Carlstrom, Fuerst, and Paustian (2017, *AEJ: Macro*)

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1 Overview

This note provides a detailed description of Carlstrom, Fuerst, and Paustian (2017, *AEJ: Macro*). Although this paper hasn't gotten too many cites (it has been eclipsed by the Gertler and Karadi 2011, 2013 papers), it's a really nice description of how QE might work and how market segmentation can get you a first order term premium.

The core of the model is a standard medium-scale New Keynesian DSGE model. Households have to issue long term debt to finance physical capital accumulation. They cannot save through long-term debt, however – they can only save through short-term deposits. This is the sense in which markets are segmented, although it would be more natural to have firms doing the investment (borrowing) instead of the households simultaneously being borrowers and savers. Long-term debt is held by financial intermediaries (i.e. banks) who finance themselves with a mix of equity and debt (the short-term deposits that households use to save). This is a more or less accurate description of financial intermediation in the real world – banks "borrow short and lend long." There are three twists to the intermediate problem. First, they are subject to a limit enforcement type constraint that in equilibrium prevents them from defaulting. This is qualitatively similar to Jermann and Quadrini (2012), although there the constraint is on firms, not on banks. Like Jermann and Quadrini (2012), there is an exogenous shock to the tightness of the constraint – a "credit shock." Intermediaries are assumed to be more impatient than households. This ensures that the constraint binds in equilibrium. This binding constraint limits their leverage, and prevents them from arbitraging away the long-short spread (the difference in returns on long bonds and short-term deposits); the households cannot arbitrage away this spread by the segmentation assumption. In addition, there is a net worth adjustment cost. This is similar to the dividend adjustment cost in Jermann and Quadrini (2012). As in Jermann and Quadrini (2012), you basically need the extra discounting to generate a steady state lending spread, and you need the adjustment cost to make it time-varying. The authors show how to measure the term premium in the model and why it shows up even to first-order and even without exotic preferences. The key insight is that the term premium is essentially perfectly correlated with an endogenous investment wedge that distorts the first order condition related to capital accumulation.

Intermediaries hold both private investment bonds (the long-bonds) and long term government bonds. The QE experiment involves reducing the amount of government bonds they hold. This frees up space on their balance sheet; and because they are balance sheet constrained, this allows them to "arbitrage away" some of the long-short lending spread. This results in an investment and aggregate demand boom, with output and inflation rising. This is more or less how policymakers think of QE as working in practice – the policy pushes up long-bond prices, pushes down their yields, lowers the term premium, and stimulates demand. The time-varying term premium also has implications for other shocks. From a normative perspective, an interesting conclusion is that *endogenous* balance sheet policies can completely mitigate the effects of credit shocks in the model and can improve the responses of aggregate variables to other shocks. This has the implication that QE-type policies perhaps ought to be used *all the time*, not just as an antidote to a binding zero lower bound.

2 Long Bonds

There will be two types of long-term debt instruments in the model. Investment bonds, issued by households, and government bonds, issued by the government. These will take an identical format, so I will only discuss things in the context of the investment bonds.

The structure of the bonds are decaying perpetuities as introduced by Woodford (2001). These offer coupon liabilities that are decaying via the parameter κ. Let CI_t be the total issuance of these bonds in period t. Denote by F_{t-1} as the coupon liability from all past issuances due in period t. It satisfies:

$$F_{t-1} = CI_{t-1} + \kappa CI_{t-2} + \kappa^2 CI_{t-3} + \dots \tag{1}$$

In other words, issuing CI_t dollars worth of bonds in period t obliges one to CI_t in coupon payments in $t+1$, κCI_t in $t+2$, $\kappa^2 CI_t$ in $t+3$, and so on. Note we can write:

$$CI_t = F_t - \kappa F_{t-1} \tag{2}$$

New issues trade at price Q_t. Because of the coupon structure, bonds issued j periods in the past simply trade at $\kappa^j Q_t$.

3 Households

Households supply differentiated labor, indexed by s to firms. There is perfect consumption insurance (though I omit that in what follows) so they are otherwise identical except for choice of labor. Households face the real budget constraint:

$$C_t + p_t^k \widehat{I}_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} \leq w_t(s) H_t(s) + R_t K_t - T_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t + \Pi_t^y + \Pi_t^k \tag{3}$$

On the expenditure side, the household can consume, purchase new capital, $\widehat{I}_t$ (at real relative price p_t^k), save through deposits, D_t, and pay off coupons on outstanding long-bond liabilities, F_{t-1}. On the income side, they earn labor and capital income, pay a lump sum tax to the government, earn interest income on their deposits, issue new long bonds at price Q_t, and get lump sum profits. div_t is their dividend from ownership in the intermediary. Π_t^y is dividend from ownership in production firms, and Π_t^k from ownership in capital producers.

Physical capital accumulates according to a standard law of motion:

$$K_{t+1} = \widehat{I}_t + (1-\delta)K_t \tag{4}$$

Households are subject to a "loan in advance constraint" that new investment must be financed by issuing long bonds:

$$p_t^k \widehat{I}_t \leq \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} \tag{5}$$

A household has fairly standard preferences, with internal habit formation over consumption measured by the parameter h. A Lagrangian for the household is:

$$\begin{aligned}\mathbb{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \ln(C_t - hC_{t-1}) - B\frac{H_t(s)^{1+\eta}}{1+\eta} + \lambda_t(\widehat{I}_t + (1-\delta)K_t - K_{t+1}) + \\ \Lambda_t \Bigg[& w_t(s)H_t(s) + R_t K_t - T_t + \frac{D_{t-1}}{P_t}R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} - C_t - p_t^k \widehat{I}_t - \frac{D_t}{P_t} - \frac{F_{t-1}}{P_t}\Bigg] + \\ & \vartheta_t \left(\frac{Q_t(F_t - \kappa F_{t-1})}{P_t} - p_t^k \widehat{I}_t \right) \Bigg\}\end{aligned}$$

Derivatives of the Lagrangian with respect to non-labor choices are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = \frac{1}{C_t - hC_{t-1}} - \Lambda_t - \beta h \,\mathbb{E}_t \frac{1}{C_{t+1} - hC_t}$$

$$\frac{\partial \mathbb{L}}{\partial \widehat{I}_t} = \lambda_t - p_t^k \Lambda_t - \vartheta_t p_t^k$$

$$\frac{\partial \mathbb{L}}{\partial D_t} = -\frac{\Lambda_t}{P_t} + \mathbb{E}_t \,\beta \Lambda_{t+1} R_t^d P_{t+1}^{-1}$$

$$\frac{\partial \mathbb{L}}{\partial K_{t+1}} = -\lambda_t + \beta \,\mathbb{E}_t(\lambda_{t+1}(1-\delta) + \Lambda_{t+1}R_{t+1})$$

$$\frac{\partial \mathbb{L}}{\partial F_t} = \Lambda_t \frac{Q}{P_t} + \vartheta_t \frac{Q_t}{P_t} - \beta \,\mathbb{E}_t \,\Lambda_{t+1} \frac{\kappa Q_{t+1}}{P_{t+1}} - \beta \,\mathbb{E}_t \,\vartheta_{t+1} \frac{\kappa Q_{t+1}}{P_{t+1}} - \beta \,\mathbb{E}_t \,\Lambda_{t+1} \frac{1}{P_{t+1}}$$

Set these equal to zero:

$$\Lambda_t = \frac{1}{C_t - hC_{t-1}} - \beta h \,\mathbb{E}_t \frac{1}{C_{t+1} - hC_t} \tag{6}$$

$$\lambda_t = p_t^k(\Lambda_t + \vartheta_t) \tag{7}$$

$$\Lambda_t = \beta\,\mathbb{E}_t\,\Lambda_{t+1}R_t^d\Pi_{t+1}^{-1} \tag{8}$$

$$\lambda_t = \beta\,\mathbb{E}_t[\Lambda_{t+1}R_{t+1} + \lambda_{t+1}(1-\delta)] \tag{9}$$

$$(\Lambda_t + \vartheta_t)Q_t = \beta\,\mathbb{E}_t\,\Pi_{t+1}^{-1}\left[(\Lambda_{t+1} + \vartheta_{t+1})\,\kappa Q_{t+1} + \Lambda_{t+1}\right] \tag{10}$$

Plug (7) into (9) to eliminate λ_t:

$$p_t^k(\Lambda_t + \vartheta_t) = \beta\,\mathbb{E}_t[\Lambda_{t+1}R_{t+1} + p_{t+1}^k(\Lambda_{t+1} + \vartheta_{t+1})(1-\delta)] \tag{11}$$

Define $M_t = 1 + \frac{\vartheta_t}{\Lambda_t}$. This then allows us to write this FOC as:

$$p_t^k\Lambda_t M_t = \beta\,\mathbb{E}_t\,\Lambda_{t+1}\left[R_{t+1} + (1-\delta)p_{t+1}^k M_{t+1}\right] \tag{12}$$

We can then similarly write (10) as:

$$\Lambda_t Q_t M_t = \beta\,\mathbb{E}_t\,\Lambda_{t+1}\Pi_{t+1}^{-1}\left[1 + \kappa Q_{t+1}M_{t+1}\right] \tag{13}$$

Stop and look at (13) to gain some intuition. If there were no "borrow in advance" constraint, then $\vartheta_t = 0$ and $M_t = 1$. But then we could write:

$$p_t^k = \mathbb{E}_t\,m_{t,t+1}\left[R_{t+1} + (1-\delta)p_{t+1}^k\right] \tag{14}$$

$$Q_t = \mathbb{E}_t\,m_{t,t+1}\Pi_{t+1}^{-1}\left[1 + \kappa Q_{t+1}\right] \tag{15}$$

Where $m_{t,t+1} = \beta\frac{\Lambda_{t+1}}{\Lambda_t}$ is the usual stochastic discount factor. (14)-(15) would be standard asset pricing conditions for capital and long-term bonds – the price you are willing to pay for an asset (left hand side) is the expected value of the product of the stochastic discount factor with the payout from the asset.

Now, turn to wage-setting. A household indexed by s can only adjust its wage with probability $1-\theta_w$. Labor input supplied to firms, H_t, is a CES aggregate of household labor:

$$H_t = \left[\int_0^1 H_t(s)^{\frac{\epsilon_w}{\epsilon_w - 1}}\,ds\right]^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{16}$$

This implies a demand function for labor of variety s:

$$H_t(s) = \left(\frac{w_t(s)}{w_t}\right)^{-\epsilon_w} H_t \tag{17}$$

And a real wage index:

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(s)^{1-\epsilon_w}\,ds \tag{18}$$

Recreate the parts of the Lagrangian corresponding to the choice of $W_t^{\#}$, which is the common reset nominal wage all households will choose who get to update. A wage chosen today will be in effect s periods into the future with probability θ_w^s. Ignore indexation. We get:

$$\mathbb{L} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Bigg\{ - B \frac{\left(W_t^{\#}\right)^{-\epsilon_w(1+\eta)} W_{t+s}^{\epsilon_w(1+\eta)} H_{t+s}^{1+\eta}}{1+\eta} + \Lambda_{t+s} \left(W_t^{\#}\right)^{1-\epsilon_w} W_{t+s}^{\epsilon_w} P_{t+s}^{-1} H_{t+s} \Bigg\}$$

The FOC is:

$$\epsilon_w (W_t^{\#})^{-\epsilon_w(1+\eta)-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s W_{t+s}^{\epsilon_w(1+\eta)} B H_{t+s}^{1+\eta} + (1-\epsilon_w)(W_t^{\#})^{-\epsilon_w} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Lambda_{t+s} W_{t+s}^{\epsilon_w} P_{t+s}^{-1} H_{t+s} = 0$$

Which may be written:

$$(W_t^{\#})^{1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{F_{1,t}}{F_{2,t}}$$

Where:

$$F_{1,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s W_{t+s}^{\epsilon_w(1+\eta)} B H_{t+s}^{1+\eta}$$

$$F_{2,t} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w)^s \Lambda_{t+s} W_{t+s}^{\epsilon_w} P_{t+s}^{-1} H_{t+s}$$

We can write these auxiliary variables recursively as:

$$F_{1,t} = W_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, F_{1,t+1}$$

$$F_{2,t} = \Lambda_t W_t^{\epsilon_w} P_t^{-1} H_t + \theta_w \beta \, \mathbb{E}_t \, F_{2,t+1}$$

We need to write these in terms of real wages (denoted with lowercase letters). So:

$$F_{1,t} = w_t^{\epsilon_w(1+\eta)} P_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, F_{1,t+1}$$

$$F_{2,t} = \Lambda_t w_t^{\epsilon_w} P_t^{\epsilon_w - 1} H_t + \theta_w \beta \, \mathbb{E}_t \, F_{2,t+1}$$

Now, we need to get rid of the price level terms. Define $f_{1,t} = F_{1,t}/P_t^{\epsilon_w(1+\eta)}$ and $f_{2,t} = F_{2,t}/P_t^{\epsilon_w - 1}$.

