_{t+1}) D_{2,t+1}]}$$

# Risk Premium

- Ratio of yields is:

$$\frac{1+i_{2,t}}{1+i_{1,t}} = \frac{\mathbb{E}[D_{2,t+1}]\,\mathbb{E}[u'(Y_{t+1})]}{\mathbb{E}[D_{2,t+1}u'(Y_{t+1})]}$$

- One would be tempted to “distribute” the expectations operator and conclude this is 1 (so no difference in yields)
- But in general cannot do this
- After using properties of expectations, one gets:

$$i_{2,t} - i_{1,t} \approx -\frac{cov(D_{2,t+1}, u'(Y_{t+1}))}{\mathbb{E}[D_{2,t+1}u'(Y_{t+1})]}$$

- Risk premium depends on covariance of bond payout with $u'(Y_{t+1})$, not variance of payout per se

# Intuition

- Assuming $u''(\cdot) < 0$ (what we call risk averse), you desire to smooth consumption
- Both across time as well as across states
- Periods where income is high: you want to save to give yourself resources in other periods
- For assets, you like assets that have comparatively high payouts in periods where income is low (i.e. periods where $u'(Y_{t+1})$ is high, so consumption is highly valued). This helps you smooth consumption across states
- Dislike assets that don't help you do this
- Assets most likely to default in "bad" periods (where $u'(Y_{t+1})$ is high): you demand a higher yield to hold them (lower price)
- Countercyclical default risk: source of bond risk premia

# Uncertainty and the Risk Premium

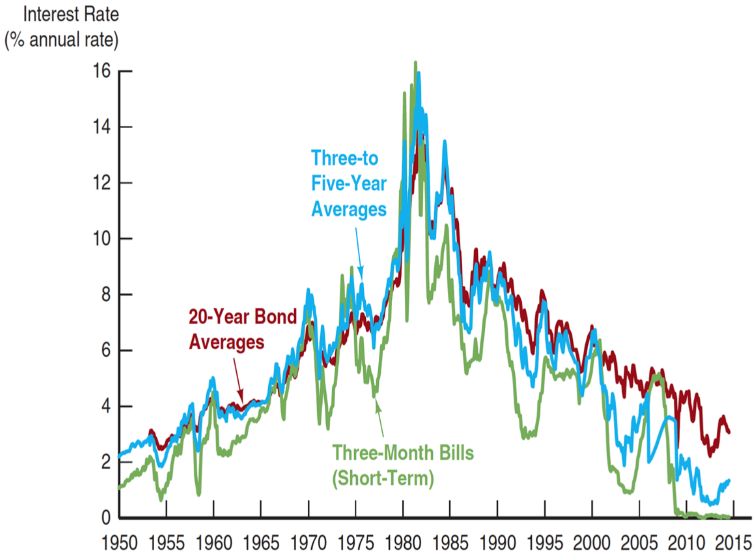
- $Y_{t+1}^h = 1.1,\ Y_{t+1}^l = 0.9,\ Y_t = 1,\ \beta = 0.95$, log utility
- Recall: state 1: income high, bond pays; state 2: income high, bond defaults; state 3, income low, bond pays; state 4: income low, bond defaults

