Advanced Macro: Bernanke, Gertler, and Gilchrist (1999, *Handbook of Macroeconomics*)

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1 Overview

This note describes the celebrated Bernanke, Gertler, and Gilchrist (1999) financial accelerator paper. The paper is in the tradition of Bernanke and Gertler (1989, *AER*) and Carlstrom and Fuerst (1997, *AER*). But it departs in a couple of important ways. First, it is a New Keynesian model with sticky prices (in contrast to the RBC model of Carlstrom and Fuerst and the simplified neoclassical OLG model in Bernanke and Gertler). Second, it applies the agency friction to the financing of the *entire* capital stock, whereas in Carlstrom and Fuerst it is only *new* investment that is subject to the agency friction. This has the effect of resulting in more amplification. A third more minor difference is that the loan over which there are agency frictions is intertemporal as opposed to intratemporal in Carlstrom and Fuerst.

Below I proceed somewhat non-linearly. BGG do not do a very good job laying out the details of their model, so it is very difficult to recreate from scratch (better expositions are in Christiano, Motto, and Rostagno 2014, *AER*, and Carlstrom, Fuerst, and Paustian 2016, *AEJ: Macro*). As such, I'm going to simply start with the linearized equilibrium conditions that BGG have. The key equation is as follows:

$$\mathbb{E}_t\, r^k_{t+1} - r_t = \nu\left[n_t - (q_t + k_{t+1})\right] \tag{1}$$

Here r_t is the safe real interest rate, n_t is net worth, q_t is the price of capital, and k_{t+1} is the capital stock accumulated in t available for production in $t+1$. $\mathbb{E}_t\, r^k_{t+1}$ is the expected return on capital. The left hand side can be interpreted as an external finance premium, and the right hand side is the negative of a leverage ratio (i.e. assets, in linearized form, are $q_t + k_{t+1}$, relative to equity, n_t. $\nu > 0$ means that there are agency frictions. The key insight, as in the earlier papers, is that increases in borrower net worth, n_t, reduce agency frictions if $\nu > 0$. This lowers the external finance premium and stimulates investment and aggregate demand. The notion of the "accelerator" effect is that expansionary shocks which much asset prices up are *more* expansionary because they accordingly improve the balance sheet condition of borrowers, which further leads to a boom and more asset price appreciation.

2 Linearized Model

Let lowercase variables with time subscripts donate log-deviations from steady state. The linearized model is as follows:

$$y_t = \frac{C}{Y}c_t + \frac{I}{Y}i_t + \frac{G}{Y}g_t + \frac{C^e}{Y}c_t^e \tag{2}$$

$$c_t = -r_t + \mathbb{E}_t\, c_{t+1} \tag{3}$$

$$c_t^e = n_t \tag{4}$$

$$\mathbb{E}_t\, r_{t+1}^k - r_t = -\nu\left[n_t - (q_t + k_{t+1})\right] \tag{5}$$

$$r_t^k = (1-\epsilon)(y_t - k_t - x_t) + \epsilon q_t - q_{t-1} \tag{6}$$

$$q_t = \varphi(i_t - k_t) \tag{7}$$

$$y_t = a_t + \alpha k_t + (1-\alpha)\Omega h_t \tag{8}$$

$$y_t - h_t - x_t - c_t = \eta^{-1} h_t \tag{9}$$

$$\pi_t = -\kappa x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} \tag{10}$$

$$k_{t+1} = \delta i_t + (1-\delta)k_t \tag{11}$$

$$n_t = \gamma \frac{RK}{N}(r_t^k - r_{t-1}) + r_{t-1} + n_{t-1} \tag{12}$$

$$r_t^n = \rho r_{t-1}^n + \zeta \pi_{t-1} + s_r \varepsilon_{r,t} \tag{13}$$

$$r_t^n = r_t + \mathbb{E}_t\, \pi_{t+1} \tag{14}$$

$$a_t = \rho_a a_{t-1} + s_a \varepsilon_{a,t} \tag{15}$$

$$g_t = \rho_g g_{t-1} + s_g \varepsilon_{g,t} \tag{16}$$

(2) is the resource constraint, where the upper-case terms are expenditure shares. (3) is the linearized Euler equation for bonds, assuming log utility. r_t is the real interest rate. Consumption of entrepreneurs, c_t^e, is just proportional to the net worth of entrepreneurs. Each period, a fixed fraction of entrepreneurs die and consume their net worth, giving rise to this expression. (5) is sort of the key relationship, showing a positive relationship between leverage, $q_t + k_{t+1} - n_t$, and the external finance premium, $\mathbb{E}_t\, r_{t+1}^k - r_t$. This is governed by the parameter ν. If $\nu = 0$, there is no external finance premium and hence no financial accelerator. (6) is the ex-post return on capital – there is sort of bad notation here with the ϵ, which is not an elasticity of substitution. (7) is the linearized first order condition for investment. (2)-(7) constitute the aggregate demand block of the model.

(8)-(10) constitute the aggregate supply block. (8) is the production function. (9) is the labor market-clearing condition after having eliminated the wage rate. x_t is the markup of price over marginal cost; equivalently, $-x_t$ is real marginal cost. (10) is the Phillips curve.

(11)-(12) show the evolution of state variables, capital and net worth. γ is the fraction of surviving entrepreneurs while R, K, and N are steady state values.

(13) is the Taylor rule, while (14) is the Fisher relationship. (15)-(16) describe the evolution of productivity and government spending.

Overall, $\left\{y_t, c_t, i_t, g_t, c_t^e, r_t, n_t, r_t^k, q_t, x_t, k_t, h_t, \pi_t, r_t^n, a_t\right\}$ constitute a linear system with 15 variables and 15 equations.

2.1 Aside: Where Do These Come From?

The following are the agents in the model:

1. Household
2. Retailers
3. Wholesale producers
4. Government

The household sector is standard. Retailers are just a trick to introduce Calvo price-setting. The government conducts policy via a Taylor rule and consumes an exogenous amount of output. The action is really on the wholesale firm side. Each period, the wholesalers have to get a loan to finance *the entirety* of next period's capital stock, subject to idiosyncratic returns to capital, as capture by a variable ω_t. This is like the setup in Carlstrom and Fuerst (1997), except the agency friction applies to the producers of output, rather than the producers of new investment goods. The loan contract is also *intertemporal* as opposed to intratemporal. In what follows below, I will *briefly* describe how to get to the linearized conditions described above. I will then spend some more time on the formal contracting problem.