We get:

$$f_{1,t} = w_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \frac{F_{1,t+1}}{P_t^{\epsilon_w(1+\eta)}}$$

$$f_{2,t} = \Lambda_t w_t^{\epsilon_w} H_t + \theta_w \beta \, \mathbb{E}_t \frac{F_{2,t+1}}{P_t^{\epsilon_w - 1}}$$

Or:

$$f_{1,t} = w_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \frac{P_{t+1}^{\epsilon_w(1+\eta)}}{P_t^{\epsilon_w(1+\eta)}} \frac{F_{1,t+1}}{P_{t+1}^{\epsilon_w(1+\eta)}}$$

$$f_{2,t} = \Lambda_t w_t^{\epsilon_w} H_t + \theta_w \beta \, \mathbb{E}_t \frac{P_{t+1}^{\epsilon_w - 1}}{P_t^{\epsilon_w - 1}} \frac{F_{2,t+1}}{P_{t+1}^{\epsilon_w - 1}}$$

So:

$$f_{1,t} = w_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w(1+\eta)} f_{1,t+1} \tag{19}$$

$$f_{2,t} = \Lambda_t w_t^{\epsilon_w} H_t + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w - 1} f_{2,t+1} \tag{20}$$

But then we have:

$$\frac{F_{1,t}}{F_{2,t}} = \frac{f_{1,t}}{f_{2,t}} P_t^{1+\epsilon_w \eta}$$

But the price level term there makes the reset wage on the left hand side of the wage-setting equal the real reset wage:

$$(w_t^{\#})^{1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{f_{1,t}}{f_{2,t}} \tag{21}$$

4 Production

4.1 Final Good

The final good is a CES aggregate of intermediates, where intermediates are indexed by l:

$$Y_t = \left[\int_0^1 Y_t(l)^{\frac{\epsilon_p - 1}{\epsilon_p}} \right]^{\frac{\epsilon_p}{\epsilon_p - 1}} \tag{22}$$

This implies a demand function and a price index:

$$Y_t(l) = \left(\frac{P_t(l)}{P_t} \right)^{-\epsilon_p} Y_t \tag{23}$$

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(l)^{1-\epsilon_p} dl \tag{24}$$

4.2 Intermediate Producers

Intermediate producers produce $Y_t(l)$ according to:

$$Y_t(l) = A_t K_t(l)^{\alpha} H_t(l)^{1-\alpha}$$

Capital and labor are hired nominal prices W_t and R_t^n. An intermediary cannot freely choose its price, but will always pick inputs to minimize cost, subject to producing enough to meet demand. A cost minimization problem written as a Lagrangian implies:

$$\mathbb{L} = -W_t H_t(l) - R_t^n K_t(l) + MC_t(l) \left[A_t K_t(l)^\alpha H_t(l)^{1-\alpha} - \left(\frac{P_t(l)}{P_t} \right)^{-\epsilon_p} Y_t \right]$$

The FOC are:

$$W_t = MC_t(l)(1-\alpha)A_t \left(\frac{K_t(l)}{H_t(l)} \right)^\alpha$$

$$R_t^n = MC_t(l)\alpha A_t \left(\frac{K_t(l)}{H_t(l)} \right)^{\alpha-1}$$

Combining these, we get:

$$\frac{W_t}{R_t^n} = \frac{1-\alpha}{\alpha} \frac{K_t(l)}{H_t(l)}$$

But since all intermediates face the same factor prices, this implies they all choose the same capital-labor ratio, which will in turn equal the aggregate capital-labor ratio. Hence, they all face the same nominal marginal cost. We can then re-write the firm's FOC in real terms, where $mc_t = MC_t/P_t$:

$$w_t = mc_t(1-\alpha)A_t \left(\frac{K_t}{H_t} \right)^\alpha \tag{25}$$

$$R_t = mc_t \alpha A_t \left(\frac{K_t}{H_t} \right)^{\alpha-1} \tag{26}$$

Intermediate flow profit, in nominal terms, is:

$$\Pi_t^{y,n} = P_t(l)Y_t(l) - W_t H_t(l) - R_t^n K_t(l)$$

But then using the cost minimization conditions, this is:

$$\Pi_t^{y,n} = P_t(l)Y_t(l) - MC_t Y_t(l)$$

Writing in real terms by dividing by P_t:

$$\Pi_t^y = \frac{P_t(l)}{P_t} Y_t(l) - mc_t Y_t(l)$$

Now plug in the demand function:

$$\Pi_t^y = \left(\frac{P_t(l)}{P_t} \right)^{1-\epsilon_p} Y_t - mc_t \left(\frac{P_t(l)}{P_t} \right)^{-\epsilon_p} Y_t$$

Firms can only adjust their prices with probability $1-\theta_p$. Ignore indexation. Their problem is to pick $P_t^{\#}$, which will be common to all intermediate producers, to maximize the PDV of flow profits measured in utils. So the problem is:

$$\max_{P_t^{\#}} \; \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_p)^s \Lambda_{t+s} \left[\left(P_t^{\#}\right)^{1-\epsilon_p} P_{t+s}^{\epsilon_p - 1} Y_t - mc_{t+s} \left(P_t^{\#}\right)^{-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} \right]$$

The FOC is:

$$(1-\epsilon_p)(P_t^{\#})^{-\epsilon_p} \, \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_p)^s P_{t+s}^{\epsilon_p - 1} Y_{t+s} + \epsilon_p (P_t^{\#})^{-\epsilon_p - 1} \, \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_p)^s mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s} = 0$$

Which can be written:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}}{X_{2,t}}$$

Where:

$$X_{1,t} = \Lambda_t mc_t P_t^{\epsilon_p} Y_t + \theta_p \beta \, \mathbb{E}_t \, X_{1,t+1}$$

$$X_{2,t} = \Lambda_t P_t^{\epsilon_p - 1} Y_t + \theta_p \beta \, \mathbb{E}_t \, X_{2,t+1}$$

Define $x_{1,t} = X_{1,t}/P_t^{\epsilon_p}$ and $x_{2,t} = X_{2,t}/P_t^{\epsilon_p - 1}$. We get:

$$x_{1,t} = \Lambda_t mc_t Y_t + \theta_p \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_p} x_{1,t+1} \tag{27}$$

$$x_{2,t} = \Lambda_t Y_t + \theta_p \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_p - 1} x_{2,t+1} \tag{28}$$

But since:

$$\frac{X_{1,t}}{X_{2,t}} = \frac{x_{1,t}}{x_{2,t}} P_t$$

We can define $\Pi_t^{\#} = P_t^{\#}/P_t$ and write:

$$\Pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{x_{1,t}}{x_{2,t}} \tag{29}$$

4.3 New Capital Producer

New investment goods, $\widehat{I}_t$, are produced using unconsumed output, I_t, according to:

$$\widehat{I}_t = \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) \right] I_t \tag{30}$$

The investment producer picks I_t to maximize the PDV of flow profits measured in utils, where p_t^k is the real price of new capital goods:

$$\max_{I_t} \;\; \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ p_t^k \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) \right] I_t - I_t \right\}$$

The FOC is:

$$\Lambda_t p_t^k \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) \right] - \Lambda_t p_t^k \mu_t S'\left(\frac{I_t}{I_{t-1}} \right) \frac{I_t}{I_{t-1}} - \Lambda_t + \beta \, \mathbb{E}_t \, \Lambda_{t+1} p_{t+1}^k \mu_{t+1} S'\left(\frac{I_{t+1}}{I_t} \right) \left(\frac{I_{t+1}}{I_t} \right)^2 = 0$$

Setting equal to zero, we get:

$$p_t^k \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) - S'\left(\frac{I_t}{I_{t-1}} \right) \frac{I_t}{I_{t-1}} \right] = 1 - \beta \, \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} p_{t+1}^k \mu_{t+1} S'\left(\frac{I_{t+1}}{I_t} \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \tag{31}$$

5 Financial Intermediaries

The financial intermediary is the sole-buyer of investment bonds, issued by the household and denoted by F_t as described above, and long-term government bonds, denoted by B_t. It finances itself with deposits from the household, D_t, and net worth/equity, N_t. Its balance sheet in real terms at any point in time is:

$$\frac{B_t}{P_t} Q_t + \frac{F_t}{P_t} Q_t = \frac{D_t}{P_t} + N_t \tag{32}$$

Government and private bonds are perfect substitutes and have the same price and cash flow details. In the problem, the FI is going to want to maximize the PDV of the utility value of dividends, where there is extra discounting given by $\zeta < 1$. As we will see, this extra discounting ensures that the FI doesn't accumulate enough net worth so as to arbitrage away lending spreads in the steady state. Second, the FI is subject to a net worth adjustment cost. This is going to keep the FI from adjusting net worth too quickly so to allow for meaningful deviations for a steady state lending spread.

I'm going to present the problem in a somewhat backwards way. In the formal problem, the FI is subject to a limited enforcement type constraint – it can only take in deposits up until the point at which it does not want to abscond with them (i.e. default). There will be an exogenous component to the amount they can run off with, akin to Jermann and Quadrini's credit shock. This constraint, plus the net worth adjustment cost, complicates things because it adds an additional term to the value function of an intermediary. To counter this, the authors are going to make the fraction of resources that the FI can abscond with in default take on a special form. This special form will be such that the FI treats its leverage ratio *as given.* I'm going to present in a backwards way because I'm first going to show the optimal choice for net worth, taking leverage as given, and then will show what we need to assume in the so-called "holdup" problem to be consistent with the idea that leverage is taken as given.

5.1 The Choice of Net Worth

Going from $t-1$ to t, the FI earns $(1+\kappa Q_t)$ on each unit of bonds it holds – 1 is the coupon payment and κQ_t is what existing bonds are worth (capital gain). It pays R^d_{t-1} on each unit of deposits. So its nominal *gross* profit (gross revenue minus gross interest expense) going from $t-1$ to t is:

$$PROF_t = (1+\kappa Q_t)(F_{t-1}+B_{t-1}) - R^d_{t-1}D_{t-1}$$

Define $R^L_t = \frac{1+\kappa Q_t}{Q_{t-1}}$ as the holding period return. Multiply and divide by Q_{t-1} to therefore get:

$$PROF_t = \frac{1+\kappa Q_t}{Q_{t-1}}\left(Q_{t-1}F_{t-1}+Q_{t-1}B_{t-1}\right) - R^d_{t-1}D_{t-1} = R^L_t(Q_{t-1}F_{t-1}+Q_{t-1}B_{t-1}) - R^d_{t-1}D_{t-1}$$

But note $D_{t-1} = (Q_{t-1}F_{t-1}+Q_{t-1}B_{t-1}) - P_{t-1}N_{t-1}$. So we get:

$$PROF_t = \left(R^L_t - R^d_{t-1}\right)\left(Q_{t-1}F_{t-1}+Q_{t-1}B_{t-1}\right) + R^d_{t-1}P_{t-1}N_{t-1}$$

Define leverage as:

$$L_t = \frac{\frac{F_t}{P_t}Q_t + \frac{B_t}{P_t}Q_t}{N_t} \tag{33}$$

We will take leverage as *given*; we shall show below why the FI will do so. Given this, it means we can write period t gross profits in nominal terms as:

$$PROF_t = P_{t-1}\left[\left(R^L_t - R^d_{t-1}\right)L_{t-1} + R^d_{t-1}\right]N_{t-1}$$

To put this in real terms, divide by P_t, noting that $\Pi_t = P_t/P_{t-1}$:

$$prof_t = \Pi_t^{-1}\left[\left(R^L_t - R^d_{t-1}\right)L_{t-1} + R^d_{t-1}\right]N_{t-1} \tag{34}$$

(34) is the same as (18) in their paper.