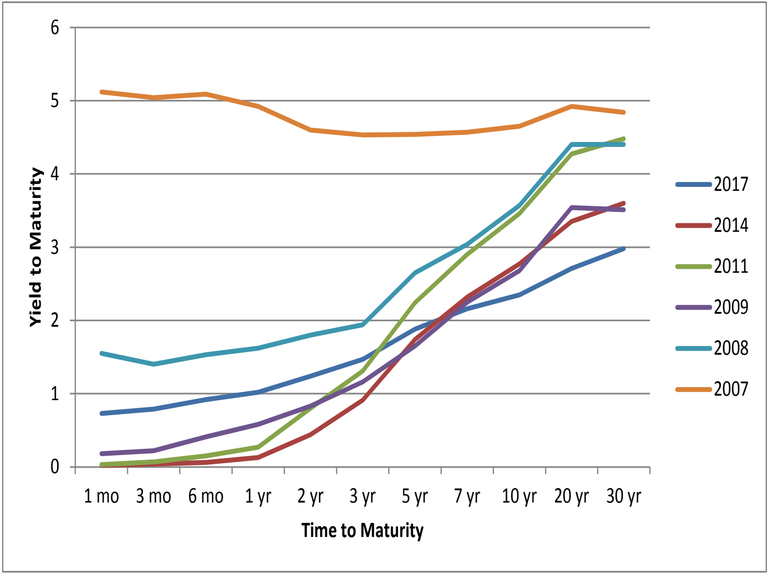
| Probabilities | $\mathbb{E}(D)$ | $\mathbb{E}(Y_{t+1})$ | $\mathbb{E}(D_{2,t+1} \mid Y_{t+1}^h)$ | $\mathbb{E}(D_{2,t+1} \mid Y_{t+1}^l)$ | $i_{2,t} - i_{1,t}$ |
|---|---|---|---|---|---|
| $p_1 = p_2 = p_3 = p_4 = 0.25$ | 0.5 | 1 | 0.5 | 0.5 | 0.00 |
| $p_1 = 0.5,\ p_2 = p_3 = 0,\ p_4 = 0.5$ | 0.5 | 1 | 1 | 0 | 0.12 |
| $p_1 = p_4 = 0,\ p_2 = p_3 = 0.5$ | 0.5 | 1 | 0 | 1 | -0.09 |
| $p_1 = 0.4,\ p_2 = 0.1,\ p_3 = 0.1,\ p_4 = 0.4$ | 0.5 | 1 | 0.8 | 0.2 | 0.07 |

- If default is mostly likely in periods where income is low, you demand a higher yield relative to the risk-free bond
- In principal risk premium could be negative if default is most likely when income is high
- Also, risk premium doesn't depend on uncertainty over default per se

# Term Structure of Interest Rates

- Bonds with otherwise identical cash flows and risk characteristics can have different times to maturity (or just maturities, for short)
- Think about (default) riskless Treasury securities with different maturities
  - Treasury bills: discount bonds with maturities of 4, 13 (three month T-Bill), 26, or 52 week maturities
  - Treasury notes: coupon bonds with maturities of 2, 3, 5, 7 and 10 years
  - Treasury bonds: coupon bonds with maturities $> 10$ years
- How do yields vary with time to maturity for a bond with otherwise identical default characteristics?
- A plot of yields on bonds against the time to maturity is known as a yield curve


Interest Rate
(% annual rate)
16
14
12
10
8
6
4
2
0
Three-to
Five-Year
Averages
20-Year Bond
Averages
Three-Month Bills
(Short-Term)
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015



Yield to Maturity
6
5
4
3
2
1
0
1 mo
3 mo
6 mo
1 yr
2 yr
3 yr
5 yr
7 yr
10 yr
20 yr
30 yr
Time to Maturity
2017
2014
2011
2009
2008
2007

# Observations

- The following observations can be made from previous two pictures
  1. Yields on bonds of different maturities tend to move together
  2. Yield curves are upward-sloping *most of the time*
     - The slope of the yield curve is often predictive of recession. Flat or downward-sloping ("inverted") yield curves often precede recessions
  3. When short term interest rates are low, yield curves are more likely to be upward-sloping

# Yield Curves Prior to Recent Recessions

# Theories of the Yield Curve

- Main theories of the term structure:
  1. Expectations hypothesis
  2. Segmented markets
  3. Liquidity premium theory
- Expectations hypothesis: bonds of different maturities are perfect substitutes
- Segmented markets: bonds with different maturities are not substitutes
- Liquidity premium: bonds with different maturities are imperfect substitutes
  - Essentially expectations hypothesis + uncertainty, where longer maturity bonds are riskier (i.e. less liquid) than short maturity bonds because their prices fluctuate more