The household problem is standard. There is an Euler equation for deposits/bonds and an intratemporal labor supply condition:

$$\frac{1}{C_t} = \beta R_t \, \mathbb{E}_t \, \frac{1}{C_{t+1}} \tag{17}$$

$$\frac{\xi}{1 - H_t} = \frac{W_t}{C_t} \tag{18}$$

Taking logs of these and letting $z_t = d \ln Z_t$ for generic variable Z_t:

$$-\ln C_t = \ln \beta + \ln R_t - \mathbb{E}_t \ln C_{t+1}$$
$$c_t = -r_t + \mathbb{E}_t \, c_{t+1}$$

Which is (3). For the labor supply condition:

$$\ln \xi - \ln(1 - H_t) = \ln W_t - \ln C_t$$
$$\frac{1}{1-H} dH_t = w_t - c_t$$
$$\frac{H}{1-H} h_t = w_t - c_t$$

Which is (9) when you define $\eta = \frac{1-H}{H}$ as the Frisch elasticity and note the definition of the wage from the wholesale producer problem (see below).

The capital accumulation equation is:

$$K_{t+1} = \Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta) K_t \tag{19}$$

The function $\Phi(\cdot)$ is defined where $\Phi(0) = 0$, $\Phi(\delta) = 1$, and $\Phi'(\delta) = 1$. Take logs and totally differentiate:

$$\begin{aligned}
\ln K_{t+1} &= \ln \left[\Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta) K_t\right] \\
k_{t+1} &= \frac{1}{K}\left[\Phi\left(\frac{I}{K}\right) dK_t + \Phi'\left(\frac{I}{K}\right) dI_t - \Phi'\left(\frac{I}{K}\right)\left(\frac{I}{K}\right) dK_t + (1-\delta) dK_t\right] \\
k_{t+1} &= \delta k_t + \frac{I}{K} i_t - \delta k_t + (1-\delta) k_t \\
k_{t+1} &= \delta i_t + (1-\delta) k_t
\end{aligned}$$

Which is (11).

Now consider the Tobin's Q relationship. This comes from the optimal choice of investment by firms subject to the adjustment cost embedded in the accumulation equation above. In nonlinear form:

$$Q_t = \left[\Phi'\left(\frac{I_t}{K_t}\right)\right]^{-1} \tag{20}$$

Take logs and totally differentiating, noting that $Q = 1$:

$$\begin{aligned}
\ln Q_t &= -\ln \left[\Phi'\left(\frac{I_t}{K_t}\right)\right] \\
q_t &= -\frac{1}{Q}\left[\Phi''\left(\frac{I}{K}\right)\left(\frac{dI_t}{K} - \frac{I}{K^2} dK_t\right)\right] \\
q_t &= -\Phi''(\delta)\delta[i_t - k_t]
\end{aligned}$$

Then $\varphi = -\Phi''(\delta)\delta$. Since $\Phi''(\cdot) < 0$, this is positive. Their notation on pg. 1362 seems bad but this seems to gives you the same thing as (7).

The expected return on holding capital from t to $t+1$ is:

$$\mathbb{E}_t[R^k_{t+1}] = \mathbb{E}_t \frac{RR_{t+1} + (1-\delta)Q_{t+1}}{Q_t} \tag{21}$$

Where RR_t is the implicit rental rate on capital/marginal product of capital. Rather simply, if you buy an additional unit of capital available for production tomorrow, K_{t+1}, you pay Q_t today. You get RR_{t+1} tomorrow and have $(1-\delta)$ left over, which is valued at Q_{t+1}. Take logs, ignoring expectation operators, and totally differentiate:

$$\begin{aligned}
\ln R^k_{t+1} &= \ln\left[RR_{t+1} + (1-\delta)Q_{t+1}\right] - \ln Q_t \\
\mathbb{E}_t\, r^k_{t+1} &= \frac{1}{RR + (1-\delta)}\left[dRR_{t+1} + (1-\delta)dQ_{t+1}\right] - q_t \\
\mathbb{E}_t\, r^k_{t+1} &= \frac{RR}{RR + (1-\delta)}\,\mathbb{E}_t\, rr_{t+1} + \frac{1-\delta}{RR + (1-\delta)}\,\mathbb{E}_t\, q_{t+1} - q_t
\end{aligned}$$

Define $\epsilon = \frac{1-\delta}{RR+(1-\delta)}$. Then we have:

$$1 - \epsilon = \frac{RR + (1-\delta)}{RR + (1-\delta)} - \frac{1-\delta}{RR + (1-\delta)} = \frac{RR}{RR + (1-\delta)}$$

Hence, we can write:

$$\mathbb{E}_t\, r^k_{t+1} = (1-\epsilon) rr_{t+1} + \epsilon\, \mathbb{E}_t\, q_{t+1} - q_t$$

Which is (5) when you take into account the definition of the rental rate as being the marginal product of capital.

The wholesale firm optimality conditions for capital and household labor are, respectively, to hire up until the point where the marginal products equal the product of the factor prices and the markup of price over marginal cost, X_t:

$$X_t RR_t = \alpha \frac{Y_t}{K_t} \tag{22}$$

$$X_t W_t = \Omega(1-\alpha)\frac{Y_t}{H_t} \tag{23}$$

Log-linearizing, we get:

$$\begin{aligned}
rr_t &= y_t - k_t - x_t \\
w_t &= y_t - h_t - x_t
\end{aligned}$$

Subbing these in for rr_t and w_t in the capital demand curve and labor market-clearing conditions give the linearized conditions in the paper.

The aggregate production function is:

$$d_t Y_t = A_t K_t^{\alpha} H_t^{\Omega(1-\alpha)} (H_t^e)^{(1-\Omega)(1-\alpha)} \tag{24}$$

d_t is price dispersion. It satisfies:

$$d_t = (1-\theta)(\Pi_t^*)^{-\epsilon} + \theta \Pi_t^{\epsilon} d_{t-1} \tag{25}$$

This is going to be second order and hence can be ignored. Linearizing therefore gives (noting that $H_t^e = 1$ is constant):

$$y_t = a_t + \alpha k_t + \Omega(1-\alpha) h_t$$

Which is (8).