The FI will pay out some of its real profits as dividends, div_t. Some it will retain as net worth. Its flow of funds constraint is:

$$div_t + N_t\left[1+f(N_t)\right] \leq \Pi_t^{-1}\left[\left(R^L_t - R^d_{t-1}\right)L_{t-1} + R^d_{t-1}\right]N_{t-1} \tag{35}$$

This simply says that next period's net worth is gross profit today minus dividends, minus an adjustment cost $f(N_t)$. This adjustment cost is convex and will keep the FI from too quickly adjusting its net worth. We could write this as a law of motion for net worth as follows:

$$N_t \leq \Pi_t^{-1}\left[\left(R^L_t - R^d_{t-1}\right)L_{t-1} + R^d_{t-1}\right]N_{t-1} - div_t - f(N_t)N_t$$

Which could be written:

$$N_t \leq \Pi_t^{-1} \left[\left(R_t^L - R_{t-1}^d \right) L_{t-1} + r_{t-1}^d \right] N_{t-1} - div_t - f(N_t) N_t + \Pi_t^{-1} N_{t-1}$$

Where $R_t^d = 1 + r_t^d$, so r_t^d is the net interest rate. The first term on the right hand side would be net or flow profit. This simply says that real net worth in period t is real flow profit, less the real dividend, less the adjustment cost, plus the real value of yesterday's net worth.

The FI wishes to maximize the PDV of dividends to households. Future dividends are put in terms of the marginal utility of consumption via Λ_t, and there is additional discounting, $\zeta < 1$. The problem is:

$$\max_{N_t} \ \mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\zeta)^t \Lambda_t div_t$$

Plugging in the law of motion for net worth assuming it binds, we have:

$$\max_{N_t} \ \mathbb{E}_0 \sum_{t=0}^{\infty} (\beta\zeta)^t \Lambda_t \left[\Pi_t^{-1} \left[\left(R_t^L - R_{t-1}^d \right) L_{t-1} + R_{t-1}^d \right] N_{t-1} - N_t \left[1 + f(N_t) \right] \right]$$

The FOC, again taking leverage as given, is:

$$-\Lambda_t \left[1 + f(N_t) + f'(N_t) N_t \right] + (\beta\zeta) \, \mathbb{E}_t \, \Lambda_{t+1} \Pi_{t+1}^{-1} \left[\left(R_{t+1}^L - R_t^d \right) L_t + R_t^d \right]$$

Setting equal to zero, we have:

$$\Lambda_t \left[1 + f(N_t) + f'(N_t) N_t \right] = \beta\zeta \, \mathbb{E}_t \, \Lambda_{t+1} \Pi_{t+1}^{-1} \left[\left(R_{t+1}^L - R_t^d \right) L_t + R_t^d \right] \tag{36}$$

In the steady state, $f(N) = f'(N) = 0$. We can see something useful in (36). If $\zeta = 1$, then in steady state we'd have to have $R^L - R^d = 0$. This is because from the household problem we have $\Lambda = \beta \Lambda \Pi^{-1} R^d$. So the additional discounting generates $R^L - R^d > 0$ in steady state – i.e. an interest rate spread. The adjustment cost, captured by $f(N_t)$, will get the spread to move around in response to shocks. This is very similar to what is done in Jermann and Quadrini (2012) – they have extra discounting to make a constraint bind in steady state, and a dividend adjustment cost to make that constraint bind dynamically. This is very similar in Carlstrom, Fuerst, and Paustian (2017).

Now let's go to this assumption of taking leverage as given. First, I'm going to do some stuff related to the value function, then we'll return to the so-called holdup problem.

5.2 FI's Value Function

For ease of notation, define:

If there were no adjustment cost, the FI's value function would be linear in net worth and would equal:

$$X_t = \Pi_t^{-1} \left[\left(R_t^L - R_{t-1}^d \right) L_{t-1} + R_{t-1}^d \right]$$

This then gives us:

$$div_t = X_t N_{t-1} - N_t \left[1 + f(N_t)\right]$$

Using this notation, as described above, the FOC would be:

$$\Lambda_t \left[f(N_t) + f'(N_t)N_t\right] = \beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} X_{t+1} - \Lambda_t \tag{37}$$

Or:

$$f(N_t) + f'(N_t)N_t = \frac{\beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} X_{t+1} - \Lambda_t}{\Lambda_t}$$

This implies that N_t is a function of:

$$N_t = h(z_t), \ \text{ where } \ z_t = \frac{\beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} X_{t+1} - \Lambda_t}{\Lambda_t} \tag{38}$$

Note that, in steady state, $f(N) = f'(N) = 0$, and $\Lambda = \beta\zeta\Lambda X$. Hence, we have $N = h(0)$. Now, conjecture that the value function is given by:

$$V_t = \Lambda_t X_t N_{t-1} + g_t \tag{39}$$

Where g_t is independent of net worth. The actual value function is:

$$V_t = \Lambda_t div_t + \beta\zeta\, \mathbb{E}_t\, V_{t+1} = \Lambda_t X_t N_{t-1} - \Lambda_t N_t \left[1 + f(N_t)\right] + \beta\zeta\, \mathbb{E}_t\, V_{t+1}$$

Now substitute in the conjectured value function:

$$\Lambda_t X_t N_{t-1} + g_t = \Lambda_t X_t N_{t-1} - \Lambda_t N_t \left[1 + f(N_t)\right] + \beta\zeta\, \mathbb{E}_t \left(\Lambda_{t+1} X_{t+1} N_t + g_{t+1}\right)$$

Some stuff cancels and we can write this as:

$$g_t = -N_t \Lambda_t \left[1 + f(N_t)\right] + N_t \beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} X_{t+1} + \beta\zeta\, \mathbb{E}_t\, g_{t+1}$$

But we know from the FOC that $\beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} X_{t+1} = \Lambda_t \left[1 + f(N_t) + f'(N_t)N_t\right]$. So plug this in:

$$g_t = -N_t \Lambda_t \left[1 + f(N_t)\right] + N_t \Lambda_t \left[1 + f(N_t) + f'(N_t)N_t\right] + \beta\zeta\, \mathbb{E}_t\, g_{t+1}$$

Which then leaves:

$$g_t = \Lambda_t N_t^2 f'(N_t) + \beta\zeta\, \mathbb{E}_t\, g_{t+1} \tag{40}$$

But then using (39), we can write this as:

$$g_t = \Lambda_t (h(z_t))^2 f'(h(z_t)) + \beta\zeta\, \mathbb{E}_t\, g_{t+1} \tag{41}$$

Which can be solved forward:

$$g_t = \mathbb{E}_t \sum_{j=0}^{\infty} (\beta\zeta)^j \Lambda_{t+j} \left[h(z_{t+j})\right]^2 f' \left[h(z_{t+j})\right] \tag{42}$$

So g_t is a function of z_t, independent of N_{t-1}, and in steady state is 0 (since $f'(h(0)) = 0$).

5.3 Returning to the Leverage Assumption

The intermediate is subject to a "holdup" problem. At the beginning of $t+1$, the FI can default on planned repayments to depositors. In this case, depositors can only seize $1 - \Psi_t$ of FI assets. This will leave the FI with $\Psi_t R^L_{t+1} L_t N_t$. An incentive compatability constraint requires that the FI does not want to default – i.e. that its value of continuing as a going concern exceeds the value of defaulting:

$$\mathbb{E}_t V_{t+1} \geq \Psi_t L_t N_t \, \mathbb{E}_t \Lambda_{t+1} R^L_{t+1} \Pi^{-1}_{t+1} \tag{43}$$

The left hand side of (43) is the value of continuing, and the right hand side is the value of defaulting (in real terms), put into utils by multiplying by the household's marginal utility of consumption. Now, use the conjecturing value function, this would be:

$$N_t \, \mathbb{E}_t \Lambda_{t+1} X_{t+1} + \mathbb{E}_t g_{t+1} \geq \Psi_t L_t N_t \, \mathbb{E}_t \Lambda_{t+1} R^L_{t+1} \Pi^{-1}_{t+1} \tag{44}$$

Now, if $\mathbb{E}_t g_{t+1} = 0$, the N_t would cancel on each side, and we would get that optimal leverage is independent of firm net worth and only a function of aggregates (which is ultimately what they want). But for the more general case, we would get:

$$\mathbb{E}_t \Lambda_{t+1} X_{t+1} + \frac{\mathbb{E}_t g_{t+1}}{N_t} \geq \Psi_t L_t \, \mathbb{E}_t \Lambda_{t+1} R^L_{t+1} \Pi^{-1}_{t+1}$$

For $\mathbb{E}_t g_{t+1} > 0$, we would have L_t decreasing in net worth. They are going to avoid this complication by picking the Ψ_t term to make net worth drop out, so that leverage is only a function of aggregates. They assume that Ψ_t depends on other variables as follows:

$$\Psi_t = \Phi_t \left[1 + \frac{1}{N_t} \frac{\mathbb{E}_t g_{t+1}}{\mathbb{E}_t \Lambda_{t+1} X_{t+1}} \right] \tag{45}$$

Note there is a typo in the paper – (23) is missing a Λ_{t+1} in the denominator inside the brackets, though it appears in the appendix (A40). Φ_t is the exogenous component of the enforcement constraint. (45) is essentially saying that the hold up problem is less severe when there is more net worth, which seems plausible. Note we can write this as:

$$\Psi_t = \Phi_t \left[\frac{N_t \, \mathbb{E}_t \Lambda_{t+1} X_{t+1} + \mathbb{E}_t g_{t+1}}{N_t \, \mathbb{E}_t \Lambda_{t+1} X_{t+1}} \right] \tag{46}$$

Now plug (46) into (44) and get:

$$\mathbb{E}_t\,\Lambda_{t+1}X_{t+1} = \Phi_t L_t\,\mathbb{E}_t\,\Lambda_{t+1}R^L_{t+1}\Pi^{-1}_{t+1}$$

Now plug in for X_{t+1} using the definition given above:

$$\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\left[\left(R^L_{t+1} - R^d_t\right)L_t + R^d_t\right] = \Phi_t L_t\,\mathbb{E}_t\,\Lambda_{t+1}R^L_{t+1}\Pi^{-1}_{t+1} \tag{47}$$

Note (47) is equivalent to (24) in the paper. To make it exact, divide both sides by R^d_t:

$$\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\left[\left(\frac{R^L_{t+1}}{R^d_t} - 1\right)L_t + 1\right] = \Phi_t L_t\,\mathbb{E}_t\,\Lambda_{t+1}\frac{R^L_{t+1}}{R^d_t}\Pi^{-1}_{t+1} \tag{48}$$

Now, (48) is exactly what appears in (24) of the paper. This says that L_t is a function only of stuff external to an individual FI. So, as long as the constraint binds, we can treat leverage as given and just think about the firm choosing N_t, as we did above. If we want to whittle (48) down further, we can. Distribute multiplications:

$$\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1} + \mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\frac{R^L_{t+1}}{R^d_t}L_t - \mathbb{E}_t\,\Lambda_{t+1}\Pi_{t+1}L_t = \Phi_t L_t\,\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\frac{R^L_{t+1}}{R^d_t}$$

Collect terms:

$$\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1} = \left[(\Phi_t - 1)\,\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\frac{R^L_{t+1}}{R^d_t} + \mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\right]L_t$$

Or:

$$L_t = \frac{\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}}{\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1} + (\Phi_t - 1)\,\mathbb{E}_t\,\Lambda_{t+1}\Pi^{-1}_{t+1}\frac{R^L_{t+1}}{R^d_t}} \tag{49}$$

(49 is what appears as (A23) in the appendix.

6 Government

A monetary authority sets the deposit rate according to a Taylor rule:

$$\ln R^d_t = (1-\rho_R)\ln R^d + \rho_R \ln R^d_{t-1} + (1-\rho_R)\left[\tau_\pi(\ln\Pi_t - \ln\Pi) + \tau_y(\ln Y_t - \ln Y_{t-1})\right] + s_R\varepsilon_{R,t} \tag{50}$$

I'm going to assume the Taylor rule responds to output growth, not a gap as they do, because solving for the gap requires solving a separate model with no price or wage stickiness.

We also need to say something about the government. It does no spending, but it does issue long bonds, B_t. Its real budget constraint is:

$$\frac{B_{t-1}}{P_t} = T_t + Q_t \frac{B_t - \kappa B_{t-1}}{P_t} \tag{51}$$

Here $CB_t = B_t - \kappa B_{t-1}$ is the new issuance of government debt, valued at Q_t (the same price as the private investment bonds, since these are exactly the same). On the expenditure side, you have the real coupon liability. This must be financed by raising taxes or issuing new debt.