# Expectations Hypothesis

- Expectations hypothesis: the yield on a long maturity bond is the average of the expected yields on shorter maturity bonds
- For example, suppose you consider 1 and 2 year bonds
- The yield on a 1 year bond is 4 percent; you expect the yield on a 1 year bond one year from now to be 6 percent
- Then the yield on a two year bond ought to be 5 percent $(0.5 \times (4 + 6) = 5)$
- Why? If you buy a two 1 year bonds in succession, your yield over the two year period is (approximately, ignoring compounding) 10 percent: $(1 + 0.04) \times (1 + 0.06) - 1 = 0.1024 \approx 0.10$
- If the 2 and 1 year bonds are perfect substitutes, the yield on the two year bond has to be the same: $(1 + i)^2 = 0.1 \Rightarrow i \approx 0.05$
- Demand and supply: if you expect future short yields to rise, this lowers expected profitability of long term bonds in the present, reducing demand, driving down price, and driving up yield: current long term yield tells you something about expected future short term yields

# Microfoundations of Expectations Hypothesis

- Suppose a household lives for three periods, receiving an exogenous endowment of income in each period
- In period $t$, can purchase one or two period maturity discount bonds:

$$C_t + P_{t,t,t+1}B_{t,t,t+1} + P_{t,t,t+2}B_{t,t,t+2} = Y_t$$

- Subscript notation. First subscript denotes date of observation (i.e. period $t$ is the "present"). Second subscript denotes date of issuance, third subscript denotes date of maturity. So $B_{t,t,t+1}$ is the quantity of a bond purchased in period $t$ (first subscript), that was issued in period $t$ (second subscript), which matures in $t+1$ (third subscript)

# Future Budget Constraints

- Assume, for now, no uncertainty over the future
- In $t+1$, household can buy/sell newly issued one period bonds or change its holdings of existing two period bonds (no new issues of two period bonds since there is no $t+3$). Receives exogenous endowment income plus maturing one period bonds

$$C_{t+1} + P_{t+1,t+1,t+2} B_{t+1,t+1,t+2} + P_{t+1,t,t+2}(B_{t+1,t,t+2} - B_{t,t,t+2}) = Y_{t+1} + B_{t,t,t+1}$$

- $B_{t+1,t,t+2} - B_{t,t,t+2}$: change in stock of two period bonds (can buy/sell on secondary market)
- In final period just consume available resources:

$$C_{t+2} = Y_{t+2} + B_{t+1,t+1,t+2} + B_{t+1,t,t+2}$$

# Preferences and Optimality Conditions

- Lifetime utility:

$$U = \ln C_t + \beta \ln C_{t+1} + \beta^2 \ln C_{t+2}$$

- Optimality conditions:

$$\begin{aligned} B_{t,t,t+1} &: \; P_{t,t,t+1} u'(C_t) = \beta u'(C_{t+1}) \\ B_{t,t,t+2} &: \; P_{t,t,t+2} u'(C_t) = \beta P_{t+1,t,t+2} u'(C_{t+1}) \\ B_{t+1,t+1,t+2} &: \; P_{t+1,t+1,t+2} u'(C_{t+1}) = \beta u'(C_{t+2}) \\ B_{t+1,t,t+2} &: \; P_{t+1,t,t+2} u'(C_{t+1}) = \beta u'(C_{t+2}) \end{aligned}$$

- Note: must have $P_{t+1,t,t+2} = P_{t+1,t+1,t+2}$. Price of bond depends on time to maturity not date of issuance