The non-linear price-setting conditions can be written as follows. There is bad notation here in that they use ϵ as the price elasticity of demand but then use it again in the expression for the return on capital. But the price elasticity of demand ends up being irrelevant for the linearized pricing condition. The optimal relative reset price satisfies:

$$\Pi_t^* = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} \tag{26}$$

$$x_{1,t} = X_t^{-1} Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon} x_{1,t+1} \tag{27}$$

$$x_{2,t} = Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon-1} x_{2,t+1} \tag{28}$$

Here $\Pi_t = P_t/P_{t-1}$ is gross inflation and $\Pi_t^* = P_t^*/P_t$ is relative reset price inflation. The aggregate price level evolves according to:

$$1 = (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta \Pi_t^{\epsilon-1} \tag{29}$$

Linearizing all of these and simplifying yields (after a decent amount of work):

$$\pi_t = -\kappa x_t + \beta \, \mathbb{E}_t \, \pi_{t+1}$$

This (10), where $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$.

The key condition relating net worth to the external finance premium is:

$$\mathbb{E}[R_{t+1}^k] = s\left(\frac{N_t}{Q_t K_{t+1}}\right) R_t \tag{30}$$

Take logs, ignoring the expectations operator:

$$\ln R_{t+1}^k = \ln \left[s\left(\frac{N_t}{Q_t K_{t+1}}\right)\right] + \ln R_t$$

Totally differentiating:

$$r_{t+1}^k = \frac{s'(\cdot)}{s(\cdot)} \left[\frac{dN_t}{QK} - \frac{N}{Q^2 K} dQ_t - \frac{N}{QK^2} dK_{t+1} \right] + r_t$$
$$r_{t+1}^k = \frac{s'(N/K)}{s(N/K)} \frac{N}{K} [n_t - q_t - k_{t+1}] + r_t$$

Letting $\nu = \frac{s'(N/K)}{s(N/K)} \frac{N}{K}$ yields (5). More on the formal contracting problem is below.

Each period, a fraction $1-\gamma$ of entrepreneurs die and consume their net worth. Hence, aggregate consumption of entrepreneurs is:

$$C_t^e = (1-\gamma) V_t \tag{31}$$

Where V_t is entrepenurial equity from the capital holdings. Ignoring the higher order terms:

$$V_t = \left(R_t^k - R_{t-1} \right) (Q_{t-1} K_t - N_{t-1}) + R_{t-1} N_{t-1}$$

Net worth at the middle of the period is:

$$N_t = \gamma V_t + W_t^e$$

Where γ is the probability of survival, and W_t^e is the entrepreneurial wage. In other words, surviving entrepreneurs inherit V_t of equity and earn some additional equity from supplying labor, W_t^e. As noted in (31), entrepreneurs who exist just consume their existing equity. Since γ is close to 1 and W_t^e is small, you can treat $V_t \approx N_{t+1}$, which is what gives(4):

$$c_t^e = n_t$$

Aggregate net worth evolves according to:

$$N_t = \gamma \left[(R_t^k - R_{t-1}) Q_{t-1} K_t + \iota_t (Q_{t-1} K_t - N_t) + R_{t-1} N_{t-1} \right] + W_t^e \tag{32}$$

This is (4.13) from the paper, with ι the term involving the integral:

$$\iota_t = \mu \int_0^{\bar{\omega}_t} \omega_t \phi(\omega_t) R_t^k Q_{t-1} K_t d\omega_t \tag{33}$$

Take logs:

$$\ln N_t = \ln \left[\gamma \left[(R_t^k - R_{t-1}) Q_{t-1} K_t + \iota_t (Q_{t-1} K_t - N_t) + R_{t-1} N_{t-1} \right] + W_t^e \right]$$

Totally differentiate, and ignore the ι term:

$$n_t = \frac{1}{N}\left[\gamma QK(dR_t^k - dR_{t-1}) + \gamma(R^k - R)dQ_{t-1}K + \gamma(R^k - R)dK_t + \gamma dR_{t-1}N + \gamma R dN_{t-1} + dW_t^e\right]$$
$$n_t = \frac{\gamma RK}{N}\left(\frac{dR_t^k}{R} - \frac{dR_t}{R}\right) + \frac{\gamma(R^k - R)}{R}\frac{RK}{N}q_{t-1} + \frac{\gamma(R^k - R)}{R}\frac{RK}{N}k_{t-1} + \gamma R r_{t-1} + \gamma R n_{t-1} + \frac{W^e}{N}w_t^e$$
$$n_t = \frac{\gamma RK}{N}(r_t^k - r_t) - \frac{\gamma RK}{N}r_t^k + \frac{\gamma K}{N}dR_t^k + \gamma\frac{K}{N}\left(\frac{R^k}{R} - 1\right)q_{t-1} + \gamma\frac{K}{N}\left(\frac{R^k}{R} - 1\right)k_t + \gamma R(r_{t-1} + n_{t-1}) + \frac{W^e}{N}w_t^e$$
$$n_t = \frac{\gamma RK}{N}(r_t^k - r_t) + \gamma R(r_{t-1} + n_{t-1}) + \gamma\frac{K}{N}\left(\frac{R^k}{R} - 1\right)(r_t^k + q_{t-1} + k_t) + \frac{W^e}{N}w_t^e$$

Now this is almost exactly what they have for (4.24) in the text. They have a coefficient of 1 multiplying $r_{t-1} + n_{t-1}$, whereas I have γR. I am assuming that they are simply approximating $\gamma R \approx 1$. γ will be slightly less than 1, and R slightly greater than 1, so this is probably fine.

The other terms relate to the "higher order terms" (which don't actually seem to be higher order but which are nevertheless small). I still think there is an error or two. If you look at ϕ_t^n on pg. 1362, this is basically what I have with a few exceptions. First, it seems there should be a γ multiplying the first term in ϕ_t^n. Second, they seem to be missing a parentheses on the $-x_t$ at the end of that expression – it should be weighted by W^e/N. But, again, quantitatively they are not missing much by keeping these terms out. $\frac{R^k}{R} = 1.02^{0.25}$ (a 200 basis point annualized spread). Hence, $\frac{R^k}{R} - 1 \approx 0$. So they are just dropping these terms, which seems fine. Finally, since W^e is very small, $W^e/N \approx 0$ so the last term drops out as well in a loose approximate sense.

The exogenous processes and policy rule are already log-linear.

2.2 The Formal Contracting Problem

Where does the formal contracting problem come from? Basically, we want to understand where the condition relating the interest rate spread to firm leverage comes from. For completeness, the linearized condition is below:

$$\mathbb{E}_t\, r_{t+1}^k - r_t = -\nu\left[n_t - (q_t + k_{t+1})\right]$$

The formal problem is not very well laid out by BBG. A better exposition can be found in Christiano, Motto, and Rostagno (2014, *AER*) or Carlstrom, Fuerst, and Paustian (2016, *AEJ: Macro*.

