Graduate Macro Theory II:
Two Period Consumption-Saving Models

Eric Sims
University of Notre Dame

Spring 2017

1 Introduction

This note works through some simple two-period consumption-saving problems. In this model households receive an exogenous stream of income and have to decide how much to consume and save. Two periods is the minimum number of periods to make problems dynamic; it also turns out that you can usually split infinite horizon problems into two periods (the present and a future continuation value), so focusing on two periods does not cost us much. The basic consumption-saving problem is really at the core of modern dynamic macroeconomics, so it's important to understand it well.

Focusing on this simple setup allows us to study a couple of different phenomena. First, it allows us to think about competitive equilibrium in a simple framework. I like to call economies such as this "endowment economies" – since total income is exogenous (and hence fixed), there is no production, and in equilibrium total consumption must equal total income. Hence, equilibrium price adjustment will undo all of the consumption smoothing motives you've previously learned about – aggregate consumption will have to equal aggregate income, and prices will adjust to make this consistent with agent optimization. Second, the two period setup allows us to think about the role of heterogeneity and risk-sharing. If markets are "complete" in a sense to be defined below, agents can have stochastic idiosyncratic income draws but their consumption will be independent of their idisoyncratic income and will be perfectly correlated with the aggregate endowment. Such a setup helps motivate the use of representative agent assumptions in macroeconomics.

2 One Type of Agent

Suppose there are sufficiently many agents who have identical preferences and endowments. As long as the number of agents is sufficiently many such that agents behave as price-takers, we can normalize the number of agents to 1.

An agent wakes up in period t and expects to live for two periods, t and $t+1$. The agent receives an exogenous and deterministic endowment of income each period, Y_t and Y_{t+1}. The agent can

borrow or save in period t by buying/selling bonds, B_t. These bonds cost q_t units of consumption (which serves as the numeraire); B_t units of bonds brought into period $t+1$ pays out B_t units of income in period $t+1$. B_t can be positive or negative; a positive value means that the agent saves, a negative value means that the agent borrows. The agent begins time with no stock of bonds, and must die with no stock of bonds either (an agent wouldn't choose to die after period $t+1$ with positive wealth left over; the agent would like to die in debt, but we rule this out). The agent faces two flow budget constraints for each period:

$$C_t + q_t B_t \leq Y_t$$

$$C_{t+1} \leq Y_{t+1} + B_t$$

The agent wants to maximize the expected present discounted value of flow utility, where consumption, C_t and C_{t+1}, maps into utils by $\ln C_t$ and $\ln C_{t+1}$. Agents discount future utility flows by $0 < \beta < 1$. The optimization problem of the agent is then:

$$\max_{C_t, B_t, C_{t+1}} \quad \ln C_t + \beta \ln C_{t+1}$$

$$\text{s.t.}$$

$$C_t + q_t B_t \leq Y_t$$

$$C_{t+1} \leq Y_{t+1} + B_t$$

We can form a Lagrangian:

$$\mathcal{L} = \ln C_t + \beta \ln C_{t+1} + \lambda_1 \left(Y_t - C_t - q_t B_t \right) + \lambda_2 \left(Y_{t+1} + B_t - C_{t+1} \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_1 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}} = 0 \Leftrightarrow \beta \frac{1}{C_{t+1}} = \lambda_2 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \Leftrightarrow q_t \lambda_1 = \lambda_2 \tag{3}$$

If we combine these, we are left with the Euler equation:

$$q_t \frac{1}{C_t} = \beta \frac{1}{C_{t+1}} \tag{4}$$

This condition has an intuitive interpretation: saving one fewer unit (i.e. reducing B_t by 1) increases current consumption by q_t, the marginal utility of which is $q_t \frac{1}{C_t}$; saving one fewer unit today means you can consume one fewer unit tomorrow, which reduces utility tomorrow by $\frac{1}{C_{t+1}}$,

which discounted into period t utils is $\beta \frac{1}{C_{t+1}}$. At an optimum, the extra benefit and extra cost of saving a little more must be equal – at the optimum, you're indifferent.

The problem as laid out above is a decision problem: it generates an optimality condition telling us how the agent should behave given prices. We are now in a position to define a competitive equilibrium for this economy. A generic definition of a competitive equilibrium is a set of prices and allocations such that agents are behaving optimally (behaving according to their decision rules) and all markets simultaneously clear. Here the price is q_t, and the allocations will be C_t, C_{t+1}, and B_t. What is the market-clearing condition in this economy? The market-clearing condition in this economy is that $B_t = 0$. Why is this the market-clearing condition? Market-clearing means that demand for saving must equal supply of borrowing. Since there is only one type of agent, if that agent wanted to save, there would be no borrowing supply; similarly, if that agent wanted to borrow, there would be no saving supply. Hence, it must be the case that $B_t = 0$. Equivalently, from the first period budget constraint, this means that $Y_t = C_t$: all income must be consumed.