# Combining These Together

- If we combine these FOC, we get:

$$P_{t,t,t+2} = P_{t,t,t+1} \times P_{t+1,t+1,t+2}$$

- Price of long bond is product of short maturity bond prices
- Yield is discount rate which equates bond price to present discounted value of cash flows:

$$\frac{1}{(1+i_{t,t+2})^2} = \frac{1}{1+i_{t,t+1}} \frac{1}{1+i_{t+1,t+2}}$$

- Note: don't need to keep track of issuance date, just date of observation and maturity date
- Approximately:

$$i_{t,t+2} \approx \frac{1}{2}\left[i_{t,t+1} + i_{t+1,t+2}\right]$$

- Extends to $n$ period maturity:

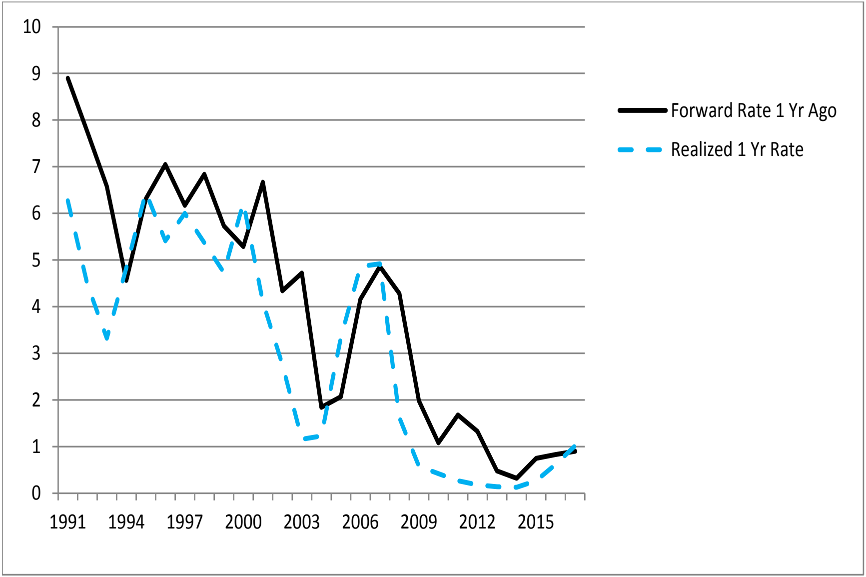
$$i_{t,t+n} \approx \frac{1}{n}\left[i_{t,t+1} + i_{t+1,t+2} + \cdots + i_{t+n-1,t+n}\right]$$

# Forward Rates

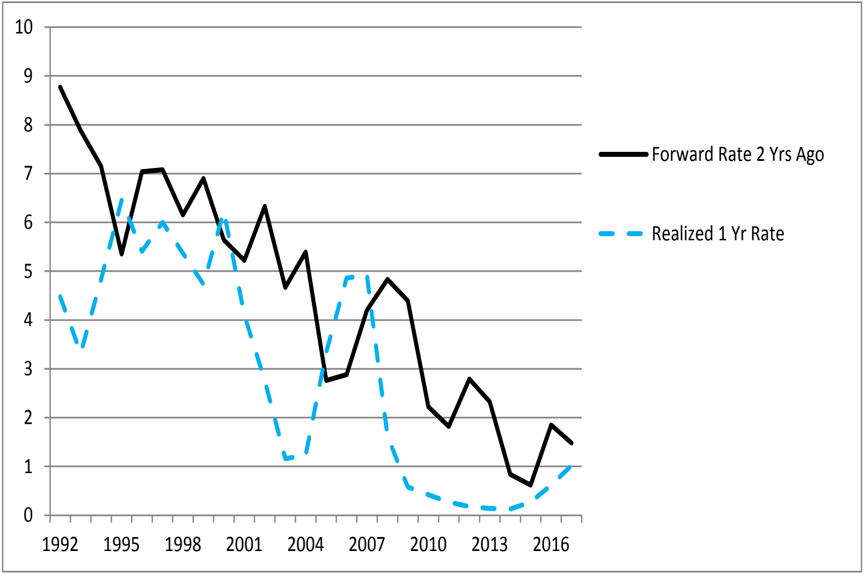
- Expectations hypothesis can be used to infer market expectations of future short maturity interest rates
- How? By using present yields of bonds on different maturities
- We call these forward rates:

$$
\begin{aligned}
i^e_{t+1,t+2} &= 2i_{t,t+2} - i_{t,t+1} \\
i^e_{t+2,t+3} &= 3i_{t,t+3} - 2i_{t,t+2} \\
&\vdots \\
i^e_{t+n-1,t+n} &= ni_{t,t+n} - (n-1)i_{t,t+n-1}
\end{aligned}
$$

# Implied 1 Year Ahead Forward Rate



# Implied 2 Year Ahead Forward Rate



# Evaluating the Expectations Hypothesis

- Expectations hypothesis can help make sense of several facts:
  1. Long and short term rates tend to move together. If current short term rates are low and people expect them to stay low for a while (interest rates are quite persistent), then long term yields will also be low
  2. Why a flattening/inversion of yield curve can predict recessions. If people expect economy to go into recession, they will expect the Fed to lower short term interest rates in the future. If short term interest rates are expected to fall, then long term rates will fall relative to current short term rates, and the yield curve will flatten.
- Where the expectations hypothesis fails: why are yield curves almost always upward-sloping? If interest rates are mean-reverting, then the typical yield curve ought to be flat, not upward-sloping

# Segmented Markets

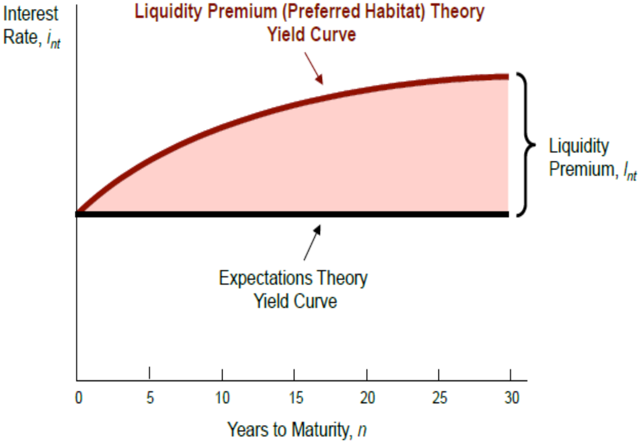
- Segmented markets: household heterogeneity
- Some households like long maturity debt, others prefer short maturity debt
  - e.g. investors match maturity to their retirement horizon
- This could result in higher yields on long-term bonds in equilibrium (or it could not)
- But because there is no relation between short- and long-term yields, it fails to make sense of other facts about the term structure

# Liquidity Premium Theory

- Segmented markets hypothesis cannot explain why interest rates of different maturities tend to move together, or account for why you can predict future short term rates from long term yields
- Liquidity premium theory: essentially expectations hypothesis with uncertainty
- Bonds of different maturities are substitutes, but not perfect substitutes
- Why? Long maturity bonds are riskier – they bear interest rate risk where bond prices will fluctuate
- Savers demand compensation for this risk
- Relationship between long and short rates:

$$i_{t,t+n} = \underbrace{\frac{i_{t,t+1} + i^e_{t+1,t+2} + \dots i^e_{t+n-1,t+n}}{n}}_{\text{Expectations Hypothesis}} + \underbrace{l_{n,t}}_{\text{Liquidity Premium}}$$

- The liquidity premium, $l_{n,t}$ (also called term premium), is increasing in the time to maturity


Interest
Rate, $i_{nt}$
Liquidity Premium (Preferred Habitat) Theory
Yield Curve
Liquidity
Premium, $l_{nt}$
Expectations Theory
Yield Curve
0
5
10
15
20
25
30
Years to Maturity, $n$