Because all firms end up with the same optimality conditions, I am going to drop firm-specific superscripts in what follows so as to ease up on the notation a bit. A firm gets a loan from an intermediary to finance *the entirety of its next-period stock of capital* (this is different than Carlstrom and Fuerst 1997, where you just finance the production of investment). The firm has net worth of N_t and wishes to purchase Q_tK_{t+1} of new capital at the end of period t. It hence borrows $Q_tK_{t+1} - N_t$ from the intermediary. Suppose that the loan rate is Z_{t+1} (gross). After the borrower makes the loan decision, he receives an idiosyncratic shock to the return, ω_{t+1} (note to be completely correct this should have a firm-specific superscript on it, but we are going to ignore that for now). Let R^k_{t+1} be the aggregate return on capital on capital, over which there is uncertainty because of aggregate shocks; the borrowers' specific return is $\omega_{t+1}R^k_{t+1}$. Average across firms, the $\omega_{t+1} = 1$. A particular firms gets to keep $\omega_{t+1}R^k_{t+1}$, and has to pay back $Z_{t+1}(Q_tK_{t+1} - N_t)$ in the event of no default. The borrower will default if his net return is negative. This implies a *cutoff* value of ω_{t+1}, call it $\bar{\omega}_{t+1}$, below which he will choose to default. This is implicitly defined by:

$$Z_{t+1}(Q_tK_{t+1} - N_t) = \bar{\omega}_{t+1}R^k_{t+1}Q_tK_{t+1} \tag{34}$$

Note that $\bar{\omega}_{t+1}$ depends on the realization of R^k_{t+1}. It is convenient to write this cutoff in terms of a leverage ratio, $L_t = \frac{Q_tK_{t+1}}{N_t}$. Then we see that the loan rate satisfies:

$$Z_{t+1} = \bar{\omega}_{t+1}R^k_{t+1}\frac{L_t}{L_t - 1} \tag{35}$$

The ω_{t+1} that each entrepreneur draws is distributed log-normal, with CDF $\Phi(\omega_{t+1})$, density $\phi(\omega_{t+1})$, and $\mathbb{E}[\omega_{t+1}] = 1$ (this expectation is across entrepreneurs; there is no aggregate uncertainty on ω_{t+1}). Let us calculate the expected shares of the payout from the project the entrepreneur and lender each get to keep, respectively. The expected entrepreneurial income from getting a loan is:

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1}R^k_{t+1}Q_tK_{t+1} - (1 - \Phi(\bar{\omega}_{t+1}))Z_{t+1}(Q_tK_{t+1} - N_t) \tag{36}$$

The first term is the expected payout *conditional on not defaulting*, i.e. drawing $\omega_{t+1} \geq \bar{\omega}_{t+1}$. The second term is the expected repayment, which is the probability of non-default, $1 - \Phi(\bar{\omega}_{t+1})$, times the repayment, $Z_{t+1}(Q_tK_{t+1} - N_t)$. But from (34), we can get rid of the Z_{t+1} term:

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1}R^k_{t+1}Q_tK_{t+1} - (1 - \Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1}R^k_{t+1}Q_tK_{t+1} \tag{37}$$

But then this reduces to:

$$\left[\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} - (1 - \Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1}\right] R^k_{t+1}Q_tK_{t+1} \tag{38}$$

Now define $f(\bar{\omega}_{t+1})$ as the term inside the brackets, which is the share of the returns the firm expects to keep:

$$f(\bar{\omega}_{t+1}) = \int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} - (1 - \Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1} \tag{39}$$

The borrower is exposing his net worth, N_t, to earn (38). The total return is the ratio. Using the definition of leverage above, we can write the firm's expected return as:

$$f(\bar{\omega}_{t+1})R^k_{t+1}L_t \tag{40}$$

Now, let's think about the lender's expected return from the project. It is:

$$\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}(1-\mu)R^k_{t+1}Q_tK_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))Z_{t+1}(Q_tK_{t+1}-N_t) \tag{41}$$

The first term is what the lender expects to keep in the event of default. He gets to keep $(1-\mu)R^k_{t+1}Q_tK_{t+1}$ times the expected value of ω_{t+1} conditional on the entrepreneur defaulting, i.e. $\omega_{t+1} < \bar{\omega}_{t+1}$. $\mu \geq 0$ is a bankruptcy cost. The second term is just the probability of no default times the return on making a loan in that case. But again using (34), we can write this as:

$$\left[(1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1}\right] R^k_{t+1}Q_tK_{t+1} \tag{42}$$

Define the term in brackets as the lender's expected share of the return:

$$g(\bar{\omega}_{t+1}) = (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1} \tag{43}$$

The entrepreneur is exposing $Q_tK_{t+1} - N_t$ (i.e. the amount of the loan), to get back (42). The expected return is therefore:

$$\frac{g(\bar{\omega}_{t+1})R^k_{t+1}Q_tK_{t+1}}{Q_tK_{t+1}-N_t} \tag{44}$$

Using the definition of leverage, this can be written:

$$g(\bar{\omega}_{t+1})R^k_{t+1}\frac{L_t}{L_t-1} \tag{45}$$

Now we can write the formal contracting problem. The entrepreneur wants to pick a leverage ratio, L_t, and cutoff value of $\bar{\omega}_{t+1}$, to maximize his expected return subject to a participation constraint for the lender. The lender is assumed risk neutral, and hence faces an opportunity cost of funds of the safe gross interest rate, R_t. Hence, the formal problem for the entrepreneur is:

$$\max_{\bar{\omega}_{t+1}, L_t} \quad \mathbb{E}_t\, R^k_{t+1}f(\bar{\omega}_{t+1})L_t$$

s.t.

$$R^k_{t+1}g(\bar{\omega}_{t+1})\frac{L_t}{L_t-1} \geq R_t$$

As Calrstrom, Fuerst, and Paustian (2016, *AEJ: Macro*) emphasize, the lender's return is predetermined. It is R_t, the safe (gross) interest rate. This means that $\bar{\omega}_{t+1}$ is state-contingent – it moves with R^k_{t+1} such that the participation constraint will always hold and the lender gets R_t.