What will make this market-clearing condition hold? The price of the bond, q_t. A generic definition of a competitive equilibrium is a set of prices and allocation such that (i) the allocations solve the agents' decision problems and (ii) all markets clear. Here the allocations are C_t and C_{t+1} and the price is q_t; the decision problem is given above, and the market-clearing condition is $B_t = 0$. Since $B_t = 0$ implies $C_t = Y_t$ and $C_{t+1} = Y_{t+1}$, and the Y_t and Y_{t+1} are exogenous, we can plug these into the Euler equation to solve for the equilibrium price of bonds:

$$q_t = \beta \frac{Y_t}{Y_{t+1}} \tag{5}$$

Hence, the equilibrium is characterized by the price $q_t = \beta \frac{Y_t}{Y_{t+1}}$, and the allocations $C_t = Y_t$, $C_{t+1} = Y_{t+1}$, and $B_t = 0$. If I were to assume that the endowment were constant across time, with $Y_t = Y_{t+1}$, we would simply have $q_t = \beta$.

We can do some interesting comparative static exercises here, and compare them to the predictions of the partial equilibrium permanent income hypothesis. First of all, consider what happens if Y_t increases. We know from basic PIH theory that households would like to smooth their consumption intertemporally – hence, they would like to save some of the increase in current income to be able to increase both C_t and C_{t+1}. But we know from the market-clearing condition that they cannot do this in equilibrium: we must have $C_t = Y_t$ and $C_{t+1} = Y_{t+1}$. In other words, in equilibrium, the household must consume all of the extra endowment in period t. What makes them willing to do this? The price of bonds increases – we can see this from (5); when Y_t goes up, q_t must also go up. Intuitively, what's going on here is that, when Y_t goes up, there is an increase in demand for B_t (households want to save so as to increase consumption in the future). But, in equilibrium here, there can be no bond holding – effectively, the supply of bonds is 0. The increase in demand therefore just drives up the price of bonds. We can see this in a simple supply-demand graph:
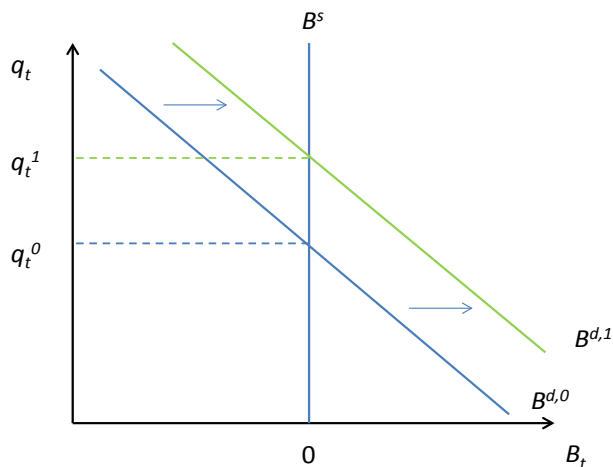
Next, consider the case of an increase in the future endowment, Y_{t+1}. We know that this makes people want to increase consumption immediately in the present. Doing so requires either borrowing or saving less. But in equilibrium, current consumption cannot change, since current Y_t is unchanged. We can see from the expression for equilibrium q_t that q_t will fall. Effectively, what is going on is that the increase in the future endowment reduces demand for bonds, which drives down their equilibrium price given that the supply of bonds in equilibrium is fixed at zero. We can see this in the figure below:
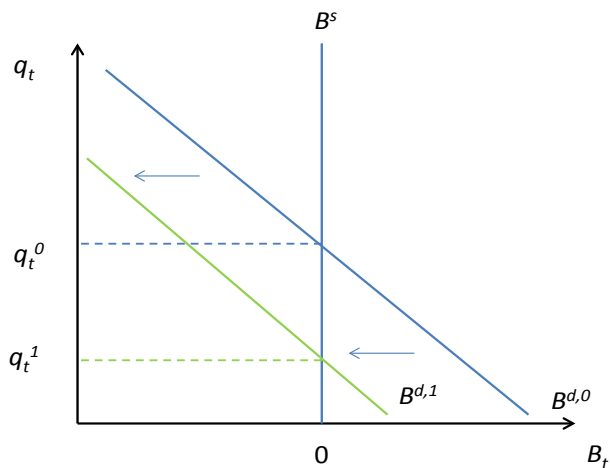
What we see here is that there is no way for agents in the model in equilibrium to smooth their consumption relative to the aggregate endowment. This happens because even though bonds exist in this model, in equilibrium they are not held if everyone is the same. In other words, agents are unable to "insure" themselves against aggregate income "risk" here – I put "insure" and "risk" in quotations because the model as laid is deterministic, but you should hopefully get the point.