# Microfoundations of Liquidity Premium

- Same setup as before, except future endowments are uncertain
- Assume no default risk
- Endowment in $t+1$ or $t+2$ can be high or low – $Y_{t+1} = Y_{t+1}^h$ or $Y_{t+1}^l$ (and similarly for $Y_{t+2}$)
- Assume probability is $p$ of high income draw and $1-p$ of low endowment; same in $t+1$ and $t+2$
- Budget constraints (significantly) more complicated – two states of the world in $t+1$, and essentially four states of the world in $t+2$
  - Why four in $t+2$? Because consumption in $t+2$ depends on decisions made in $t+1$. So $2 \times 2 = 4$.
- Objective function is to maximize expected lifetime utility
- We will skip a bunch of (nasty) steps but it is laid out for you in GLS Ch. 34

# Optimality Conditions

- Optimality conditions look the same as above, but with expectations operators
- Once again, price of bonds in $t+1$ depends only on time to maturity, not date of issuance
- Using these facts, we get:

$$P_{t,t,t+1}u'(C_t) = \beta\,\mathbb{E}[u'(C_{t+1})]$$
$$P_{t,t,t+2}u'(C_t) = \beta\,\mathbb{E}[P_{t+1,t+1,t+2}u'(C_{t+1})]$$

- Intuitive MB = MC interpretation, just with expectations operators

## Combining These Conditions

- Divide second FOC by first and re-arrange to get:

$$P_{t,t,t+2} = P_{t,t,t+1} \frac{\mathbb{E}[P_{t+1,t+1,t+2} u'(C_{t+1})]}{\mathbb{E}[u'(C_{t+1})]}$$

- One is tempted to "distribute" the expectations operator in the numerator, which would yield:

$$P_{t,t,t+2} = P_{t,t,t+1} \, \mathbb{E}[P_{t+1,t+1,t+2}]$$

- This would be the natural analog of what we just had, allowing for uncertainty over the future bond price
- But in general you cannot do this unless (i) $u'(C_{t+1})$ is a constant (i.e. $u(\cdot)$ is linear) or (ii) $P_{t+1,t+1,t+2}$ is uncorrelated with $u'(C_{t+1})$
- Note: this is very much like what we saw when studying risk premia – you cannot "distribute" the expectations operator through a product

# Re-Arranging

- Can write this expression as:

$$P_{t,t,t+2} = P_{t,t,t+1}\,\mathbb{E}[P_{t+1,t+1,t+2}]\frac{\mathbb{E}[P_{t+1,t+1,t+2}u'(C_{t+1})]}{\mathbb{E}[P_{t+1,t+1,t+2}]\,\mathbb{E}[u'(C_{t+1})]}$$

- If income equals consumption (bonds in zero net supply), we get:

$$P_{t,t,t+2} = \underbrace{P_{t,t,t+1}\,\mathbb{E}[P_{t+1,t+1,t+2}]}_{\text{Expectations Hypothesis}}\underbrace{\left(1 + \frac{cov(P_{t+1,t+1,t+2}, u'(Y_{t+1}))}{\mathbb{E}[P_{t+1,t+1,t+2}]\,\mathbb{E}[u'(Y_{t+1})]}\right)}_{\text{Term Premium}}$$

- We would expect covariance term to be negative
- When $Y_{t+1}$ is low, $u'(Y_{t+1})$ is high and demand for one period bonds is low (people don't want to save), so $P_{t+1,t+1,t+2}$ is low
- Negative covariance $\Rightarrow$ two period bond trades at a discount relative to what expectations hypothesis would predict

# Prices to Yields

- Written in terms of yields, approximately we get:

$$(1+i_{t,t+2})^2 = (1+i_{t,t+1})\,\mathbb{E}[1+i_{t+1,t+2}]\left(1+\frac{cov(P_{t+1,t+1,t+2},u'(Y_{t+1}))}{\mathbb{E}[P_{t+1,t+1,t+2}]\,\mathbb{E}[u'(Y_{t+1})]}\right)^{-1}$$

