Advanced Macro: Carlstrom and Fuerst (1997, *AER*)

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1 Introduction

This note describes the model environment and recreates impulse responses to shocks from "Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis," by Charles Carlstrom and Timothy Fuerst, published in the *American Economic Review* 87(5) in 1997. The paper embeds the agency problem described in Bernanke and Gertler (1989) into a dynamic general equilibrium model. It shows that the agency cost introduces persistence and hump-shaped output dynamics to a productivity shock that a standard RBC model cannot. In this way, the agency costs are somewhat isomorphic to investment adjustment costs.

2 Partial Equilibrium Contracting Problem

Financial contracts are intra-period and hence can be separated out from the rest of the model. The price of capital, q, and net worth of entrepreneurs, n, are relevant but we can take these as given. There is a risk-neutral capital market mutual fund (CMF) and risk-neutral entrepreneur.

The entrepreneur transforms i consumption goods into ωi units of capital within period. ω is iid across entrepreneurs, with distribution Φ and density ϕ, and satisfies $\mathbb{E}\omega = 1$. Lenders can't observe realized ω. They have to pay a monitoring cost μi to observe it. This incentivizes entrepreneurs to misreport. An optimal contract ensures that they truthfully report ω.

To make this all interesting, net worth has to be sufficiently small so that entrepreneurs need external funds. The contract will consist of an intra-period interest rate, r^k, where the entrepreneur borrows $i-n$ consumption goods, agreeing to repay $(1+r^k)(i-n)$ capital goods. The entrepreneur can default. He will default if the proceeds from investment, ωi, are less than what must be repaid to the lender, $(1+r^k)(i-n)$. This implies a cutoff:

$$\omega < \frac{(1+r^k)(i-n)}{i} = \bar{\omega} \tag{1}$$

The lender will monitor only in the event of default, thereby paying the μi monitoring cost, and confiscates the remaining returns from the project. Note that, once i and ω are solved for, re-arranging (1) yields:

$$1 + r^k = \frac{\bar{\omega} i}{i - n} \tag{2}$$

Expected entrepreneurial income from getting an intra-period loan is:

$$q \left[\int_{\bar{\omega}}^{\infty} \omega i \phi(\omega) d\omega - (1 - \Phi(\bar{\omega}))(1 + r^k)(i - n) \right] \tag{3}$$

The first part, inside the integral, is the expected revenue from the project if there is no default. Note that $\Phi(\bar{\omega})$ is the probability of being in the default range, $1 - \Phi(\bar{\omega})$ is the probability of no default. The second part is the probability of no default, $1 - \Phi(\bar{\omega})$, times the repayment. But we can re-arrange using (1):

$$q \left[\int_{\bar{\omega}}^{\infty} \omega i \phi(\omega) d\omega - (1 - \Phi(\bar{\omega}))\bar{\omega} i \right] = qi \left[\int_{\bar{\omega}}^{\infty} \omega \phi(\omega) d\omega - (1 - \Phi(\bar{\omega}))\bar{\omega} \right] = qif(\bar{\omega}) \tag{4}$$

In (4), we are simply defining $f(\bar{\omega})$ as the expected fraction of net capital goods received by the entrepreneur/borrower.

The expected income of the lender, the capital market fund, is:

$$q \left[\int_{0}^{\bar{\omega}} \omega i \phi(\omega) d\omega - \Phi(\bar{\omega})\mu i + (1 - \Phi(\bar{\omega}))(1 + r^k)(i - n) \right] \tag{5}$$

The first term in (5) is what the entrepreneur claims in the event of default; $\Phi(\bar{\omega})\mu i$ is what they expect to pay in default; and the last term is what they expect to get when there is no default. Note that what the lender pays in default is independent of the realization of ω; they simply pay μi in default, the probability of which is $\Phi(\bar{\omega})$. Again, since $1 + r^k = \frac{\bar{\omega} i}{i - n}$, we can write this as:

$$q \left[\int_{0}^{\bar{\omega}} \omega i \phi(\omega) d\omega - \Phi(\bar{\omega})\mu i + (1 - \Phi(\bar{\omega}))\bar{\omega} i \right] = qi \left[\int_{0}^{\bar{\omega}} \omega \phi(\omega) d\omega - \Phi(\bar{\omega})\mu + (1 - \Phi(\bar{\omega}))\bar{\omega} \right] = qig(\bar{\omega}) \tag{6}$$

Where, in (6), $g(\bar{\omega})$ is the expected net capital output received by the lender. We must have:

$$g(\bar{\omega}) + f(\bar{\omega}) = 1 - \Phi(\bar{\omega})\mu \tag{7}$$

In (7), if there were no monitoring cost, $\mu = 0$, the shares would sum to one. With a monitoring cost, there is in effect a deadweight loss.