We can characterize the optimum using a Lagrangian. Let Λ_{t+1} be the multiplier on the constraint. The Lagrangian is:

$$\mathbb{L} = \mathbb{E}_t \left\{ R^k_{t+1}f(\bar{\omega}_{t+1})L_t + \Lambda_{t+1}\left[R^k_{t+1}g(\bar{\omega}_{t+1})L_t - R_t(L_{t-1})\right] \right\}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial \bar{\omega}_{t+1}} = \mathbb{E}_t \left\{ R^k_{t+1} f'(\bar{\omega}_{t+1}) L_t + \Lambda_{t+1} R^k_{t+1} g'(\bar{\omega}_{t+1}) L_t \right\}$$

$$\frac{\partial \mathbb{L}}{\partial L_t} = \mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) + \Lambda_{t+1} \left[R^k_{t+1} g(\bar{\omega}_{t+1}) - R_t \right] \right\}$$

$$\frac{\partial \mathbb{L}}{\partial \Lambda_{t+1}} = R^k_{t+1} g(\bar{\omega}_{t+1}) L_t - (L_t - 1) R_t$$

Setting these equal to zero and simplifying somewhat yields:

$$\mathbb{E}_t \left\{ R^k_{t+1} f'(\bar{\omega}_{t+1}) + \Lambda_{t+1} R^k_{t+1} g'(\bar{\omega}_{t+1}) \right\} = 0 \tag{46}$$

$$\mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) + \Lambda_{t+1} \left[R^k_{t+1} g(\bar{\omega}_{t+1}) - R_t \right] \right\} = 0 \tag{47}$$

$$R^k_{t+1} g(\bar{\omega}_{t+1}) L_t = (L_t - 1) R_t \tag{48}$$

Note that (48) holds for all possible realization of R^k_{t+1} – i.e. $\bar{\omega}_{t+1}$ is state-contingent and adjusts to ensure that the lender's return is always predetermined.

Now let's linearize these about the steady state. I will use no time subscripts to denote steady state values. To first order, we needn't worry about the expectations operator either. Start with (46). Note that we can drop the R^k_{t+1} now:

$$f''(\bar{\omega}) d\bar{\omega}_{t+1} + g'(\bar{\omega}) d\Lambda_{t+1} + \Lambda g''(\bar{\omega}) d\bar{\omega}_{t+1} = 0$$

To ease notation, define $\widehat{\omega}_{t+1} = \frac{d\bar{\omega}_{t+1}}{\bar{\omega}}$. For other variables, lowercase letters denote percentage deviations. We then have:

$$\bar{\omega} f''(\bar{\omega}) \widehat{\omega}_{t+1} + \Lambda g'(\bar{\omega}) \lambda_{t+1} + \Lambda \bar{\omega} g''(\bar{\omega}) \widehat{\omega}_{t+1} = 0$$

We know that, in steady state, we must have:

$$\Lambda = -\frac{f'(\bar{\omega})}{g'(\bar{\omega})} \tag{49}$$

Hence:

$$\bar{\omega} f''(\bar{\omega}) \widehat{\omega}_{t+1} - f'(\bar{\omega}) \lambda_{t+1} - \bar{\omega} f'(\bar{\omega}) \frac{g''(\bar{\omega})}{g'(\bar{\omega})} \widehat{\omega}_{t+1} = 0$$

Divide both sides by $f'(\bar{\omega})$. We then have:

$$\lambda_{t+1} = \left[\frac{\bar{\omega} f''(\bar{\omega})}{f'(\bar{\omega})} - \frac{\bar{\omega} g''(\bar{\omega})}{g'(\bar{\omega})} \right] \widehat{\omega}_{t+1}$$

Define $\Psi = \frac{\bar{\omega} f''(\bar{\omega})}{f'(\bar{\omega})} - \frac{\bar{\omega} g''(\bar{\omega})}{g'(\bar{\omega})}$. We can then write the log-linear version of (46) as:

$$\Psi\bar{\omega}_{t+1} = \lambda_{t+1} \tag{50}$$

Before log-linearizing the (47), combine (48), noting that $R^k_{t+1}g(\bar{\omega}_{t+1}) = \frac{L_t - 1}{L_t}R_t$, with it to write:

$$\mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) - \Lambda_{t+1}\frac{R_t}{L_t} \right\} = 0$$

But ignoring the expectations operator, we can write this as:

$$R^k_{t+1} f(\bar{\omega}_{t+1}) = \Lambda_{t+1}\frac{R_t}{L_t}$$

Take logs:

$$\ln R^k_{t+1} + \ln f(\bar{\omega}_{t+1}) = \ln \Lambda_{t+1} + \ln R_t - \ln L_t$$

Totally differentiate:

$$r^k_{t+1} + \frac{f'(\bar{\omega})}{f(\bar{\omega})} d\bar{\omega}_{t+1} = \lambda_{t+1} + r_t - L_t$$

Which can be written:

$$r^k_{t+1} + \frac{\bar{\omega} f'(\bar{\omega})}{f(\bar{\omega})} \widehat{\omega}_{t+1} = \lambda_{t+1} + r_t - L_t$$

Define $\Theta_f = \frac{\bar{\omega} f'(\bar{\omega})}{f(\bar{\omega})}$. We can then write:

$$r^k_{t+1} - r_t + l_t + \Theta_f \widehat{\omega}_{t+1} = \lambda_{t+1} \tag{51}$$

Now let's linearize (48). Take logs first:

$$\ln R^k_{t+1} + \ln g(\bar{\omega}_{t+1}) + \ln L_t = \ln R_t + \ln(L_t - 1)$$

Totally differentiate:

$$r^k_{t+1} + \frac{g'(\bar{\omega})}{g(\bar{\omega})} d\bar{\omega}_{t+1} + l_t = r_t + \frac{1}{L-1} dL_t$$

Which can be written:

$$r^k_{t+1} + \frac{\bar{\omega} g'(\bar{\omega})}{g(\bar{\omega})} \widehat{\omega}_{t+1} + l_t = r_t + \frac{L}{L-1} l_t$$

Now define $\Theta_g = \frac{\bar{\omega} g'(\bar{\omega})}{g(\bar{\omega})}$. We therefore have:

$$r^k_{t+1} - r_t + \Theta_g \widehat{\omega}_{t+1} = \frac{1}{L-1} l_t \tag{52}$$

The three linearized conditions are then repeated here for convenience:

$$\Psi \bar{\omega}_{t+1} = \lambda_{t+1} \tag{53}$$

$$r^k_{t+1} - r_t + l_t + \Theta_f \widehat{\omega}_{t+1} = \lambda_{t+1} \tag{54}$$

$$r^k_{t+1} - r_t + \Theta_g \widehat{\omega}_{t+1} = \frac{1}{L-1} l_t \tag{55}$$

Now let's combine these in such a way as to eliminate ω_{t+1} and λ_{t+1}.