7 Aggregation

We need to think about aggregate profits. Aggregate nominal profit earned by intermediate firms is:

$$\Pi_t^{y,n} = \int_0^1 \left[P_t(l) Y_t(l) - W_t H_t(l) - R_t^n K_t(l) \right] dl$$

Divide both sides by P_t to put in real terms, and plug in the demand function for intermediate output:

$$\Pi_t^y = P_t^{\epsilon_p - 1} Y_t \int_0^1 P_t(l)^{1-\epsilon_p} dl - w_t H_t - R_t K_t$$

Here I have used the fact that $H_t = \int_0^1 H_t(1)$. But since $\int_0^1 P_t(l)^{1-\epsilon} dl = P_t^{1-\epsilon_p}$, this simply becomes:

$$\Pi_t^y = Y_t - w_t H_t - R_t K_t \tag{52}$$

Profit from the investment goods producer is :

$$\Pi_t^k = p_t^k \widehat{I}_t - I_t \tag{53}$$

Now integrate over the household's flow budget constraint:

$$C_t + p_t^k \widehat{I}_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = \\ \int_0^1 w_t(s) H_t(s) ds + R_t K_t - T_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t + \Pi_t^y + \Pi_t^k$$

Now, plugging in the demand function for each household's labor, we get:

$$C_t + p_t^k \widehat{I}_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = \\ w_t^{\epsilon_w} H_t \int_0^1 w_t(s)^{1-\epsilon_w} ds + R_t K_t - T_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t + \Pi_t^y + \Pi_t^k$$

But since $w_t^{1-\epsilon_w} = \int_0^1 w_t(s)^{1-\epsilon_w} ds$, this is just:

$$C_t + p_t^k \widehat{I}_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = w_t H_t + R_t K_t - T_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t + \Pi_t^y + \Pi_t^k$$

Now use (52)-(53) to simplify this further:

$$C_t + I_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = Y_t - T_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t$$

Now plug in for (51) to eliminate T_t:

$$C_t + I_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = Y_t - \frac{B_{t-1}}{P_t} + Q_t \frac{B_t - \kappa B_{t-1}}{P_t} + \frac{D_{t-1}}{P_t} R_{t-1}^d + \frac{Q_t(F_t - \kappa F_{t-1})}{P_t} + div_t$$

Now distribute the Q_t terms on the RHS:

$$C_t + I_t + \frac{D_t}{P_t} + \frac{F_{t-1}}{P_t} = Y_t - \frac{B_{t-1}}{P_t} + Q_t \frac{B_t}{P_t} - \kappa Q_t \frac{B_{t-1}}{P_t} + \frac{D_{t-1}}{P_t} R_{t-1}^d + Q_t \frac{F_t}{P_t} - \kappa Q_t \frac{F_{t-1}}{P_t} + div_t$$

This can be written:

$$C_t + I_t + \frac{D_t}{P_t} + (1 + \kappa Q_t) \frac{F_{t-1}}{P_t} + (1 + \kappa Q_t) \frac{B_{t-1}}{P_t} = Y_t + Q_t \frac{B_t}{P_t} + Q_t \frac{F_t}{P_t} + \frac{D_{t-1}}{P_t} R_{t-1}^d + div_t$$

By multiplying and dividing by Q_{t-1} and P_{t-1}, on the left hand side, we can get:

$$C_t + I_t + \frac{D_t}{P_t} + R_t^L \Pi_t^{-1} \left[\frac{F_{t-1}}{P_{t-1}} + \frac{B_{t-1}}{P_{t-1}} \right] = Y_t + Q_t \frac{B_t}{P_t} + Q_t \frac{F_t}{P_t} + \frac{D_{t-1}}{P_t} R_{t-1}^d + div_t$$

But note that we have defined $L_{t-1} = \frac{\frac{F_{t-1}}{P_{t-1}} + \frac{B_{t-1}}{P_{t-1}}}{N_{t-1}}$. Hence:

$$C_t + I_t + \frac{D_t}{P_t} + R_t^L \Pi_t^{-1} L_{t-1} N_{t-1} = Y_t + Q_t \frac{B_t}{P_t} + Q_t \frac{F_t}{P_t} + \frac{D_{t-1}}{P_t} R_{t-1}^d + div_t$$

From the balance sheet condition, we know that $Q_t \frac{B_t}{P_t} + Q_t \frac{F_t}{P_t} = N_t + \frac{D_t}{P_t}$. Hence, we have further still:

$$C_t + I_t + R_t^L \Pi_t^{-1} L_{t-1} N_{t-1} = Y_t + N_t + \frac{D_{t-1}}{P_t} R_{t-1}^d + div_t$$

Note we can write:

$$C_t + I_t + R_t^L \Pi_t^{-1} L_{t-1} N_{t-1} = Y_t + N_t + \frac{D_{t-1}}{P_{t-1}} \Pi_t^{-1} R_{t-1}^d + div_t$$

From the definition of leverage, we know further still that:

$$\frac{D_{t-1}}{P_{t-1}} = (L_{t-1} - 1)N_{t-1}$$

Now plug this in:

$$C_t + I_t + R_t^L \Pi_t^{-1} L_{t-1} N_{t-1} = Y_t + N_t + (L_{t-1} - 1)N_{t-1}\Pi_t^{-1} R_{t-1}^d + div_t$$

But this then gives us:

$$C_t + I_t + \Pi_t^{-1}\left[(R_t^L - R_{t-1}^d)L_{t-1} + R_{t-1}^d\right] N_{t-1} = Y_t + N_t + div_t$$

But note $div_t = \Pi_t^{-1}\left[(R_t^L - R_{t-1}^d)L_{t-1} + R_{t-1}^d\right] N_{t-1} - N_t\left[1 + f(N_t)\right]$. Plugging this in, we get:

$$C_t + I_t + f(N_t)N_t = Y_t \tag{54}$$

Other than the adjustment cost, (54) is the standard resource constraint.

Using properties of Calvo pricing, the aggregate price level in terms of inflation evolves according to:

$$1 = (1 - \theta_p)\left(\Pi_t^{\#}\right)^{1-\epsilon_p} + \theta_p \Pi_t^{\epsilon_p - 1} \tag{55}$$

The aggregate real wage evolves according to:

$$w_t^{1-\epsilon_w} = (1 - \theta_w)(w_t^{\#})^{1-\epsilon_w} + \theta_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{56}$$

The aggregate production function is:

$$d_t^p Y_t = A_t K_t^{\alpha} H_t^{1-\alpha} \tag{57}$$

Where d_t^p is price dispersion:

$$d_t^p = (1 - \theta_p)(\Pi_t^{\#})^{-\epsilon_p} + \theta_p \Pi_t^{\epsilon_p} d_{t-1}^p \tag{58}$$

Let $f_t = F_t/P_t$ and $b_t = B_t/P_t$ be the real values of debt. They assume government debt held by FIs is exogenous (which could be affected by either the central bank buying debt or the fiscal authority issuing less of it). So we have:

$$\ln b_t = (1 - \rho_{1,b} - \rho_{2,b}) \ln b + \rho_{1,b} \ln b_{t-1} + \rho_{2,b} \ln b_{t-2} + s_b \varepsilon_{b,t} \tag{59}$$

Like the authors, I'm going to allow bond issuance to follow an AR(2) process.

It turns out we don't need to keep track of deposits. We can write the FI's balance sheet using the leverage term:

$$Q_t f_t + Q_t b_t = L_t N_t \tag{60}$$

Aggregate productivity, the MEI shock, and the credit shock all follow AR(1)s in the log, with the former two normalized to non-stochastic steady state values of unity:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{61}$$

$$\ln \mu_t = \rho_\mu \ln \mu_{t-1} + s_\mu \varepsilon_{\mu,t} \tag{62}$$

$$\ln \Phi_t = (1-\rho_\Phi)\ln \Phi + \rho_\Phi \ln \Phi_{t-1} + s_\Phi \varepsilon_{\Phi,t} \tag{63}$$

8 Bond Returns, Yields, and the Term Premium

The price of long bonds, Q_t, is the same for both private investment and government bonds. The holding period return, which has been introduced before, is:

$$R_t^L = \frac{1+\kappa Q_t}{Q_{t-1}} \tag{64}$$

The yield to maturity on the long bond is the (gross) discount rate that equates the price of the bond to the PDV of cash flows. In other words:

$$Q_t = \frac{1}{R_{y,t}} + \frac{\kappa}{R_{y,t}^2} + \frac{\kappa^2}{R_{y,t}^3} + \dots$$

Here, $R_{y,t}$ is the gross yield to maturity. This may be written:

$$Q_t = \frac{1}{R_{y,t}}\left[1 + \frac{\kappa}{R_{y,t}} + \frac{\kappa^2}{R_{y,t}^2} + \dots\right] = \frac{1}{R_{y,t}}\frac{1}{1-\frac{\kappa}{R_{y,t}}} = \frac{1}{R_{y,t}-\kappa}$$

This then gives us an expression for the yield:

$$R_{y,t} = Q_t^{-1} + \kappa \tag{65}$$

Define a hypothetical expectations hypothesis bond as having the same payout structure as the long bond, but instead discounting by the safe gross interest rate as opposed to the stochastic discount factor:

$$Q_t^{EH} = \frac{1+\kappa\,\mathbb{E}_t\, Q_{t+1}^{EH}}{R_t^d} \tag{66}$$

The yield on the expectations bond takes the same form as (65):

$$R_{y,t}^{EH} = \left(Q_t^{EH}\right)^{-1} + \kappa \tag{67}$$

The gross term premium is then the ratio of the actual yield to maturity to the hypothetical expectations hypothesis bond:

$$TP_t = \frac{R_{y,t}}{R^{EH}_{y,t}} \tag{68}$$

9 All Equilibrium Conditions

- Household (non-wage):

$$\Lambda_t = \frac{1}{C_t - hC_{t-1}} - \beta h \, \mathbb{E}_t \, \frac{1}{C_{t+1} - hC_t} \tag{69}$$

$$\Lambda_t = \beta \, \mathbb{E}_t \, \Lambda_{t+1} R^d_t \Pi^{-1}_{t+1} \tag{70}$$

$$p^k_t \Lambda_t M_t = \beta \, \mathbb{E}_t \, \Lambda_{t+1} \left[R_{t+1} + (1-\delta) p^k_{t+1} M_{t+1} \right] \tag{71}$$

$$\Lambda_t Q_t M_t = \beta \, \mathbb{E}_t \, \Lambda_{t+1} \Pi^{-1}_{t+1} \left[1 + \kappa Q_{t+1} M_{t+1} \right] \tag{72}$$

$$K_{t+1} = \widehat{I}_t + (1-\delta) K_t \tag{73}$$

$$p^k_t \widehat{I}_t = Q_t \left(f_t - \kappa \Pi^{-1}_t f_{t-1} \right) \tag{74}$$

- Household (wage-setting):

$$(w^{\#}_t)^{1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{f_{1,t}}{f_{2,t}} \tag{75}$$

$$f_{1,t} = w_t^{\epsilon_w(1+\eta)} B H_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w(1+\eta)} f_{1,t+1} \tag{76}$$

$$f_{2,t} = \Lambda_t w_t^{\epsilon_w} H_t + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w - 1} f_{2,t+1} \tag{77}$$

- Production firm:

$$w_t = mc_t (1-\alpha) A_t \left(\frac{K_t}{H_t} \right)^{\alpha} \tag{78}$$

$$R_t = mc_t \alpha A_t \left(\frac{K_t}{H_t} \right)^{\alpha - 1} \tag{79}$$

$$\Pi^{\#}_t = \frac{\epsilon_p}{\epsilon_p - 1} \frac{x_{1,t}}{x_{2,t}} \tag{80}$$

$$x_{1,t} = \Lambda_t mc_t Y_t + \theta_p \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_p} x_{1,t+1} \tag{81}$$

$$x_{2,t} = \Lambda_t Y_t + \theta_p \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_p - 1} x_{2,t+1} \tag{82}$$

- New Capital producer:

$$\widehat{I}_t = \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) \right] I_t \tag{83}$$

$$p^k_t \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}} \right) - S'\left(\frac{I_t}{I_{t-1}} \right) \frac{I_t}{I_{t-1}} \right] = 1 - \beta \, \mathbb{E}_t \, \frac{\Lambda_{t+1}}{\Lambda_t} p^k_{t+1} \mu_{t+1} S'\left(\frac{I_{t+1}}{I_t} \right) \left(\frac{I_{t+1}}{I_t} \right)^2 \tag{84}$$

- Financial intermediary:

$$Q_t f_t + Q_t b_t = L_t N_t \tag{85}$$

$$\Lambda_t \left[1 + f(N_t) + f'(N_t) N_t\right] = \beta\zeta\, \mathbb{E}_t\, \Lambda_{t+1} \Pi_{t+1}^{-1} \left[\left(R_{t+1}^L - R_t^d\right) L_t + R_t^d\right] \tag{86}$$

$$L_t = \frac{\mathbb{E}_t\, \Lambda_{t+1} \Pi_{t+1}^{-1}}{\mathbb{E}_t\, \Lambda_{t+1} \Pi_{t+1}^{-1} + (\Phi_t - 1)\, \mathbb{E}_t\, \Lambda_{t+1} \Pi_{t+1}^{-1} \frac{R_{t+1}^L}{R_t^d}} \tag{87}$$