The optimal contract maximizes the entrepreneur's take subject to the constraint that the lender at least breaks even:

$$\max \quad qif(\bar{\omega})$$

$$\text{s.t.}$$

$$qig(\bar{\omega}) \geq (i - n)$$

In addition, you have to ensure that the entrepreneur always needs external funds – i.e. that $qif(\bar{\omega}) \geq n$. Imposing that the constraint holds with equality, we have:

$$i = \frac{1}{1 - qg(\bar{\omega})} n \tag{8}$$

The unconstrained problem is then:

$$\max \quad \frac{qnf(\bar{\omega})}{1 - qg(\bar{\omega})}$$

The qn is just multiplicative, so we can drop this from the maximization and resulting FOC. Furthermore, we know that $g(\bar{\omega}) = 1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})$. So we can write the problem as:

$$\max \quad \frac{f(\bar{\omega})}{1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right]} = f(\bar{\omega})\left[1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right]\right]^{-1}$$

The FOC is:

$$f'(\bar{\omega})\left[1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right]\right]^{-1} - f(\bar{\omega})\left[1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right]\right]^{-2}\left(q\Phi'(\bar{\omega})\mu + qf'(\bar{\omega})\right) = 0$$

Which is:

$$f'(\omega) = \frac{qf(\bar{\omega})\left(\Phi'(\bar{\omega})\mu + f'(\bar{\omega})\right)}{1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right]}$$

Which can be written:

$$1 - q\left[1 - \Phi(\bar{\omega})\mu - f(\bar{\omega})\right] = qf(\bar{\omega}) + q\Phi'(\bar{\omega})\mu\frac{f(\bar{\omega})}{f'(\bar{\omega})}$$

Grouping terms, we get:

$$1 = q\left[1 - \Phi(\bar{\omega})\mu + \Phi'(\bar{\omega})\mu\frac{f(\bar{\omega})}{f'(\bar{\omega})}\right] \tag{9}$$

(8)-(9) are the same FOC as in the paper. Now what is $f'(\bar{\omega})$? Here we need to use Leibniz's rule. Differentiating (4), we have:

$$f'(\bar{\omega}) = -\bar{\omega}\phi(\bar{\omega}) - (1 - \Phi(\bar{\omega})) + \bar{\omega}\Phi'(\bar{\omega})$$

Since $\phi(\bar{\omega}) = \Phi'(\bar{\omega})$, we are simply left with:

$$f'(\bar{\omega}) = -(1 - \Phi(\bar{\omega})) \tag{10}$$

So we can re-write (9) as:

$$1 = q\left[1 - \Phi(\bar{\omega})\mu - \Phi'(\bar{\omega})\mu\frac{f(\bar{\omega})}{1 - \Phi(\bar{\omega})}\right] \tag{11}$$

(9) defines an implicit function $\bar{\omega}(q)$, which is increasing in q – in other words, higher asset prices raises the default cutoff. Call this $\bar{\omega}(q)$. Subbing into (8), we have:

$$i(q,n) = \frac{n}{1 - qg(\bar{\omega}(q))} \tag{12}$$

Expected capital output is therefore:

$$I^s(q,n) = i(q,n)\left(1 - \mu\Phi(\bar{\omega}(q))\right) = \frac{1 - \mu\Phi(\bar{\omega}(q))}{1 - qg(\bar{\omega}(q))} n = \Lambda(q)n \tag{13}$$

You can think of this more generally as a function, $\Lambda(q)$, times net worth. $\Lambda'(q) > 0$, so the capital supply curve is upward-sloping. It shifts right whenever net worth increases. Since this is linear in net worth and $\Lambda(q)$ doesn't depend on anything specific to entrepreneurs, it aggregates well.

3 General Equilibrium Model

The general equilibrium model embeds the contracting problem from above into an infinitely-lived RBC model. There is a mass $1-\eta$ of households and a mass η of entrepreneurs. Entrepreneurs are risk-neutral but discount the future more heavily than households via the parameter $\gamma < 1$. They supply one unit of labor inelastically. There is also a representative production firm.