4

2.1 Alternative Treatment of Bond Payouts

In writing the problem above, I assumed that you purchased bonds in period t at a price q_t; these bonds pay out 1 unit of consumption in period $t+1$. We saw in the example economy above with a constant endowment across time that $q_t = \beta < 1$; with the price less than one and the future payout one unit of consumption, sometimes we call these instruments "discount" bonds – they (typically) sell at a discount relative to "face" (the amount the bond will pay out in period $t+1$).

The alternative setup (and actually the setup I use most often), is to instead normalize the price of the bond in period t to be 1; the bond then pays out $1 + r_t$ units of consumption in period $t+1$, where we interpret r_t as the interest rate. These setups turn out to be identical, and we'll have $1 + r_t = \frac{1}{q_t}$, but it's worth seeing them both. In this setup, we would write the budget constraints as:

$$C_t + B_t \leq Y_t$$

$$C_{t+1} \leq Y_{t+1} + (1 + r_t)B_t$$

Setting up the optimization problem leads to the following Euler equation:

$$\frac{1}{C_t} = \beta(1 + r_t)\frac{1}{C_{t+1}} \tag{6}$$

As noted above, this is the same as we had before if you interpret $1 + r_t = \frac{1}{q_t}$. The equilibrium allocations will be identical. If you consider the two thought experiments I considered above – separately, an increase in Y_t or Y_{t+1} – we can figure out what ought to happen to r_t. In the case of an increase in Y_t, we saw that q_t would go up, which means r_t falls. The opposite happens in the case of an increase in Y_{t+1}. Basically, what is happening in the case of an increase in Y_t is that people want to save more (they want to increase their consumption by less than their income) – since consumption must equal income, the real interest rate must fall in equilibrium to dissuade this extra saving. In the case of an increase in Y_{t+1}, the real interest rate rises in equilibrium – households want to reduce saving/increase borrowing, but this cannot happen in equilibrium, so the real interest rate adjusts to dissuade this. A useful rule of thumb that emerges here is that the equilibrium real interest rate ends up being a measure of how "good" the future looks relative to the present – if the future looks brighter (bigger endowment), the real interest rate will rise in equilibrium to offset this (fully in this endowment economy case, partially in a model with some means to transfer resources intermporally like capital accumulation). Also, we see that the price, q_t, and interest rate, r_t, on bonds move opposite one another.

2.2 Stochastic Aggregate Endowment

Suppose instead of being deterministic the future endowment is stochastic. Suppose there are two states of the world: $Y_{t+1}(1)$ and $Y_{t+1}(2)$ (i.e. think of state 1 as being the high state, 2 being the low state). These states will occur with probabilities p and $1 - p$, with $0 \leq p \leq 1$.

The problem in this setup is not fundamentally different but does need to be modified. The household still has access to the bond, but assume that its payout does not depend on the future state of the world (more on this below). First, with stochastic future income, future consumption will be stochastic. This means that the household will want to maximize *expected* utility. Second, with two states of the world in the future, there will effectively be two budget constraints for period $t + 1$. Note that the budget constraints must hold in an ex-post sense (not just ex-ante), so we need to write separate budget constraints for each state of the world. Call $C_{t+1}(1)$ and $C_{t+1}(2)$ the amount of consumption the agent will do in the two states of the world. We can write the problem of the agent as:

$$\max_{C_t, B_t, C_{t+1}(1), C_{t+1}(2)} \ln C_t + p\beta \ln C_{t+1}(1) + (1 - p)\beta \ln C_{t+1}(2)$$

$$\text{s.t.}$$

$$C_t + q_t B_t \leq Y_t$$

$$C_{t+1}(1) \leq Y_{t+1}(1) + B_t$$

$$C_{t+1}(2) \leq Y_{t+1}(2) + B_t$$

Form a Lagrangian:

$$\mathcal{L} = \ln C_t + p\beta \ln C_{t+1}(1) + (1 - p)\beta \ln C_{t+1}(2) + \lambda_1 (Y_t - C_t - q_t B_t) + \dots$$
$$\dots + \lambda_2 (Y_{t+1}(1) + B_t - C_{t+1}(1)) + \lambda_3 (Y_{t+1}(2) + B_t - C_{t+1}(2))$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_1 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \Leftrightarrow q_t \lambda_1 = \lambda_2 + \lambda_3 \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}(1)} = 0 \Leftrightarrow p\beta \frac{1}{C_{t+1}(1)} = \lambda_2 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}(2)} = 0 \Leftrightarrow (1 - p)\beta \frac{1}{C_{t+1}(2)} = \lambda_3 \tag{10}$$