- Monetary Policy:

$$\ln R_t^d = (1-\rho_R)\ln R^d + \rho_R \ln R_{t-1}^d + (1-\rho_R)\left[\tau_\pi(\ln \Pi_t - \ln \Pi) + \tau_y(\ln Y_t - \ln Y_{t-1})\right] + s_R \varepsilon_{R,t} \tag{88}$$

- Aggregate Conditions:

$$Y_t = C_t + I_t + f(N_t) N_t \tag{89}$$

$$1 = (1-\theta_p)\left(\Pi_t^{\#}\right)^{1-\epsilon_p} + \theta_p \Pi_t^{\epsilon_p - 1} \tag{90}$$

$$w_t^{1-\epsilon_w} = (1-\theta_w)(w_t^{\#})^{1-\epsilon_w} + \theta_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{91}$$

$$d_t^p Y_t = A_t K_t^\alpha H_t^{1-\alpha} \tag{92}$$

$$d_t^p = (1-\theta_p)(\Pi_t^{\#})^{-\epsilon_p} + \theta_p \Pi_t^{\epsilon_p} d_{t-1}^p \tag{93}$$

$$\ln b_t = (1-\rho_{1,b}-\rho_{2,b})\ln b + \rho_{1,b}\ln b_{t-1} + \rho_{2,b}\ln b_{t-2} + s_b \varepsilon_{b,t} \tag{94}$$

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{95}$$

$$\ln \mu_t = \rho_\mu \ln \mu_{t-1} + s_\mu \varepsilon_{\mu,t} \tag{96}$$

$$\ln \Phi_t = (1-\rho_\Phi)\ln \Phi + \rho_\Phi \ln \Phi_{t-1} + s_\Phi \varepsilon_{\Phi,t} \tag{97}$$

- Bond returns and yields:

$$R_t^L = \frac{1 + \kappa Q_t}{Q_{t-1}} \tag{98}$$

$$R_{y,t} = Q_t^{-1} + \kappa \tag{99}$$

$$Q_t^{EH} = \frac{1 + \kappa\, \mathbb{E}_t\, Q_{t+1}^{EH}}{R_t^d} \tag{100}$$

$$R_{y,t}^{EH} = \left(Q_t^{EH}\right)^{-1} + \kappa \tag{101}$$

$$TP_t = \frac{R_{y,t}}{R_{y,t}^{EH}} \tag{102}$$

This is 34 equations and $\left\{C_t, H_t, I_t, \widehat{I}_t, K_t, Y_t, mc_t, N_t, R_t^d, R_t, Q_t, p_t^k, w_t, w_t^{\#}, \Pi_t, \Pi_t^{\#}, f_t, b_t, f_{1,t}, f_{2,t}, x_{1,t}, x_{2,t}, L_t, d_t^p, A_t, \mu_t, \Phi_t, R_t^L, R_{y,t}, Q_t^{EH}, R_{y,t}^{EH}, TP_t, \Lambda_t, M_t\right\}$ and 34 variables.

Note that it is useful to re-write the wage-setting equations in a slightly different form. This is because the exponent on the LHS of (75) gets very big. Define $\widehat{f}_{1,t} = f_{1,t}/(w_t^{\#})^{\epsilon_w(1+\eta)}$. We get:

$$\widehat{f}_{1,t} = \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} BH_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon(1+\eta)} \frac{f_{1,t+1}}{(w_t^{\#})^{\epsilon_w(1+\eta)}}$$

Or:

$$\widehat{f}_{1,t} = \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} BH_t^{1+\eta} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon(1+\eta)} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w(1+\eta)} \widehat{f}_{1,t+1} \tag{103}$$

Similarly, define $\widehat{f}_{2,t} = f_{2,t}/(w_t^{\#})^{\epsilon_w}$. We get:

$$\widehat{f}_{2,t} = \Lambda_t H_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w - 1} \frac{f_{2,t+1}}{(w_t^{\#})^{\epsilon_w}}$$

Or:

$$\widehat{f}_{2,t} = \Lambda_t H_t \left(\frac{w_t}{w_t^{\#}}\right)^{\epsilon_w} + \theta_w \beta \, \mathbb{E}_t \, \Pi_{t+1}^{\epsilon_w - 1} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\epsilon_w} \widehat{f}_{2,t+1} \tag{104}$$

Now, since $\frac{f_{1,t}}{f_{2,t}} = \frac{\widehat{f}_{1,t}}{\widehat{f}_{2,t}} (w_t^{\#})^{\epsilon_w \eta}$, we can write the reset wage condition as (75) as:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\widehat{f}_{1,t}}{\widehat{f}_{2,t}} \tag{105}$$

We can simply replace (75)-(77) with (105) and (103)-(104). Both sets of equations are correct. The former is less prone to numerical issues.

10 Steady State

As noted above, we have $A = \mu = 1$ in steady state. Let us implicitly pick B so that $H = 1$ in steady state as well. We will focus on a zero inflation steady state, which means $\Pi = 1$ in gross terms, and hence $\Pi^{\#} = v^p = 1$. We also have $p^k = 1$ from the investment FOC, and hence $\widehat{I} = I$.

Much of the rest of the steady state calculation will proceed by picking some financial targets. From the household's Euler equation, (70) we have in steady state that:

$$R^d = \beta^{-1} \tag{106}$$

But then we can solve for the steady state expectations hypothesis bond price as:

$$\frac{1}{\beta} = \frac{1}{Q^{EH}} + \kappa$$

So:

$$Q^{EH} = \left(\frac{1}{\beta} - \kappa\right)^{-1} \tag{107}$$

But then this gives us the steady state expectations hypothesis yield, which is the same as the steady state deposit rate:

$$R_y^{EH} = \beta^{-1} \tag{108}$$

Now, let's target a term premium of 100 basis points annualized, so $TP = 1.0025$ at a quarterly frequency. This then allows us to solve for the yield on the long bond as:

$$R_y = \beta^{-1} TP \tag{109}$$

We then see that the steady state long bond return is equal to the steady state yield on the long bond, $R^L = R_y$. From (99) we can solve for the steady state long bond price:

$$Q = \left(\beta^{-1} TP - \kappa\right)^{-1} \tag{110}$$

Now go to (86). Since $f(N) = f'(N) = 0$, this can be written:

$$1 = \beta\zeta R^d \left[\left(\frac{R^L}{R^d} - 1\right) L + 1\right]$$

Since $R^d = \beta^{-1}$ and the ratio of the return to the deposit rate equals the term premium in the steady state, this is:

$$1 = \zeta\left(TP - 1\right) L + \zeta$$

Take L as given; they target $L = 6$. This implies a restriction on ζ:

$$\zeta = [(TP - 1)L + 1]^{-1} \tag{111}$$

Note that if $TP = 1$, we would have $\zeta = 1$. So it is $\zeta < 1$ that is generating a term premium / excess bond return in the steady state. In effect, if the FI were as patient as the household, it would generate enough net worth to arbitrage away the excess bond return in steady state.

Now that we know Q, we can figure out M from (72):

$$QM = \beta(1 + \kappa QM)$$

Which implies:

$$M = \frac{\beta}{1-\beta\kappa}\frac{1}{Q} \tag{112}$$

Now stop and not something. Suppose that the term premium was 1 in steady state, so that $Q = \left(\beta^{-1} - \kappa\right)^{-1}$. Then we'd have:

$$M = \frac{\beta(\beta^{-1} - \kappa)}{1-\beta\kappa} = 1$$

This is useful to point out because we defined $M = 1 + \vartheta/\Lambda$, where ϑ is the multiplier on the "loan in advance" constraint. With no term premium, this constraint is not binding, i.e. $\vartheta = 0$. With a term premium, the constraint is binding – given the bond premium, the household would not like to issue debt to finance investment, but it is required to, which makes it worse off.

Now, let's divert to price-setting conditions. From (80)-(82), we see that:

$$mc = \frac{\epsilon_p - 1}{\epsilon_p} \tag{113}$$

Why is this useful? From (71), we can solve for the steady state rental rate on capital know that we know M:

$$M = \beta\left[R + (1-\delta)M\right]$$

So:

$$R = M\left[\frac{1}{\beta} - (1-\delta)\right] \tag{114}$$

Note that $M > 1$ distorts the standard expression for the rental rate in steady state (given in the brackets). But then we can use the capital demand condition, (79), to solve for K (assuming $H = 1$):

$$K = \left(\frac{\alpha mc}{R}\right)^{\frac{1}{1-\alpha}} \tag{115}$$

Note that this is distorted for two reasons: $mc < 1$ (the standard monopolistic competition distortion) and R being too high because $M > 1$. This in turn gives us the wage as well as Y (since $d^p = 1$ in a zero inflation steady state):

$$w = mc(1-\alpha)K^(\alpha) \tag{116}$$

$$Y = K^{\alpha} \tag{117}$$

Since the adjustment function $S(1) = 0$ and $\mu = 1$, we have $\widehat{I} = I = \delta K$. This then gives us steady state consumption:

$$C = K^{\alpha} - \delta K \tag{118}$$

Given C, we can solve for Λ from (69):

$$\Lambda = \frac{1 - h\beta}{C(1-h)} \tag{119}$$

With zero steady state inflation and no trend growth, the optimal reset wage equals the steady state real wage. Now turn to wage-setting. We have:

$$f_1 = \frac{w^{\epsilon_w(1+\eta)} B H^{1+\eta}}{1 - \theta_w \beta}$$

$$f_2 = \frac{\Lambda w^{\epsilon_w} H}{1 - \theta_w \beta}$$

Hence:

$$\frac{f_1}{f_2} = \Lambda^{-1} w^{\epsilon_w \eta} B H^{\eta}$$

Hence, making use of the fact that $w = w^{\#}$, we have:

$$\frac{\epsilon_w - 1}{\epsilon_w} \Lambda w = B H^{\eta}$$

This looks almost like the standard efficiency condition for labor supply in a standard model, which would be $BH^{\eta} = \Lambda w$. It is distorted by the wage markup. Given our target of $H = 1$, we can then solve for the requisite value of B:

$$B = \frac{\epsilon_w - 1}{\epsilon_w} \Lambda w \tag{120}$$

From the loan in advance constraint, (74), we can solve for steady state real investment bonds:

$$f = \frac{I}{Q(1-\kappa)} \tag{121}$$

But given a steady state value of b (which is assumed), and a target for L, we can then solve for net worth in steady state from (85):

$$N = \frac{Qf + Qb}{L} \tag{122}$$

Lastly, we need to calculate the steady state value of Φ to be consistent with $L = 6$. We can write (87) in the steady state as:

$$\frac{\Lambda + (\Phi - 1)\Lambda TP}{\Lambda} = \frac{1}{L}$$

Or:

$$1 + (\Phi - 1)TP = \frac{1}{L}$$

Which implies:

$$\Phi = 1 - \frac{L-1}{L}\frac{1}{TP} \tag{123}$$

11 Log-Linearization

To gain intuition, it is useful to log-linearize some of the model. The authors of the paper do the whole model, but I will only do parts.

Let's start with (87), the condition determining leverage. This is easier to write as:

$$\frac{\mathbb{E}_t\,\Lambda_{t+1}\Pi_{t+1}^{-1} + (\Phi_t - 1)\,\mathbb{E}_t\,\Lambda_{t+1}\Pi_{t+1}^{-1}\frac{R_{t+1}^L}{R_t^d}}{\mathbb{E}_t\,\Lambda_{t+1}\Pi_{t+1}^{-1}} = \frac{1}{L_t}$$

To a first order approximation we can ignore things we usually can't do with expectations operators. In particular, we can ignore the expectation operators greatly simplify the left hand side:

$$1 + (\Phi_t - 1)\frac{R_{t+1}^L}{R_t^d} = \frac{1}{L_t}$$

Take logs:

$$\ln\left[1 + (\Phi_t - 1)\frac{R_{t+1}^L}{R_t^d}\right] = -\ln L_t$$

Now, totally differentiate about the steady state:

$$-\frac{dL_t}{L} = L\left[(\Phi - 1)\frac{1}{R^d}dR_{t+1}^L - (\Phi - 1)\frac{R^L}{R^d}\frac{dR_t^d}{R^d} + d\Phi_t\frac{R^L}{R^d}\right]$$

Use lowercase or hatted variables to denote percentage deviations. We have:

$$l_t = -L\left[(\Phi - 1)\,\frac{R^L}{R^d}(\mathbb{E}_t\,r_{t+1}^L - r_t^d) + \Phi\frac{R^L}{R^d}\phi_t\right]$$

Note from above, we know that $\frac{R^L}{R^d} = TP$ and $(\Phi - 1)TP = \frac{1-L}{L}$. Using this, we have:

$$l_t = -(1-L)(\mathbb{E}_t\,r_{t+1}^L - r_t^d) - \Phi TPL\phi_t$$

We can then write:

$$\mathbb{E}_t\,r_{t+1}^L - r_t^d = \frac{1}{L-1}l_t + \frac{\Phi TPL}{L-1}\phi_t$$

From above, note that $\Phi TPL = L(TP - 1) + 1$. So we can write this as:

$$\mathbb{E}_t\,r_{t+1}^L - r_t^d = \frac{1}{L-1}l_t + \frac{1 + L(TP-1)}{L-1}\phi_t \tag{124}$$

(124) is the same as (25) in their paper. This is pretty cool, because it's basically exactly the same as Bernanke, Gertler, and Gilchrist (1999) – the lending spread is increasing in leverage, l_t. There there is a positive term related to ϕ_t – when ϕ_t increases, credit conditions tighten, which moves the spread up independently of leverage.