There are some subtle timing issues in the model with respect to the entrepreneurs. They wake up with a stock of capital, $k_{e,t}$, and are endowed with a unit of labor. They rent these to the production firm at factor prices r_t and x_t *before* choosing i_t and observing their draw of ω_t. Entrepreneurs who draw $\omega < \bar{\omega}$ default and do no consumption. Those who do not default choose consumption and future capital in a typical dynamic problem. See Table 1 in the paper for a discussion of the timing.

3.1 Households

Household's solve the following problem:

$$\max \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln c_t + \nu(1 - l_t) \right\}$$

s.t.

$$c_t + q_t(k_{c,t+1} - (1-\delta)k_{c,t}) = r_t k_{c,t} + w_t l_t$$

A Lagrangian is:

$$\mathbb{L} = \mathbb{E}_0\, \beta^t \left\{ \ln c_t + \nu(1 - l_t) + \lambda_{c,t}\left[r_t k_{c,t} + w_t l_t - c_t - q_t(k_{c,t+1} - (1-\delta)k_{c,t})\right] \right\}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_{c,t}$$

$$\frac{\partial \mathbb{L}}{\partial c_t} = -\nu + \lambda_{c,t} w_t$$

$$\frac{\partial \mathbb{L}}{\partial k_{c,t+1}} = -\lambda_{c,t} q_t + \beta\, \mathbb{E}_t\, \lambda_{c,t+1} \left(r_{t+1} + (1-\delta)\right)$$

Setting these equal to zero and eliminating the multiplier yields:

$$\nu = \frac{w_t}{c_t} \tag{14}$$

$$q_t = \beta\, \mathbb{E}_t\, \frac{c_t}{c_{t+1}} \left(r_{t+1} + (1-\delta) q_{t+1}\right) \tag{15}$$

3.2 Production Firms

Output is produced according to:

$$Y_t = \theta_t K_t^{\alpha_1} H_t^{\alpha_2} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{16}$$

Where K_t is aggregate capital, $K_t = (1-\eta)k_{c,t} + \eta k_{e,t}$, H_t is aggregate labor from households, $(1-\eta)l_t$, and $H_{e,t}$ is aggregate labor from entrepreneurs, η. Profit maximization implies factor prices:

$$r_t = \alpha_1 \theta_t K_t^{\alpha_1 - 1} H_t^{\alpha_2} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{17}$$

$$w_t = \alpha_2 \theta_t K_t^{\alpha_1} H_t^{\alpha_2 - 1} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{18}$$

$$x_t = (1-\alpha_1-\alpha_2) \theta_t K_t^{\alpha_1} H_t^{\alpha_2} H_{e,t}^{-\alpha_1-\alpha_2} \tag{19}$$

3.3 Entrepreneurs

Entrepreneurs have the following lifetime utility function:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\gamma)^t c_{e,t}$$

Entrepreneurs supply one unit of labor inelastically and earn wage x_t. They wake up with $k_{e,t}$ units of capital, which they also rent to the firm at r_t. After production takes place, this capital is worth $(1-\delta)q_t$. They do all this before the realization of their ω_t. This means that their "middle of period t" net worth satisfies:

$$n_t = x_t + \left[r_t + (1-\delta)q_t\right] k_{e,t} \tag{20}$$

Entrepreneurs get a loan from the CMF of $i_t - n_t$ to finance the purchase of new capital goods.

If $\omega_t < \bar{\omega}$, entrepreneurs default and have $c_{e,t} = k_{e,t+1} = 0$. The contracting problem and associated FOC are as above, just with time subscripts. Defaulting entrepreneurs start $t+1$ with x_{t+1} of net worth from supplying labor. Non-defaulting entrepreneurs then face the budget constraint:

$$c_{e,t} + q_t k_{e,t+1} = \omega_t i_t - (1 + r_t^k)(i_t - n_t)$$

Recall, this is the budget constraint for solvent entrepreneurs – i_t and n_t are predetermined on the right hand side. Going forward in time one period, their budget constraint in the next period in expectation will look like:

$$c_{e,t+1} + q_{t+1} k_{e,t+2} = [x_{t+1} + k_{e,t+1}(r_{t+1} + q_{t+1}(1-\delta))] \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$$

The term $\frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$ is the expected return, in $t+1$, from investing net worth given by the term in brackets. This comes from the contracting problem above. The entrepreneur gets $q_{t+1} i_{t+1} f(\bar{\omega}_{t+1})$ for each unit of net worth invested in expectation – the expected share of profit from a successful project, $f(\bar{\omega}_{t+1})$, times the price of capital, q_{t+1}, times how much investment there is (which is proportional to new worth via $\frac{1}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$. The lender breaking even requires that $i_{t+1} = \frac{n_{t+1}}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$ as above. Thus, the expected return on tomorrow's net worth is $\frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$.