- Which is approximately:

$$i_{t,t+2} \approx \frac{1}{2}\left[i_{t,t+1} + i^{e}_{t+1,t+2}\right] + \frac{1}{2}tp_t$$

- Where:

$$tp_t = -\ln\left(1+\frac{cov(P_{t+1,t+1,t+2},u'(Y_{t+1}))}{\mathbb{E}[P_{t+1,t+1,t+2}]\,\mathbb{E}[u'(Y_{t+1})]}\right)$$

- Where we would expect $tp_t > 0$ if $cov(\cdot) < 0$

# General Case

- For an $n$ period bond, we get:

$$i_{t,t+n} \approx \frac{1}{n}\left[i_{t,t+1} + i^e_{t+1,t+2} + \ldots i^e_{t+n-1,t+n}\right] + \frac{n-1}{n} tp_t$$

- Where $tp_t$ is exactly the same as before – based on covariance between future one period bond price and future marginal utility
- So as $n$ gets bigger, $\frac{n-1}{n} tp_t$ gets bigger (at a decreasing rate)
- This delivers positive slope of typical yield curve, but expectations hypothesis logic drives movements in short and long term yields

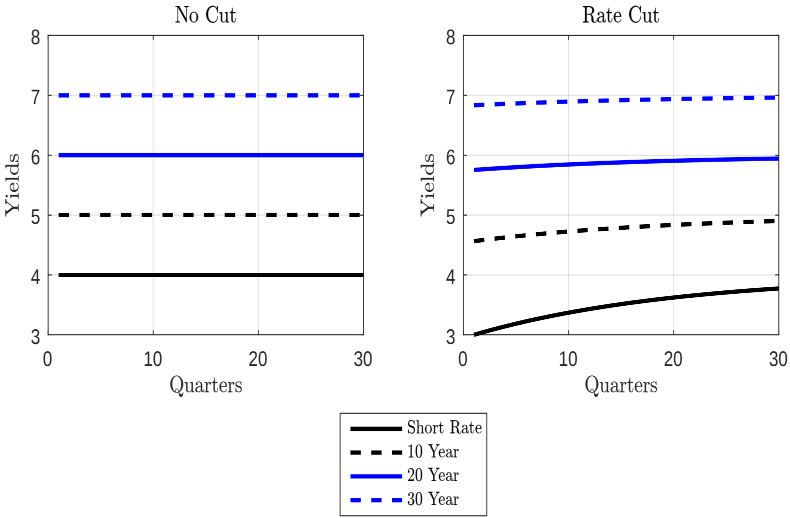
# A First Look at Quantitative Easing and Unconventional Monetary Policy

- The interest rates relevant for most investment and consumption decisions are long term (e.g. mortgage) and risky (e.g. Baa corporate bond)
- Conventional monetary policy targets short term and riskless interest rates (e.g. Fed Funds Rate)
- In practice, something like a 10 Year Treasury yield serves as a benchmark interest rate for all other kinds of interest rates – mortgage rates, credit card rates, student loan rates
- These are the rates relevant for economic decisions on spending and saving
- Out theory of bond pricing helps us understand the connection between the these different kinds of interest rates