If we combine these, we get:

$$q_t \frac{1}{C_t} = \beta \left(p \frac{1}{C_{t+1}(1)} + (1 - p) \frac{1}{C_{t+1}(2)} \right) \tag{11}$$

The term inside the parentheses is just the expected marginal utility of future consumption (which is not the same thing as the marginal utility of expected consumption). Hence, we can write this Euler equation as:

$$q_t \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \tag{12}$$

This looks identical to (4), but it is the expected future marginal utility of consumption on the right hand side instead of the realized marginal utility of future consumption.

It turns out that we could re-write the problem in the first place and will arrive at the same place. In particular, note that $p\beta \ln C_{t+1}(1) + (1-p)\beta \ln C_{t+1}(2) = \beta E_t \ln C_{t+1}$. Note also that the budget constraint must hold in both states of the world, but it also must hold in expectation. Hence, we could write the problem as:

$$\max_{C_t, B_t, C_{t+1}} \quad \ln C_t + \beta E_t \ln C_{t+1}$$

$$\text{s.t.}$$

$$C_t + q_t B_t \leq Y_t$$

$$E_t C_{t+1} \leq E_t Y_{t+1} + B_t$$

A Lagrangian is:

$$\mathcal{L} = \ln C_t + \beta E_t \ln C_{t+1} + \lambda_t (Y_t - C_t - q_t B_t) + E_t \lambda_{t+1} (Y_{t+1} + B_t - C_{t+1})$$

Here instead of numbering the Lagrange multipliers I instead have dated them, and I have an expectation operator in front of future flow utility and on front of the future flow budget constraint (noting that the multiplier in $t + 1$ is not known at time t either). The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = 0 \Leftrightarrow q_t \lambda_t = E_t \lambda_{t+1} \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial C_{t+1}} = 0 \Leftrightarrow \beta E_t \frac{1}{C_{t+1}} = E_t \lambda_{t+1} \tag{15}$$

Combing these yields:

$$q_t \frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \tag{16}$$

This is the same as before. So when dealing with aggregate uncertainty, I typically won't specific specific states of the world and probabilities and such, but it will just be implicit that the future is uncertain, and therefore we will write problems as I have done here.

The implications for equilibrium prices and allocations are not fundamentally affected by the presence of uncertainty in the aggregate endowment. In equilibrium, consumption will have to equal income – there can be no bond-holding in equilibrium. An increase in the current endowment will result in q_t rising – the agents would like to save, but can't in equilibrium, so q_t rises to offset

this. An expected increase in the future endowment will lower the expected marginal utility of future consumption; since current C_t cannot adjust, this will result in q_t falling. In this setup we can also talk about the effect of uncertainty. Suppose there were a mean-preserving spread on future endowment realizations – by this I mean that say $Y_{t+1}(1)$ gets bigger and $Y_{t+1}(2)$ gets smaller, but $E_t Y_{t+1}$ is unaffected. Given that third derivative of utility is positive (so marginal utility is convex), this will induce a precautionary motive – agents would like to increase saving in the present. This is not possible in equilibrium, so q_t will rise. Also, just as in the deterministic case, agents cannot smooth their consumption relative to the aggregate endowment. Hence, there is in a sense uninsurable aggregate risk in this model.

3   Two Types of Agents

Instead of just a large number of one type of perfectly identical agents, suppose now that there are two types of agents. Let's call them type 1 and type 2 (I will index them by $i = 1$ or $i = 2$). They have identical preferences, but differ somehow in their endowments. The key insight going forward is that the differences in endowments expose agents to idiosyncratic risk, which can potentially be gotten rid of through the one period bonds we've discussed.

For this whole section, we again focus on a two period setup; this can easily be generalized. We assume that both types of agents discount the future by β and have log utility over consumption. There are sufficiently many agents that they behave as price-takers; for simplicity, assume that there are an equal number of each type of agent (one can relax this). You can call the number of each type of agent N, but since they're all the same I'll normalize to $N = 1$.