So think about the dynamic problem conditioning on a solvent entrepreneur at the end of period t (after the draw of ω_t has been realized). We can think about a two period problem

$$\begin{aligned} \mathbb{L} = c_{e,t} + \lambda_{e,t} \left[(1 + r_t^k)(i_t - n_t) - \omega_t i_t - c_{e,t} - q_t k_{e,t+1}\right] + \\ \beta\gamma\, \mathbb{E}_t \left(c_{e,t+1} + \lambda_{e,t+1} \left([x_{t+1} + k_{e,t+1}(r_{t+1} + q_{t+1}(1-\delta))] \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})} - c_{e,t+1} - q_{t+1} k_{e,t+2} \right]\right) \end{aligned}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial c_{e,t}} = 1 - \lambda_{e,t}$$

$$\frac{\partial \mathbb{L}}{\partial k_{e,t+1}} = -q_t + \beta\gamma\, \mathbb{E}_t\, \lambda_{e,t+1} \left[r_{t+1} + q_{t+1}(1-\delta)\right] \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})}$$

$$\frac{\partial \mathbb{L}}{\partial c_{e,t+1}} = 1 - \mathbb{E}_t\, \lambda_{e,t+1}$$

Setting equal to zero and eliminating the multiplier yields:

$$q_t = \beta\gamma\, \mathbb{E}_t \left[r_{t+1} + q_{t+1}(1-\delta)\right] \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})} \tag{21}$$

3.4 Aggregation

The expected new production of new capital goods by an entrepreneur is:

$$i_t \int_0^{\infty} \omega_t \phi(\omega_t) d\omega_t - \mu i_t \int_0^{\bar{\omega}_t} \phi(\omega_t) d\omega_t$$

The first term is how much new capital is produced, which sense there is no aggregate uncertainty is just 1. The second term is how much capital is lost due to bankruptcy, which reduces to $\Phi(\bar{\omega}_t)$. So we are left with $i_t(1 - \mu\Phi(\bar{\omega}_t))$. Aggregating across entrepreneurs, we simply scale this by η, so $\eta i_t(1 - \mu\Phi(\bar{\omega}_t))$. Then we define $I_t = \eta i_t$ as aggregate investment. Recall that $K_t = (1 - \eta)k_{c,t} + \eta k_{e,t}$. Existing capital depreciates at δ. New capital must be produced by entrepreneurs and equals $\eta i_t(1 - \mu\Phi(\bar{\omega}_t)$. So the total capital stock evolves according to:

$$K_{t+1} = (1 - \delta)K_t + \eta i_t(1 - \mu\Phi(\bar{\omega}_t)) \tag{22}$$

We can further define aggregate investment as:

$$I_t = \eta i_t \tag{23}$$

Aggregate household labor input is:

$$H_t = (1 - \eta)l_t \tag{24}$$

Aggregate entrepreneurial labor input is:

$$H_{e,t} = \eta \tag{25}$$

Now *average* the budget constraint across successful entrepreneur's only:

$$c_{e,t} + q_t k_{e,t+1} = q_t f(\bar{\omega}_t) i_t \tag{26}$$

Where $f(\bar{\omega}_t)$ is expected entrepreneurial income for each unit of investment, which is in turn valued at q_t. This must be split between consumption and new owned capital. We also know that i_t satisfies:

$$i_t = \frac{n_t}{1 - q_t g(\bar{\omega}_t)} \tag{27}$$

And q_t satisfies:

$$q_t = \frac{1}{1 - \mu\Phi(\bar{\omega}_t) - \frac{\mu\Phi'(\bar{\omega}_t)f(\bar{\omega}_t)}{1 - \Phi(\bar{\omega}_t)}} \tag{28}$$