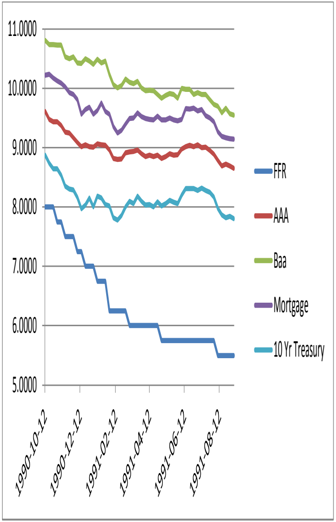
# Monetary Policy in Normal Times

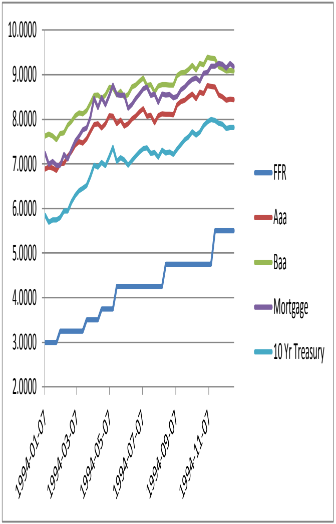
- Think of a world where the liquidity premium theory holds
- A central bank lowers (raises) short term interest rates and is expected to keep these low (high) for some time
- This ought to also lower (raise) longer term rates to the extent to which longer term rates are the average of expected shorter term rates
- Holding risk factors constant, substitutability between bonds means that riskier yields also ought to fall (increase)
- Simulation:
  - Consider 1 period, 10 year, 20 year, and 30 year bonds
  - Fixed liquidity premia of 1, 2, and 3
  - Short term rate is cut from 4 to 3 persistently
  - Simulate yields for 30 quarters

# Theoretical Simulation

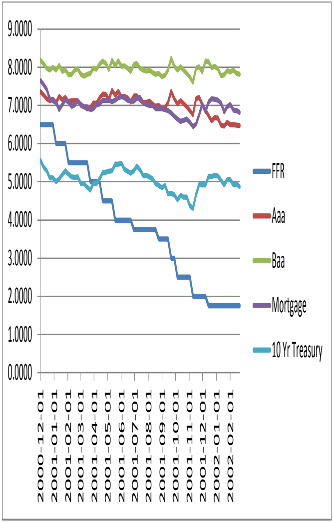


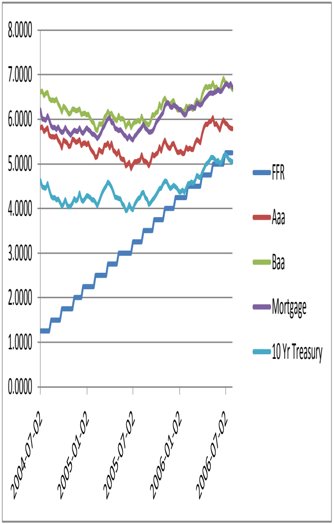
# Monetary Loosening and Tightening in Historical Episodes



# Monetary Loosening and Tightening in Historical Episodes



# Term Structure Puzzle?

- If you look at last two pictures, in the 1990s other rates tracked the FFR reasonably closely, but this falls apart in the 2000s
- This has been referred to as the term structure puzzle and is discussed in Poole (2005)
- From perspective of expectations hypothesis, would expect longer maturity, riskier rates to move along with shorter maturity rates. Not what we see in 2000s
- But two key issues:
  - Forecastability: if people expected the Fed to raise rates starting in 2004/2005 from the perspective of 2002/2003, this would have been incorporated into long rates before the FFR started to move. Most people think Fed policy has become more forecastable
  - Persistence: behavior of long rates depends on how persistent changes in short rates are expected to

# Unconventional Monetary Policy

- In a world where Fed Funds Rate is at or very near zero, conventional monetary loosening isn't on the table
- Unconventional monetary policy:
  1. Quantitative Easing (or Large Scale Asset Purchases): purchases of longer maturity government debt or risky private sector debt. Idea: raise demand for this debt, raise price, and lower yield. See here
  2. Forward Guidance: promises to keep future short term interest rates low. Idea is to work through expectations hypothesis and to lower long term yields immediately. See here
- Ben Bernanke: "The problem with quantitative easing is that it works in practice but not in theory"
  - Under expectations hypothesis, cannot affect long term yields without impacting path of short term yields
  - Not clear why it ought to be able to impact liquidity/term premium, which depends on a covariance term
  - Segmented markets?