Plug (53) into (54). This gives:

$$r^k_{t+1} - r_t + l_t = (\Psi - \Theta_f)\, \widehat{\omega}_{t+1}$$

Now, from (55), we can solve for $\widehat{\omega}_{t+1}$ as:

$$\widehat{\omega}_{t+1} = \frac{1}{\Theta_g (L-1)} l_t - \frac{1}{\Theta_g}(r^k_{t+1} - r^d_t)$$

Combine this with the above expression:

$$r^k_{t+1} - r_t + l_t = (\Psi - \Theta_f) \left[\frac{1}{\Theta_g (L-1)} l_t - \frac{1}{\Theta_g}(r^k_{t+1} - r^d_t) \right]$$

Which can be written:

$$\left[1 + \frac{\Psi - \Theta_f}{\Theta_g}\right](r^k_{t+1} - r_t) = \left[\frac{\Psi - \Theta_f}{\Theta_g (L-1)} - 1\right] l_t$$

Which can be written:

$$\frac{\Theta_g - \Theta_f + \Psi}{\Theta_g}(r^k_{t+1} - r_t) = \frac{\Psi - \Theta_f - \Theta_g (L-1)}{\Theta_g (L-1)} l_t$$

Or:

$$r^k_{t+1} - r_t = \frac{\Psi - \Theta_f - \Theta_g (L-1)}{(\Psi + \Theta_g - \Theta_f)(L-1)} l_t \tag{56}$$

Now, before stopping, we can note that there is a relationship between Θ_f and Θ_g. In the steady state, combining (48) with (47), we have:

$$R^k f(\bar{\omega}) + \Lambda \left[R^k g(\bar{\omega}) - R^k g(\bar{\omega}) \frac{L}{L-1} \right] = 0$$

The R^k drop, leaving:

$$f(\bar{\omega}) + \Lambda g(\bar{\omega}) \left(1 - \frac{L}{L-1}\right) = 0$$

Which is:

$$f(\bar{\omega}) + -\frac{\Lambda g(\bar{\omega})}{L-1} = 0$$

But we know that $\Lambda = -\frac{f'(\bar{\omega})}{g'(\bar{\omega})}$. So:

$$f(\bar{\omega}) + \frac{f'(\bar{\omega})g(\bar{\omega})}{g'(\bar{\omega})}\frac{1}{L-1} = 0$$

Divide both sides by $f'(\bar{\omega})$:

$$\frac{f(\bar{\omega})}{f'(\bar{\omega})} + \frac{g(\bar{\omega})}{g'(\bar{\omega})}\frac{1}{L-1} = 0$$

But $\frac{f(\bar{\omega})}{f'(\bar{\omega})} = \frac{\bar{\omega}}{\Theta_f}$ and similarly for the terms involving $g(\cdot)$. Hence:

$$\frac{\bar{\omega}}{\Theta_f} + \frac{\bar{\omega}}{\Theta_g}\frac{1}{L-1} = 0$$

Which implies:

$$\Theta_g(L-1) = -\Theta_f$$

But making use of this in (56), we get:

$$r^k_{t+1} - r_t = \frac{\Psi}{\Psi(L-1) - \Theta_f L} l_t \tag{57}$$

Since $l_t = q_t + k_{t+1} - n_t$, (57) is the same (5), where $\nu = \frac{\Psi}{\Psi(L-1)-\Theta_f L}$. The important point here is that there is a positive relationship between entrepreneur leverage, l_t, and the lending spread, $r^k_{t+1} - r_t$.

Now, recall from above that:

$$\Psi = \bar{\omega}\left(\frac{f''(\bar{\omega})}{f'(\bar{\omega})} - \frac{g''(\bar{\omega})}{g'(\bar{\omega})}\right)$$

We have that these shares must sum to (this is always, not just at steady state, but I'm evaluating it at steady state):

$$f(\bar{\omega}) + g(\bar{\omega}) = 1 - \mu \int_0^{\bar{\omega}} \omega \phi(\omega) d\omega$$

Note: if $\mu = 0$ (no bankruptcy cost), then we have $f(\bar{\omega}) = -g(\bar{\omega})$. But this then would mean that $\Psi = 0$ – i.e. $\nu = 0$, and there would be no relationship between leverage and external finance premium!

2.3 Calibration

I'm not going to go into great depth on calibrating the model. For the purposes of the linearization, all that really matters are a few steady state ratios and a few key parameters (such as ν, the sensitivity of the interest rate spread to leverage).

I'm going to follow most of what they report. Some parameters they don't fully report, so I'm just going to pick values that are reasonable and will look at sensitivity of the model's IRFs to those parameters. First, set "standard" parameters as follows: $\beta = 0.99$, $\eta = 3.0$ (the Frisch elasticity), $\alpha = 0.35$, and $(1-\alpha)(1-\Omega) = 0.64$ (this implies that the entrepreneurial labor share is very low at 0.01). We have $\delta = 0.025$. They set $\frac{G}{Y} = 0.2$. They set the capital adjustment cost parameter to $\varphi = 0.25$. They have $1 - \gamma = 0.0272$.

They set the Calvo pricing parameter to $\theta = 0.75$, the smoothing parameter in the interest rate rule to $\rho = 0.9$, and the coefficient on inflation of 0.11 (the long run response of the interest rate to inflation is $\zeta/(1-\rho)$, so 1.1, consistent with the Taylor principle). The parameter ϵ which shows up in the linearized capital Euler equation is $\epsilon = \frac{1-\delta}{RR+(1-\delta)}$, where RR is the steady state value of the marginal product of capital. This parameter is therefore 0.96. I'm going to assume a consumption share of income of 0.51 and an investment share of 0.18, which implies an entrepreneurial share of consumption of 0.12.