The aggregate resource constraint works out to:

$$(1 - \eta)c_t + \eta c_{e,t} + I_t = Y_t \tag{29}$$

Now, how do we get this? First, sum the budget constraint across houeholds. We get:

$$(1-\eta)c_t + (1-\eta)q_t(k_{c,t+1} - (1-\delta)k_{c,t}) = r_t(1-\eta)k_{c,t} + (1-\eta)w_t l_t$$

Now, we also know that:

$$k_{c,t} = \frac{1}{1-\eta}(K_t - \eta k_{e,t})$$

So we can plug this in to above, getting:

$$(1-\eta)c_t + q_t\left[K_{t+1} - \eta k_{e,t+1} - (1-\delta)(K_t - \eta k_{e,t})\right] = r_t K_t + (1-\eta)w_t l_t - \eta r_t k_{e,t}$$

This can be separated out to be:

$$(1-\eta)c_t + q_t(K_{t+1} - (1-\delta)K_t) - q_t\eta k_{e,t+1} + \eta(1-\delta)q_t k_{e,t} = r_t K_t + (1-\eta)w_t l_t - \eta r_t k_{e,t}$$

Which can be written further:

$$(1-\eta)c_t + q_t I_t(1-\mu\Phi(\bar{\omega}_t)) - \eta q_t k_{e,t+1} = r_t K_t + (1-\eta)w_t l_t - (r_t + (1-\delta))k_{e,t}$$

Because the production function is constant returns to scale, we must have $Y_t = r_t K_t + (1-\eta)w_t l_t + \eta x_t$. Hence, the right hand side can be written:

$$(1-\eta)c_t + q_t I_t(1-\mu\Phi(\bar{\omega}_t)) - \eta q_t k_{e,t+1} = Y_t - \eta\left[x_t + (r_t + (1-\delta))k_{e,t}\right]$$

But the term in brackets on the RHS is n_t, and we know that $n_t = (1 - q_t g(\bar{\omega}_t))i_t$. Plugging this in, we have:

$$(1-\eta)c_t + q_t I_t(1-\mu\Phi(\bar{\omega}_t)) - \eta q_t k_{e,t+1} = Y_t - \eta i_t(1 - q_t g(\bar{\omega}_t))$$

Which, since $I_t = \eta i_t$, is:

$$(1-\eta)c_t + q_t I_t(1-\mu\Phi(\bar{\omega}_t)) - \eta q_t k_{e,t+1} = Y_t - \eta i_t + q_t I g(\bar{\omega}_t))$$

Moving the last term on the RHS to the LHS, we get:

$$(1-\eta)c_t + q_t I_t(1-\mu\Phi(\bar{\omega}_t) - g(\bar{\omega}_t)) - \eta q_t k_{e,t+1} = Y_t - \eta i_t$$

But now the term multiplying $q_t I_t$ is just $f(\bar{\omega}_t)$:

$$(1-\eta)c_t + q_t I_t f(\bar{\omega}_t) - \eta q_t k_{e,t+1} = Y_t - I_t$$

We know further that $k_{e,t+1} = f(\bar{\omega}_t)i_t - c_{e,t}/q_t$. Hence, plugging this in, we have:

$$(1-\eta)c_t + q_t I_t f(\bar{\omega}_t) - \eta i_t q_t f(\bar{\omega}_t) + \eta c_{e,t} = Y_t - I_t$$

But then since $I_t = \eta i_t$, the terms involving $f(\bar{\omega}_t)$ cancel. Then we get the resource constraint given above, (29).

We assume an AR(1) process for θ_t, the productivity shock. We also consider an iid wealth shock which impacts the net worth of entrepreneurs. Note that if you write out the household's budget constraint separately, the net worth shock needs to show up in there – think of the net worth shock as a redistribution from households to entrepreneurs.