Now go to (71), the first order condition for capital. Take logs, and ignore expectations operators until we are finished.

$$\ln p_t^k + \ln \Lambda_t + \ln M_t = \ln \beta + \ln \Lambda_{t+1} + \ln \left[R_{t+1} + (1-\delta) p_{t+1}^k M_{t+1} \right]$$

Totally differentiate about the steady state:

$$\frac{dp_t^k}{p^k} + \frac{d\Lambda_t}{d\Lambda} + \frac{dM_t}{M} = \frac{d\Lambda_{t+1}}{\Lambda} + \frac{1}{R + (1-\delta)p^k M} \left[dR_{t+1} + (1-\delta)p^k dM_{t+1} + (1-\delta) M dp_{t+1}^k \right]$$

and note that $p^k = 1$. We have:

$$\widehat{p}_t^k + \lambda_t + m_t = \mathbb{E}_t \widehat{\lambda}_{t+1} + \frac{R}{R+(1-\delta)M} \mathbb{E}_t r_{t+1} + \frac{(1-\delta)M}{R+(1-\delta)M} \left(\mathbb{E}_t m_{t+1} + \mathbb{E}_t p_{t+1}^k \right)$$

Note that from above that $R = M \left[\beta^{-1} - (1-\delta) \right]$ and $M = \beta \left[R + (1-\delta)M \right]$. This means we can write $(R + (1-\delta)M)^{-1} = \beta / M$. So we get:

$$\widehat{p}_t^k + \lambda_t + m_t = \mathbb{E}_t \widehat{\lambda}_{t+1} + \frac{R\beta}{M} \mathbb{E}_t r_{t+1} + (1-\delta)\beta \left(\mathbb{E}_t m_{t+1} + \mathbb{E}_t p_{t+1}^k \right)$$

But then we can go further since $\frac{R\beta}{M} = 1 - (1-\delta)\beta$:

$$\widehat{p}_t^k + \lambda_t + m_t = \mathbb{E}_t \widehat{\lambda}_{t+1} + (1 - (1-\delta)\beta) \mathbb{E}_t r_{t+1} + (1-\delta)\beta \left(\mathbb{E}_t m_{t+1} + \mathbb{E}_t p_{t+1}^k \right) \tag{125}$$

This is the same as (41) in their paper. The important point here is that m_t distorts this linearized condition relative to what we would ordinarily see. If we linearized the Euler equation for deposits, we would get:

$$\lambda_t = \mathbb{E}_t \lambda_{t+1} + r_t^d - \mathbb{E}_t \pi_{t+1}$$

Plug this into (125):

$$\widehat{p}_t^k + m_t = -r_t^d + \mathbb{E}_t \pi_{t+1} + (1 - (1-\delta)\beta) \mathbb{E}_t r_{t+1} + (1-\delta)\beta \left(\mathbb{E}_t m_{t+1} + \mathbb{E}_t p_{t+1}^k \right) \tag{126}$$

Let's solve (126) forward one period:

$$\widehat{p}_t^k + m_t = -r_t^d + \mathbb{E}_t\, \pi_{t+1} + (1-(1-\delta)\beta)\, \mathbb{E}_t\, r_{t+1} +$$
$$(1-\delta)\beta\, \mathbb{E}_t \left[-r_{t+1}^d + \pi_{t+2} + (1-(1-\delta)\beta)\, r_{t+2} + (1-\delta)\beta \left(m_{t+2} + \mathbb{E}_t\, p_{t+2}^k \right) \right]$$

Going forward further and imposing a terminal condition that the economy returns to steady state, we can see that this satisfies:

$$p_t^k + m_t = \sum_{j=0}^{\infty} [\beta(1-\delta)]^j\, \mathbb{E}_t \left[(1-\beta(1-\delta))\, r_{t+j+1} - \left(r_{t+j}^d - \pi_{t+j+1} \right) \right] \tag{127}$$

(127) is the same as (56) in the paper, though they have the time subscript off on the return to physical capital. We can see that m_t is functionally like a tax on the price of new capital goods. Let's go further to see what m_t is. Log-linearize the pricing condition for the long-bond, (72). Take logs ignoring expectations operators:

$$\ln \Lambda_t + \ln Q_t + \ln M_t = \ln \beta + \ln \Lambda_{t+1} - \ln \Pi_{t+1} + \ln(1 + \kappa Q_{t+1} M_{t+1})$$

Totally differentiate:

$$\lambda_t + q_t + m_t = \mathbb{E}_t\, \lambda_{t+1} - \mathbb{E}_t\, \pi_{t+1} + \frac{1}{1+\kappa QM} \left(\kappa M dQ_{t+1} + \kappa Q dM_{t+1} \right)$$

So:

$$\lambda_t + q_t + m_t = \mathbb{E}_t\, \lambda_{t+1} - \mathbb{E}_t\, \pi_{t+1} + \frac{\kappa QM}{1+\kappa QM} \left(\mathbb{E}_t\, q_{t+1} + \mathbb{E}_t\, m_{t+1} \right)$$

From the steady state solution, we know that $\frac{QM}{1+\kappa QM} = \beta$. Hence, we have:

$$\lambda_t + q_t + m_t = \mathbb{E}_t\, \lambda_{t+1} - \mathbb{E}_t\, \pi_{t+1} + \kappa\beta \left(\mathbb{E}_t\, q_{t+1} + \mathbb{E}_t\, m_{t+1} \right) \tag{128}$$

(128) is the same as (42) in the paper. Now again use the linearized Euler equation for deposits for the household to eliminate λ_t and λ_{t+1}:

$$q_t + m_t = -r_t^d + \kappa\beta\, \mathbb{E}_t\, q_{t+1} + \kappa\beta\, \mathbb{E}_t\, m_{t+1}$$

Which can be written:

$$m_t = \kappa\beta\, \mathbb{E}_t\, q_{t+1} - q_t - r_t^d + \kappa\beta\, \mathbb{E}_t\, m_{t+1}$$

This can be solved forward:

$$m_t = \mathbb{E}_t \sum_{j=0}^{\infty} (\kappa\beta)^j\, \mathbb{E}_t \left(\kappa\beta q_{t+j+1} - q_{t+j} - r_{t+j}^d \right) \tag{129}$$

(129) is (57) in the paper.

Now, let's linearize the bond pricing conditions. Start with the pricing condition for the hypothetical expectations hypothesis bond. We have:

$$\begin{aligned}
\ln Q_t^{EH} &= \ln(1+\kappa Q_{t+1}^{EH}) - \ln R_t^d \\
q_t^{EH} &= \frac{1}{1+\kappa Q^{EH}} \kappa d Q_{t+1}^{EH} - r_t^d \\
q_t^{EH} &= \frac{Q^{EH}}{1+\kappa Q^{EH}} \kappa q_{t+1}^{EH} - r_t^d
\end{aligned}$$

But $\frac{Q^{EH}}{1+\kappa Q^{EH}} = 1/R^d = \beta$. So we have:

$$q_t^{EH} = \beta\kappa\, \mathbb{E}_t\, q_{t+1}^{EH} - r_t^d \tag{130}$$

(130) is the same as (59) in their paper. Now linearize the expression for the yield to maturity on the expectations hypothesis bond:

$$\begin{aligned}
\ln R_{y,t}^{EH} &= \ln\left[\left(Q_t^{EH}\right)^{-1} + \kappa\right] \\
r_{y,t}^{EH} &= \frac{-(Q^{EH})^{-2}}{R_y^{EH}} dQ_t^{EH} \\
r_{y,t}^{EH} &= \frac{-1}{Q^{EH} R_y^{EH}} q_t^{EH}
\end{aligned}$$

But from the steady state analysis, this is:

$$r_{y,t}^{EH} = -(1-\kappa\beta) q_t^{EH} \tag{131}$$

(131) is the same as (60) in the paper.

Now, let's linearize the bond pricing conditions for the long bond. We have:

$$\begin{aligned}
\ln R_t^L &= \ln(1+\kappa Q_t) - \ln Q_{t-1} \\
r_t^L &= \frac{\kappa d Q_t}{1+\kappa Q} - q_{t-1} \\
r_t^L &= \frac{\kappa Q}{1+\kappa Q} q_t - q_{t-1}
\end{aligned}$$

But from the steady state expressions, this is:

$$r_t^L = \frac{\kappa\beta}{TP} q_t - q_{t-1} \tag{132}$$

The expression for the yield on the long bond is the same as (131), but the steady state stuff is different due to the steady state term premium. We have:

$$r_{y,t} = \frac{-1}{QR_y} q_t$$

Which works out to:

$$r_{y,t}^{EH} = -(1 - \beta\kappa/TP) q_t \tag{133}$$

Then the term premium is:

$$tp_t = -(1 - \kappa\beta/TP) q_t + (1 - \kappa\beta) q_t^{EH} \tag{134}$$

Note that we can solve forward for the bond prices using (132) and (130):

$$q_t^{EH} = -\sum_{j=0}^{\infty} (\beta\kappa)^j r_{t+j}^d \tag{135}$$

$$q_t = -\sum_{j=0}^{\infty} \left(\frac{\beta\kappa}{TP}\right)^j r_{t+1+j}^L \tag{136}$$

But then we can write the term premium as:

$$tp_t = (1 - \kappa\beta/TP) \sum_{j=0}^{\infty} \left(\frac{\beta\kappa}{TP}\right)^j r_{t+1+j}^L - (1 - \kappa\beta) \sum_{j=0}^{\infty} (\beta\kappa)^j r_{t+j}^d \tag{137}$$

(138) is (62) in the paper. Since TP is close to 1 (1.0025 in the baseline parameterization), we can ignore these small differences in what appears above and approximate the term premium as:

$$tp_t \approx (1 - \beta\kappa) \sum_{j=0}^{\infty} (\beta\kappa)^j \, \mathbb{E}_t (r_{t+1+j}^L - r_{t+j}^d) \tag{138}$$

In other words, (138) tells us that the term premium is approximately proportional to the present discounted value of excess returns (the difference between the returns on the long bond and the safe short rate). But then looking at (132), we se that $\mathbb{E}_t \left[\kappa\beta q_{t+j+1} - q_{t+j}\right] \approx \mathbb{E}_t \, r_{t+1+j}^L$. But then go back to (129), and we see;

$$m_t \approx \mathbb{E}_t \sum_{j=0}^{\infty} (\kappa\beta)^j \, \mathbb{E}_t (r_{t+j+1}^L - r_{t+j}^d) \approx \frac{tp_t}{1 - \beta\kappa} \tag{139}$$

In other words, the investment wedge in the capital demand condition, (127), is approximately proportional to the term premium. This is sort of the central insight of the paper (which admittedly took a long while to get to). QE policies will be able to influence the term premium, and hence this wedge. This is the mechanism through which QE "works" in the model. Furthermore, this framework has the pretty clear implication that optimal policy would like to eliminate the investment wedge, and hence eliminate the term premium. This has implications for optimal balance sheet policies.