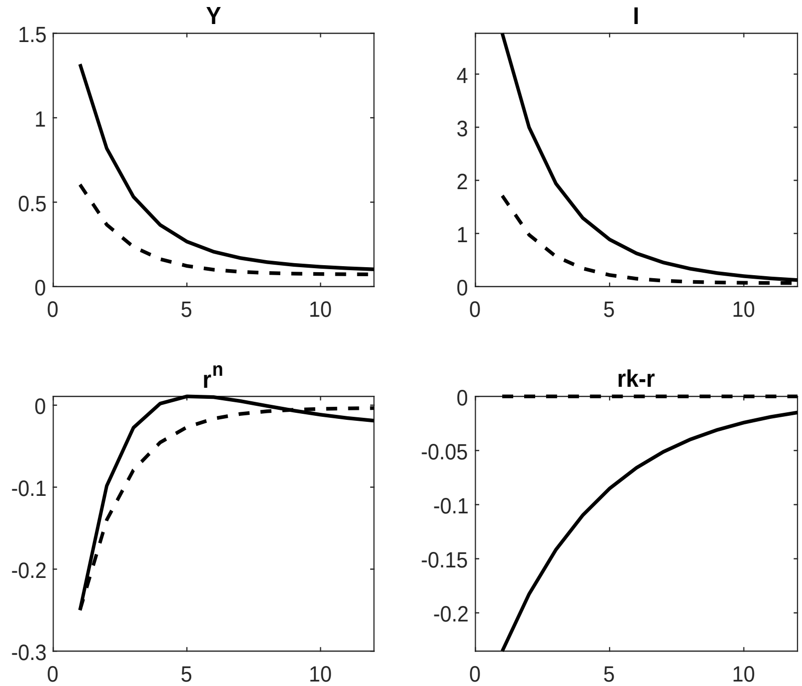
They discuss the parameters related to the financial frictions in depth on pg. 1368. The key parameter is $\nu = 0.2$. I'll show responses under different values. We can think about the no frictions case as being $\nu = 0$ – in this case, the risk spread would just be constant.

The government spending shock is calibrated with $\rho_g = 0.95$. They assume that $\rho_a = 1$ – this turns out to be important for whether the financial accelerator amplifies or dampens the productivity shock response.

2.4 Impulse Response Functions

My overall parameterization sort of loosely matches what they report in the paper, albeit not perfectly. First, consider the responses to the monetary policy shock. Note the scale of the shock – I am shocking the policy rule by 0.25/4 in the model (which is quarterly), which in turn produces an *annualized* policy rate response of 25 basis points on impact.

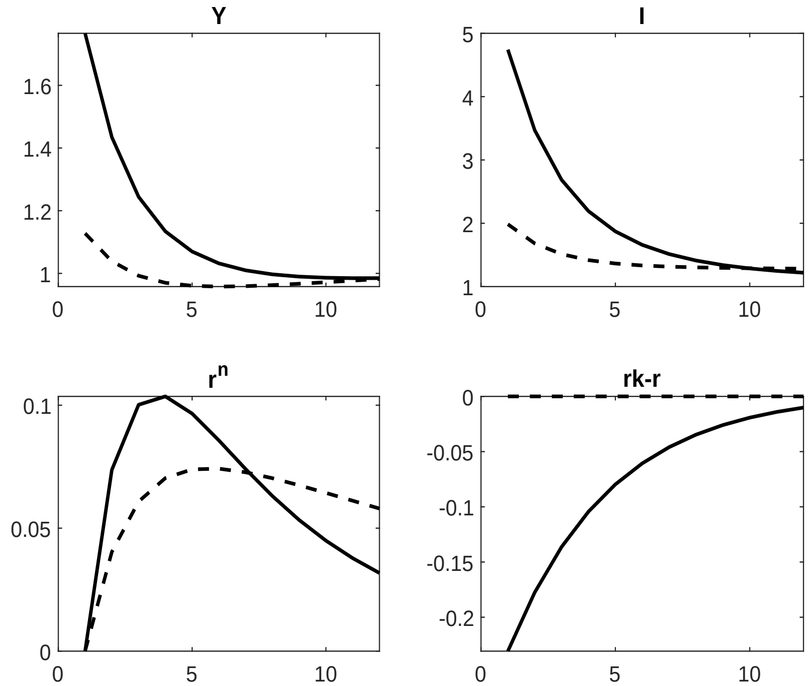
Figure 1: IRFs to Policy Shock



The responses shown above are very similar (if not exact) to the responses shown in Figure 3 of their paper. Solid lines show responses when $\nu = 0.2$, so that there is a financial accelerator mechanism. Dashed lines fix $\nu = 0$, so that this mechanism is absent. Output and investment go up (and revert) after an exogenous cut in the policy rate. The financial accelerator in fact amplifies the effects of the policy shock – both output and investment go up significantly more. The interest rate spread, or perhaps more precisely the external finance premium, shown in the bottom right of the figure, declines. This is the source of the amplification.

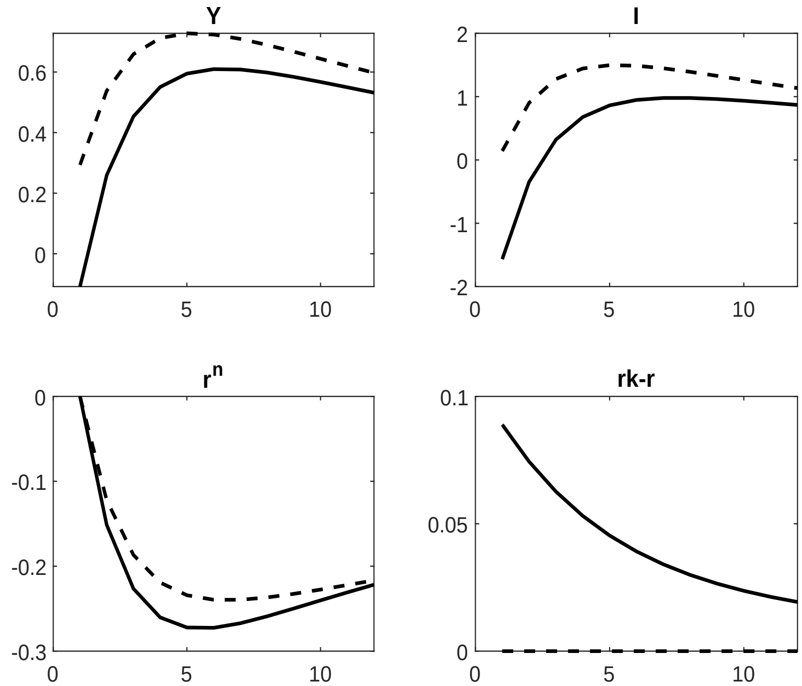
There is a kind of multiplier effect. The stimulative monetary policy raises the demand for capital, which raises investment and the price of capital, q_t. This increase in asset prices raises net worth. Higher net worth lowers the external finance premium. But this further stimulates investment and the price of capital, which further stimulates net worth. This is the "accelerator" idea – the change in asset prices lowers the external finance premium, which in turn further stimulates asset prices and real activity.