The full set of equilibrium conditions are therefore:

$$\nu = \frac{w_t}{c_t} \tag{30}$$

$$q_t = \beta\,\mathbb{E}_t \frac{c_t}{c_{t+1}} \left(r_{t+1} + (1-\delta)q_{t+1}\right) \tag{31}$$

$$r_t = \alpha_1 \theta_t K_t^{\alpha_1 - 1} H_t^{\alpha_2} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{32}$$

$$w_t = \alpha_2 \theta_t K_t^{\alpha_1} H_t^{\alpha_2 - 1} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{33}$$

$$x_t = (1-\alpha_1-\alpha_2)\theta_t K_t^{\alpha_1} H_t^{\alpha_2} H_{e,t}^{-\alpha_1-\alpha_2} \tag{34}$$

$$n_t = x_t + \left[r_t + (1-\delta)q_t\right] k_{e,t} + s_n e_{n,t} \tag{35}$$

$$q_t = \beta\gamma\,\mathbb{E}_t \left[r_{t+1} + q_{t+1}(1-\delta)\right] \frac{q_{t+1} f(\bar{\omega}_{t+1})}{1 - q_{t+1} g(\bar{\omega}_{t+1})} \tag{36}$$

$$K_{t+1} = (1-\delta)K_t + I_t(1 - \mu\Phi(\bar{\omega}_t)) \tag{37}$$

$$I_t = \eta i_t \tag{38}$$

$$H_t = (1-\eta) l_t \tag{39}$$

$$H_{e,t} = \eta \tag{40}$$

$$(1-\eta)c_t + \eta c_{e,t} + I_t = Y_t \tag{41}$$

$$c_{e,t} + q k_{e,t+1} = q_t f(\bar{\omega}_t) i_t \tag{42}$$

$$i_t = \frac{n_t}{1 - q_t g(\bar{\omega}_t)} \tag{43}$$

$$q_t = \frac{1}{1 - \mu\Phi(\bar{\omega}_t) - \frac{\mu\Phi'(\bar{\omega}_t) f(\bar{\omega}_t)}{1-\Phi(\bar{\omega}_t)}} \tag{44}$$

$$Y_t = \theta_t K_t^{\alpha_1} H_t^{\alpha_2} H_{e,t}^{1-\alpha_1-\alpha_2} \tag{45}$$

$$\theta_t = (1-\rho) + \rho\theta_{t-1} + s_\theta e_{\theta,t} \tag{46}$$

$$g(\bar{\omega}_t) = \Phi\left(\frac{\ln\bar{\omega}_t - \mu - \sigma^2}{\sigma}\right) - \Phi(\bar{\omega}_t)\mu + (1 - \Phi(\bar{\omega}_t))\bar{\omega}_t \tag{47}$$

$$f(\bar{\omega}_t) = 1 - \Phi(\bar{\omega}_t)\mu - g(\bar{\omega}_t) \tag{48}$$

$$\Phi'(\bar{\omega}_t) = \phi(\bar{\omega}_t)\frac{1}{\bar{\omega}_t \sigma} \tag{49}$$

3.5 Calibration

We set $\beta = 0.99$, $\alpha_1 = 0.36$, and $\alpha_2 = 0.6399$. We set $\delta = 0.02$ and $\eta = 0.1$ (the latter of which is just a normalization). We set $\mu = 0.25$.

We need to parameterize the distribution from which ω is draw to be consistent with $\mathbb{E}\,\omega = 1$. Since for a log normal distribution we have:

$$\mathbb{E}\,\omega = \exp(M + \frac{1}{2}\sigma^2) \tag{50}$$

We need M to satisfy:

$$M = -\frac{1}{2}\sigma^2 \tag{51}$$

There are two financial targets. First, the spread of:

$$(1 + r^k)q - 1 = 0.0187/4 \tag{52}$$

Why is this a spread? The lender earns $1 + r^k$ from making a loan, and this is denominated in units of capital (hence multiplication by q). The opportunity cost for the lender (since this is an intra-period loan) is 1, not the safe intertemporal rate. Note also because the targeted spread is at an annualized frequency of 187 basis points, we have to divide by four. Furthermore, we target a bankruptcy rate to satisfy:

$$\Phi(\bar{\omega}) = 0.00974 \tag{53}$$

I am going to guess values of σ and $\bar{\omega}$. Then I'm going to solve for the steady state, including an implied normalization on γ to be consistent with $\frac{\gamma q f(\bar{\omega})}{1 - q g(\bar{\omega})} = 1$. Then I iterate on my guesses of σ and $\bar{\omega}$ to hit the two targets.