12 Impulse Responses

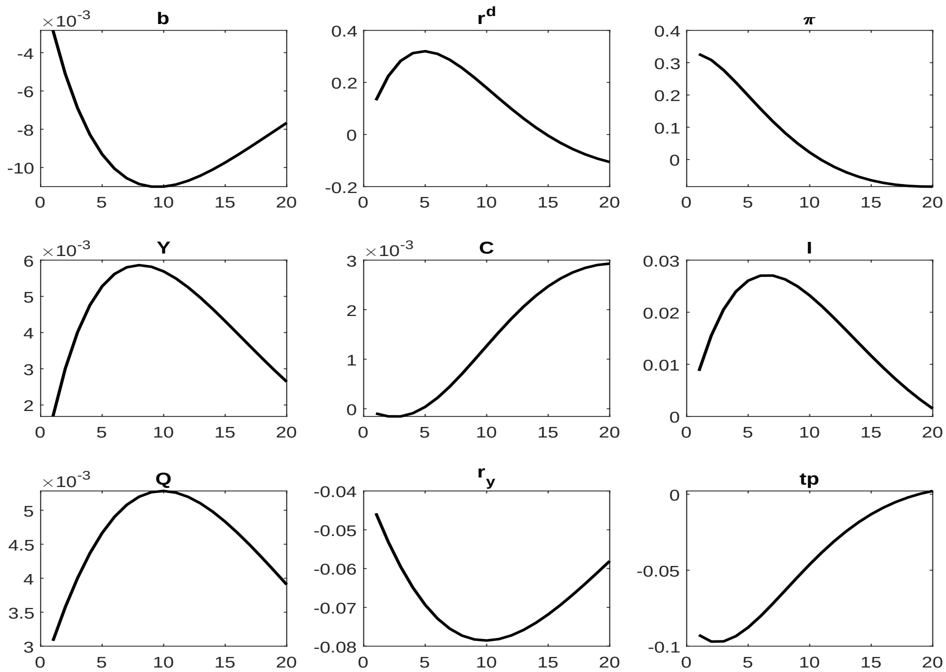
I calibrate the model and solve it to generate impulse responses. They estimate the model, but there are some weird issues with the estimation – such as the extremely high wage rigidity parameter (they estimate $\theta_w = 0.97$). I set $\beta = 0.99$ and pick $\kappa = 1 - 40^{-1}$; this corresponds to a duration of long bonds of 10 years (40 quarters). I set $\epsilon_p = \epsilon_w = 11$, $\alpha = 1/3$, and $\delta = 0.025$. I set $\theta_w = \theta_p = 0.75$. The parameters of the Taylor rule are $\rho_R = 0.8$, $\tau_\pi = 1.5$, and $\tau_y = 0.25$. I set $\eta = 1$ for a Frisch elasticity of unity and set the habit formation parameter to $h = 0.8$. The parameter B is chosen so as to normalize $H = 1$ in the steady state.

Turning to financial parameters, I target a steady state term premium of 100 basis points annualized, or, in gross terms at a quarterly frequency, 1.0025. I target a steady state leverage ratio of 6. This calibration implies a value of $\zeta = 0.9852$. I set the net worth adjustment cost to $\psi_n = 2$. This is also what I set the capital adjustment cost to.

For the shocks, I set $\rho_A = 0.95$, $\rho_\Phi = 0.90$, and $\rho_\mu = 0.8$. The standard deviation of shocks are not interesting othe than to scale the impulse responses. I set the AR(2) parameters of the exogenous bond process following the authors, with $\rho_1 = 1.8$ and $\rho_2 = -0.81$.

Consider first the IRFs to a QE shock. As noted above, this is an exogenous *reduction* in the real quantity of bonds that FIs must hold. As written, this is really kind of a *fiscal shock* – it's like the government reducing its debt issuance. The authors actually don't model what QE was in practice – which involved the monetary authority creating reserves to purchase government bonds from FIs. At the end of the day, in the model the FIs are restricted in terms of how many private investment bonds they can hold. Reducing government bonds (either via a fiscal intervention or the central bank buying them, effectively frees up space on FI balance sheets to buy more investment bonds. This will result in reducing the investment wedge friction and will be expansionary.

Figure 1: IRFs to QE Shock

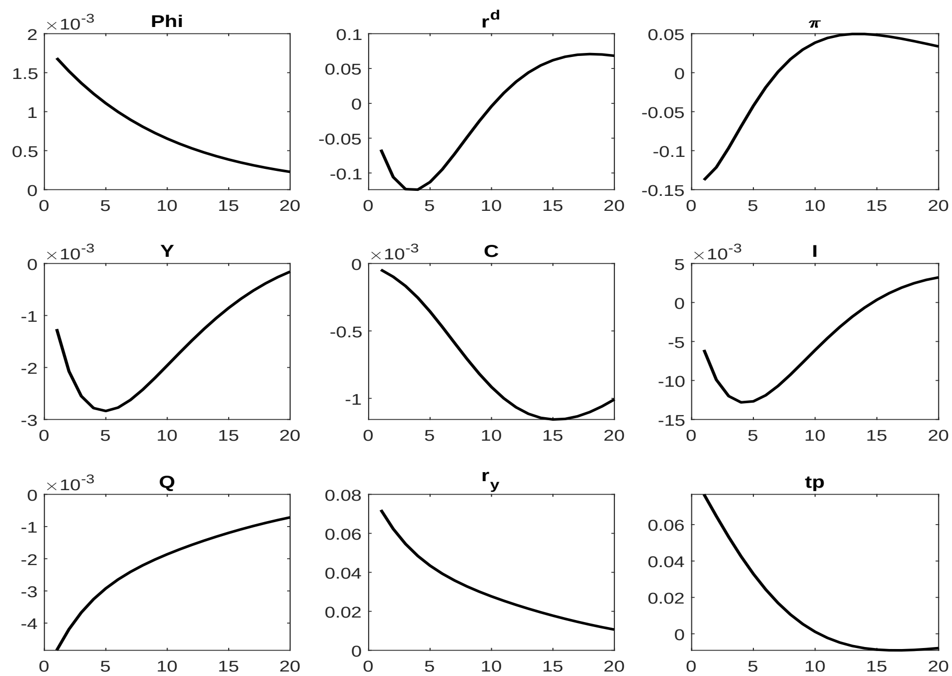


The QE shock involves a very persistent and hump-shaped reduction in government bonds held by FIs (by construction). This results in an increase in long bond prices, a reduction in yield, and a reduction in the term premium. As a result, output and investment go up. Inflation increases. This causes the central bank to raise the short-term rate via the Taylor rule. Consumption initially slightly decreases, but it ultimately increases, following the path of output. The initial decline in consumption makes sense. Because the QE intervention lowers the long bond yield and eases the investment wedge, this incentivizes the households to shift from consumption to investment. Then through general equilibrium effects (i.e. output increases) consumption gets indirectly stimulated. But the initial movement is a slight decline in consumption.

To gather intuition for what is going on, it is useful to look at the linearized model, in particular (124). In the instant b_t goes down, holding all other prices fixed, l_t (linearized leverage) goes down – firm net worth is the same, but now FIs are holding fewer total assets. Temporarily lower leverage puts downward-pressure on the long-short spread – basically, as I hinted at above, holding fewer government bonds allows FIs to buy more private investment bonds. This pushes up their price and hence pushes down the yield.

Then how does this transmit to the real economy? Well, look at (138). Lower long-short spreads puts downward pressure on the term premium. Then from (139), we see that the term premium is close to proportional to the investment wedge in the Euler equation for capital (in fact, numerically, the correlation between the tp_t and m_t in my simulations is 0.999). But a lower term premium, and hence a lower m_t, is like a lower tax on investment. This stimulates private investment demand, which in turn raises aggregate demand. Because of price and wage stickiness, this increase in demand results in output and inflation rising.

Figure 2: IRFs to Credit Shock



Now let's look at the IRFs to a credit shock (exogenous increase in Φ). These are shown above. The credit shock is *basically* the inverse of the QE shock. When Φ goes up, FIs have to cut back on their purchases of private investment bonds because they are not allowed to have as much leverage – a higher Φ is exacerbating the holdup problem facing the FI. This causes long bond prices to fall and the yield to rise, along with it the term premium. This exacerbates the investment wedge, which leads to a reduction in aggregate demand, resulting in a decline in inflation. This is met by the central bank lowering the policy rate. At the end of the day, the effects of this shock are basically the mirror image of the QE shock (not exactly as shown above because the AR processes are different). But this fact will be useful when thinking about optimal policy.

The remaining figures show impulse responses to other shocks. These are close but not exact to what the authors report – again, there are some parameterization differences (e.g. the wage stickiness parameter, the AR parameter on the productivity shock). We can see how the financial friction matters by looking at the response of the term premium. So, for example, the rising term premium *dampens* the responses to the productivity and MEI shocks. It also dampens the contractionary effects to the conventional monetary policy shock.