Figure 2: IRFs to Productivity Shock



Impulse responses to the productivity shock are shown above. These are very similar (at least for output) to what is shown in Figure 4 (where they only show the output response). But there is a bit of slight of hand going on. This result turns out to be very sensitive to the assumed autocorrelation of the productivity process. What happens if I assume a more mean-reverting value, such as $\rho_a = 0.95$? The IRFs are shown below:

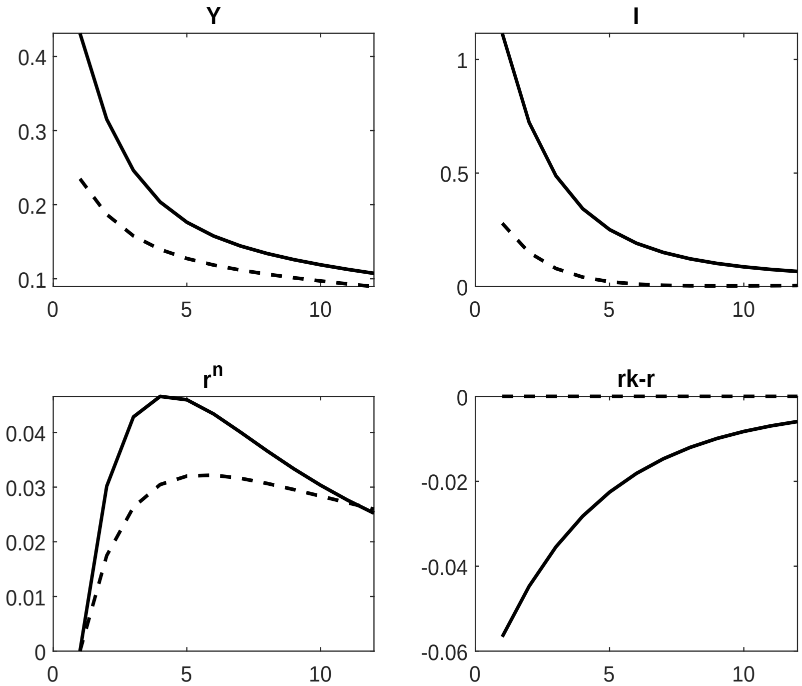
Figure 3: IRFs to Productivity Shock, $\rho_a = 0.95$ instead of $\rho_a = 1$



In this specification, the financial accelerator actually *dampens* the responses to the productivity shock relative to the unconstrained model. What's driving this is again the price of capital. When $\rho_a \to 1$, the productivity shock is much more of a "demand" shock than a supply shock, and with sticky prices, output is at least partially demand determined. There is a big demand for output, which puts upward pressure on q_t and net worth, and consequently lowers the external finance premium. But when the shock is (just a little) less persistent, things flip – demand doesn't rise by much, q_t doesn't change by much, and the external finance premium actually goes up, not down. It's not a formal proof, but in lots of these models, you see that financial frictions *amplify* demand shocks but often *weaken* supply shocks. That is what we see at play here.

Next, consider the government spending shock. The responses are shown below. These are similar to what they report in the paper. There is amplification from the financial accelerator mechanism and the external finance premium.

Figure 4: IRFs to Government Spending Shock

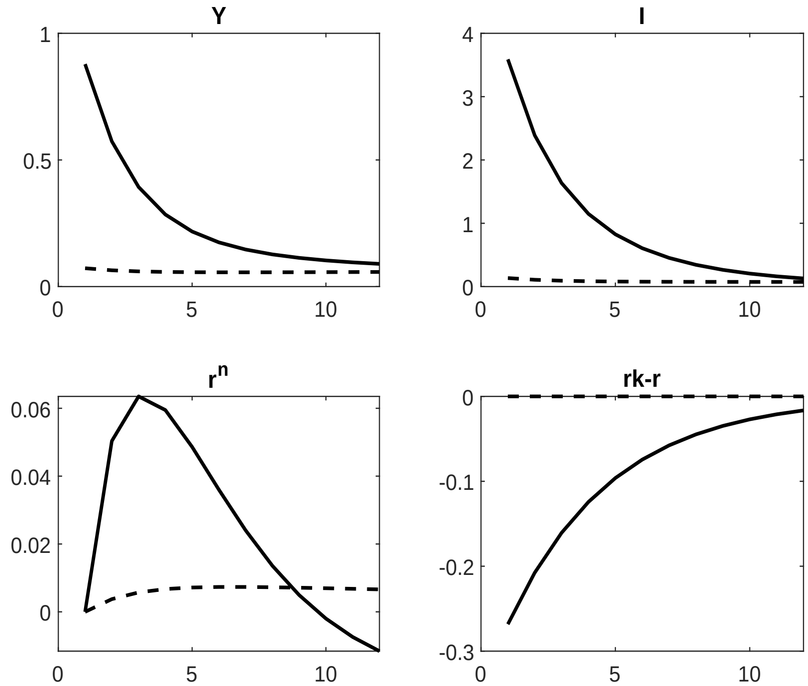


Finally, consider an exogenous shock to net worth. This is introduced via a shock to the net worth evolution expression (and an offsetting transfer from households, which does not otherwise show up in the linearized equilibrium conditions). In particular:

$$n_t = \gamma \frac{RK}{N}(r_t^k - r_{t-1}) + r_{t-1} + n_{t-1} + \varepsilon_{n,t} \tag{58}$$

The next figure shows the impulse responses to the net worth shock

Figure 5: IRFs to Net Worth Shock



The effects of the shock are pretty easy to understand. When entrepreneurs exogenously get more net worth, agency frictions decline. This lowers the external finance premium and leads to a boom. Note that this redistribution would have small, non-zero effects even when $\nu = 0$ (so no financial accelerator mechanism); this is because more net worth stimulates entrepreneurial consumption.

2.5 Differences Relative to Carlstrom and Fuerst (1997)

In Carlstrom and Fuerst (1997), the agency friction tends to dampen the response to a productivity shock but increases propagation. In the Bernanke, Gertler, Gilchrist (1999) setup, we don't see the hump-shaped propagation but instead see amplification.

There are some differences in the two setups that end up driving these results. For one, Bernanke, Gertler, and Gilchrist (1999) have sticky prices and a capital adjustment cost (which, even absent agency frictions, would result in time-variation in the price of capital, q_t). But there is another subtle difference. In Carlstrom and Fuerst (1997), the agency friction only applies to entrepreneurs who produce new investment goods. In BGG, the agency friction applies to production firms who own the entire capital stock. A simple way to think about this is that in CF agency frictions apply to producers of new investment goods, whereas in BGG agency frictions apply to the whole capital stock (which is much bigger than the flow of new investment). See the discussion above about the formal contracting problem. Fluctuations in the supply price of capital therefore have much bigger effects on net worth in the BGG framework and end up being a source of amplification.