Given my guesses, I can first calculate the steady state values of Φ and Φ' as:

$$\Phi(\bar{\omega}) = NN\left(\frac{\ln \bar{\omega} - M}{\sigma}\right) \tag{54}$$

$$\Phi'(\bar{\omega}) = nn\left(\frac{\ln \bar{\omega} - M}{\sigma}\right)(\bar{\omega}\sigma)^{-1} \tag{55}$$

Where NN and nn are the normal cdf and pdf, respectively. The division by $\bar{\omega}\sigma$ in (55) is application of the chain rule when taking the derivative of the normal CDF in (54). But then I can solve for $f(\bar{\omega})$ and $g(\bar{\omega})$ using facts about partial expectations with the log-normal distribution:

$$g(\bar{\omega}) = \Phi\left(\frac{\ln \bar{\omega} - M - \sigma^2}{\sigma}\right) - \Phi(\bar{\omega})\mu + (1 - \Phi(\bar{\omega}))\bar{\omega} \tag{56}$$

$$f(\bar{\omega}) = 1 - \mu\Phi(\bar{\omega}) - g(\bar{\omega}) \tag{57}$$

But then I can get steady state price of capital:

$$q = \frac{1}{1 - \Phi(\bar{\omega})\mu - \frac{\Phi'(\bar{\omega})\mu f(\bar{\omega})}{1-\Phi(\bar{\omega})}} \tag{58}$$

This then gives me the normalization on γ:

$$\gamma = \frac{1 - qg(\bar{\omega})}{qf(\bar{\omega})} \tag{59}$$

But then I can solve for r:

$$r = \frac{q(1 - \beta(1 - \delta))}{\beta} \tag{60}$$

Using this, I can get K. And knowing K as well as H and H_e, I can get Y and the factor prices w and x. Then knowing K, I can get I, and hence i, from the accumulation equation. But then I can solve for steady state net worth and consumption. Everything else falls out.

I end up with values of $\gamma = 0.947$ and $\sigma = 0.205$. The former is exactly what they report, the σ value is very slightly off (they report 0.207). There is actually no difference – I am finding the variance of the normal distribution; they are reporting the variance of the log-normal. The relationship between the two is given by:

$$\sigma_2^2 = \exp[2M + \sigma^2][\exp(\sigma^2) - 1]$$

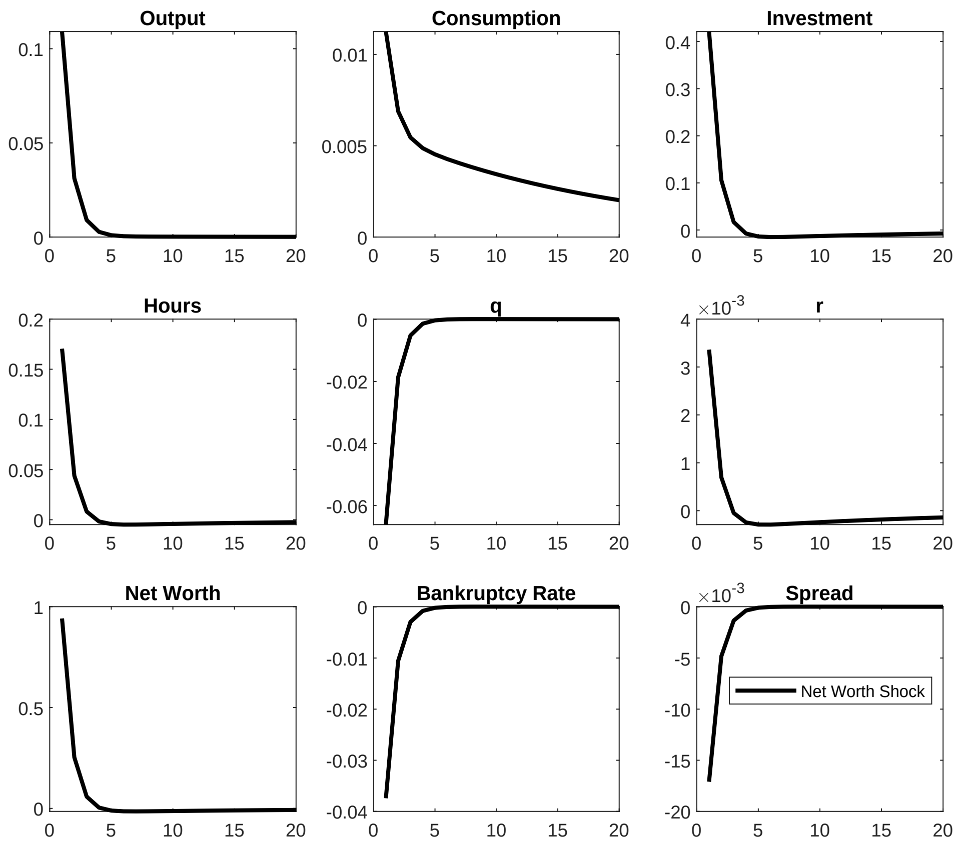
If I plug in my value of M and σ and then take the square root, I get a value of $\sigma_2 = 0.207$, which is exactly what they report in the paper.