Figure 3: IRFs to Productivity Shock

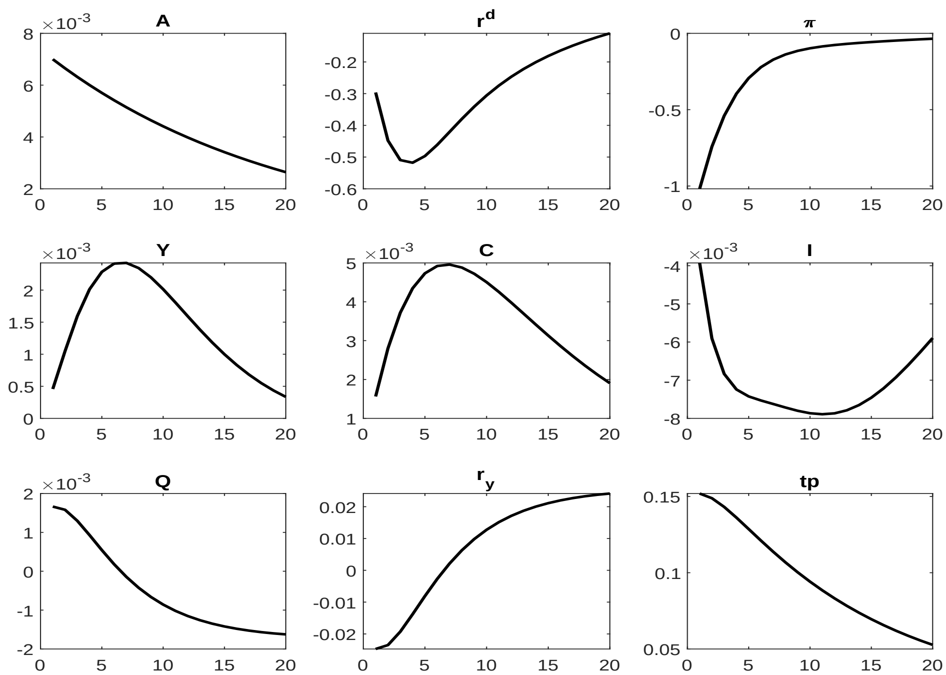


Figure 4: IRFs to MEI Shock

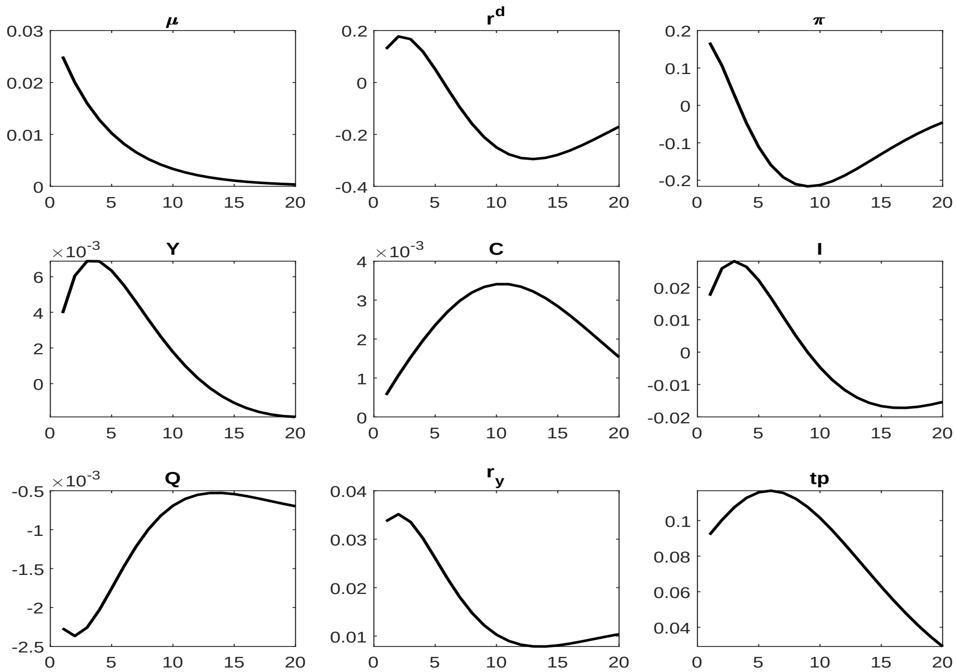
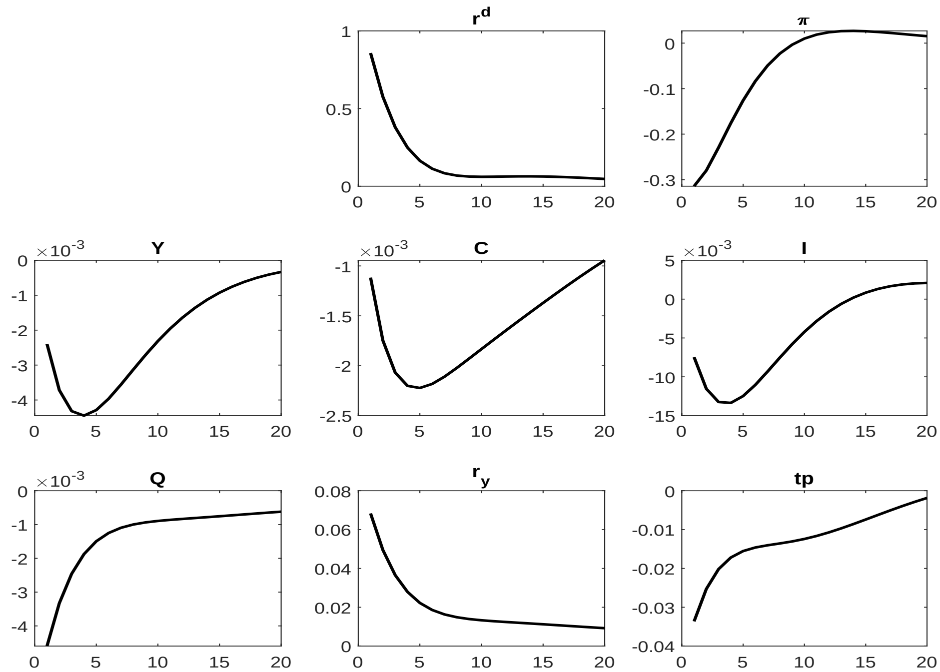


Figure 5: IRFs to Monetary Policy Shock



Before moving on, it is useful to note the role of the net worth adjustment cost, captured by the parameter ψ_n. When I set $\psi_n = 0$, the QE shock is completely irrelevant. Basically, going back to the chain of logic discussed above, if there is no cost to adjusting net worth, net worth just adjusts the reduction in government bonds in such way as to leave leverage unaffected. This results in no addition bond buying by the FIs, and hence no economic affects. This can be seen quite easily by looking at (47) in their paper, which is a log-linearized expression relating net worth to the interest rate spread and leverage. When $\psi_n = 0$, leverage is collinear with the lending spread. But then plugging that into (124), you get that the spread is only affected by the credit shock, and nothing else. So QE is completely irrelevant without the net worth adjustment cost in this model.

Credit shocks still do have small effects even without the net worth adjustment cost – but they are just that, small. This is similar to the results in Jermann and Quadrini (2012) about the effects of credit shocks without their dividend adjustment cost.

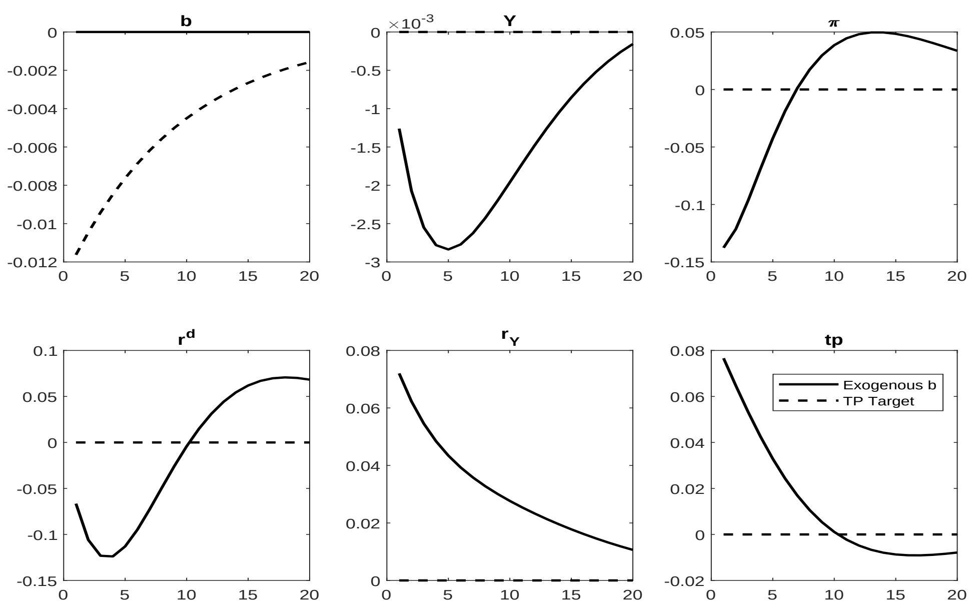
12.1 Endogenous Balance Sheet Policy: Targeting the Term Premium

There is an interesting policy implication related to the amount of government bonds in circulation that pops out of the model. This is the following. Since QE and credit shocks are roughly the mirror image of one another, one ought to be able to use QE to offset credit shocks one-for-one. It turns out you can do exactly this in the model.

To see this, you can replace the exogenous process for government bonds with a term premium targeting rule, $TP_t = TP$ (i.e. target the gross term premium at the steady state value). This will cause the amount of government bonds held by FIs to endogenously react to shocks so as to keep the term premium fixed. Based on our discussion above, since the term premium is essentially the investment wedge, this makes sense. Optimal policy is all about undoing wedges relative to an efficient allocation.

First, consider the responses to a credit shock under a term premium target, shown in the figure below. To counter a contractionary credit shock (i.e. an increase in Φ), the central bank / fiscal authority needs to reduce the amount of government bonds being held by intermediaries. We can see this in the dashed line of the figure, which plots responses under a term premium target (the solid lines show responses under the exogenous debt rule). We see here that a term premium target *completely neutralizes the effects of a credit shock.*

Figure 6: IRFs to Credit Shock, Term Premium Target



Below I also show responses under a term premium target for the other shocks. In response to each of these shocks, government bonds adjust to keep the term premium fixed. Since the term premium *dampens* the responses to these shocks under the exogenous debt rule considered above, the term premium target results in output and other variables reacting more to these shocks. And from an efficiency perspective, that is good in this model.

Figure 7: IRFs to Productivity Shock, Term Premium Target

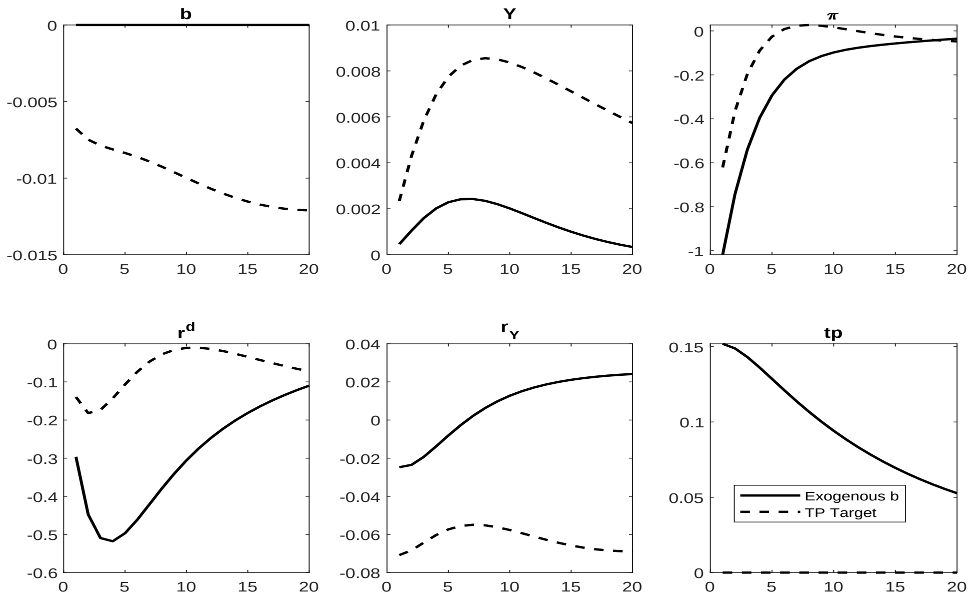


Figure 8: IRFs to MEI Shock, Term Premium Target

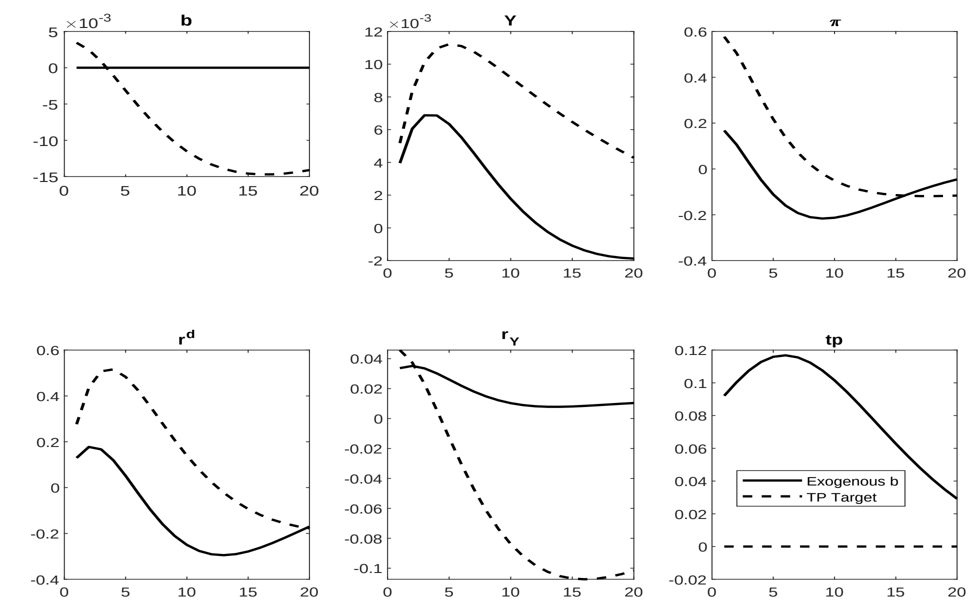
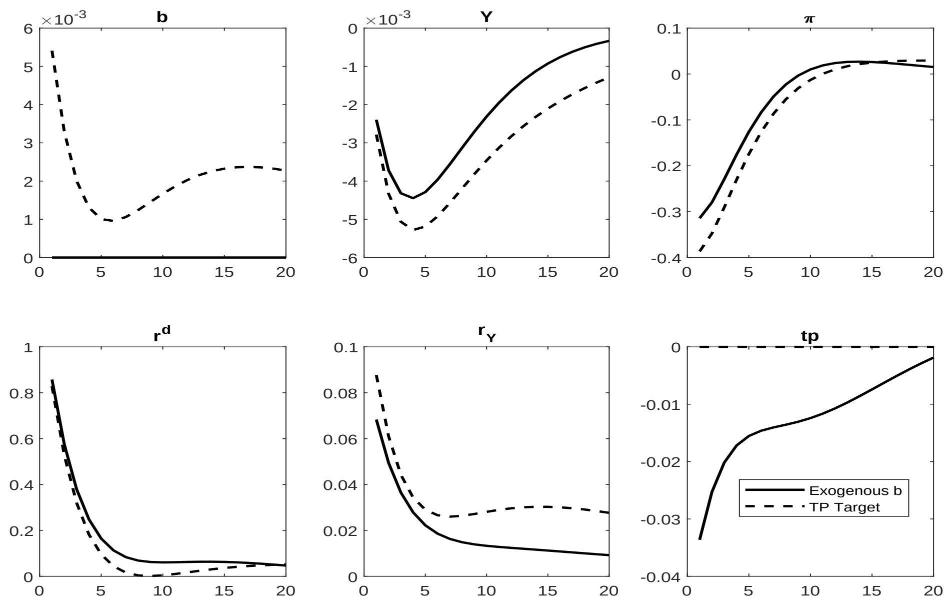


Figure 9: IRFs to Monetary Policy Shock, Term Premium Target



The conclusion here is that "balance sheet" policies are well-suited to "undo" frictions associated in credit markets. Hence, the normative implication is that balance sheet policies potentially ought to be used all the time in response to all shocks – not just at the ZLB as a substitute for conventional policy rate adjustment. This same point is made in a simpler model with similar core frictions by Sims and Wu (2019).