I get exactly the same steady state ratios as reported in the paper: $ce/n = 0.067$, $n/i = 0.38$, and $q = 1.024$. I parameterize the stochastic process for θ_t with $\rho = 0.95$. And I can consider one unit shocks to productivity and net worth in computing impulse responses.

4 Impulse Responses and Analysis

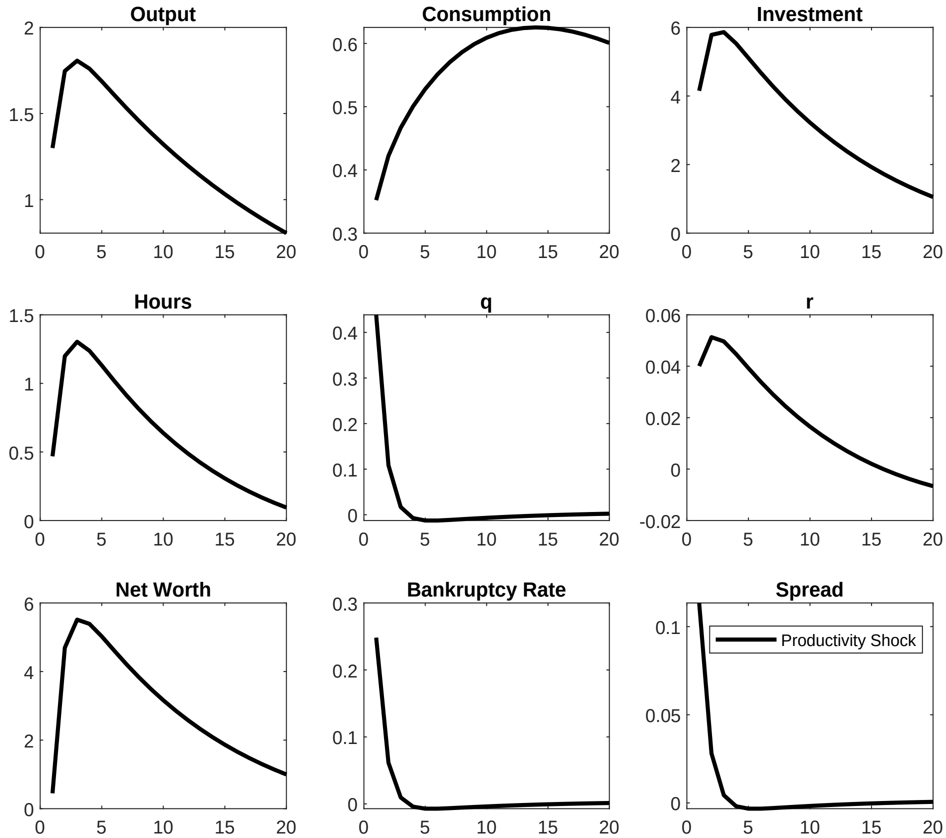
I show responses of logged variables, focusing on aggregate consumption, $C_t = (1 - \eta)c_t + \eta c_{e,t}$. I first show responses to a net worth shock, and then to a productivity shock.

Figure 1: Net Worth Shock



The responses to a net worth shock are virtually identical to what is reported in the paper. Output and investment rise, albeit only temporarily. Similarly with hours. The price of capital, the bankruptcy rate, and the interest rate spread all fall. The responses are not particularly persistent, but recall we are hitting the economy with an iid shock – so all of this persistence is endogenous.

Figure 2: Productivity Shock



The key fact about the productivity impulse responses is that the responses of output, investment, and hours are hump-shaped. This is endogenous persistence that a standard RBC model cannot generate; it is isomorphic to an investment adjustment cost. The model is problematic in that the bankruptcy rate and spread both rise. Although I do not show the response without the agency problem, note that in their paper output under-responds with the agency cost relative to the RBC model. That is, agency costs generate some persistence at the expense of amplification.