3.1   Deterministic but Heterogeneous Endowments

Suppose that type 1 agents receive one unit of the endowment in period t and none in period $t+1$, while the opposite is true for type 2 agents. This is perfectly deterministic. That is, the endowment patterns are:

$$(Y_{1,t}, Y_{1,t+1}) = (1, 0)$$

$$(Y_{2,t}, Y_{2,t+1}) = (0, 1)$$

The optimization problem faced by each type of agents is identical to the problem faced by the single agent in the above section. They face a common price of bonds, q_t. Hence, the Euler equation must hold for each type:

$$\frac{1}{C_{i,t}} = \beta \frac{1}{q_t} \frac{1}{C_{i,t+1}} \quad i = 1, 2 \tag{17}$$

A useful insight from the Euler equation is the following – each type of agent will have the same *ratio* of consumption in the future to consumption in the present. This holds because they face

the same price on the bond, q_t, and have the same discount factor. We can manipulate the Euler equations for both types to write:

$$\frac{C_{1,t+1}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{2,t}} \tag{18}$$

We can re-arrange this to get:

$$\frac{C_{2,t}}{C_{1,t}} = \frac{C_{2,t+1}}{C_{1,t+1}} \tag{19}$$

This tells us something neat right away – the ratio of type 2 to type 1 consumption is the same in both periods. Note that this holds regardless of the endowment patterns. Type 1 agents may have more (or less) consumption than type 2 agents, but their *relative* consumption will be constant across time. For the particular endowment pattern I've assumed here, this means that type 2's relative consumption in period t is no different than his consumption relative to type 1 in period $t + 1$, even though type 2 has no income in period 1. Effectively, what is going on is a sort of risk-sharing between the two types of agents, allowing them to smooth their consumption across states of the world.

This setup of the model has a slightly different market-clearing condition. Since there are two types, it is conceivable that they could borrow/lend with one another (indeed, since their relative consumption is constant across time, if their endowment patterns are not the same across time, they will have to borrow/lend with one another). On net, *aggregate* borrowing/lending must be zero, but the individual agents can borrow/lend. Hence, we have:

$$B_{1,t} + B_{2,t} = 0 \tag{20}$$

Intuitively, if type 1 saves ($B_{1,t} > 0$), then type 2 must borrow in an equal amount ($B_{2,t} = -B_{1,t}$). If we impose this condition and sum the flow budget constraints together, we get:

$$C_{1,t} + C_{2,t} = Y_{1,t} + Y_{2,t} = 1 \tag{21}$$

$$C_{1,t+1} + C_{2,t+1} = Y_{1,t+1} + Y_{2,t+1} = 1 \tag{22}$$

In other words, aggregate consumption must equal the aggregate endowment, which I've set up to be fixed. You can see where this is going – since aggregate consumption is fixed, and the Euler equations tell us that relative consumption is also fixed, it's going to be the case that consumption is going to be constant across time for both agents. In particular, solve for $C_{2,t}$ and $C_{2,t+1}$ in terms of $C_{1,t}$ and $C_{1,t+1}$ from the constraints:

$$C_{1,t} = 1 - C_{2,t}$$

$$C_{1,t+1} = 1 - C_{2,t+1}$$

Plug these in to the condition above that consumption is the same ratio across agents in both periods:

$$\frac{1 - C_{2,t}}{C_{2,t}} = \frac{1 - C_{2,t+1}}{C_{2,t+1}}$$

Simplifying this equation implies that $C_{2,t} = C_{2,t+1}$ – as predicted, consumption will be equal across time. Similarly, we will have $C_{1,t} = C_{1,t+1}$. If you go to the Euler equation, this then implies that $q_t = \beta$. To solve for the values of consumption of each type, combine the budget constraints, yielding:

$$C_{i,t} + q_t C_{i,t+1} = Y_{i,t} + q_t Y_{i,t+1}$$

Imposing that $q_t = \beta$ and that consumption is equal across time, we have:

$$C_{i,t} = \frac{1}{1 + \beta} \left(Y_{i,t} + \beta Y_{i,t+1} \right)$$

Plugging in the endowment streams, we get:

$$C_{1,t} = \frac{1}{1 + \beta} = C_{1,t+1} \tag{23}$$

$$C_{2,t} = \frac{\beta}{1 + \beta} = C_{2,t+1} \tag{24}$$

We can use these to back out saving behavior to check our work. Noting that $B_{i,t} = \frac{1}{q}(Y_{i,t} - C_{i,t})$, we have:

$$B_{1,t} = \frac{1}{\beta} \left(1 - \frac{1}{1 + \beta} \right) = \frac{1}{1 + \beta}$$

$$B_{2,t} = \frac{1}{\beta} \left(-\frac{\beta}{1 + \beta} \right) = -\frac{1}{1 + \beta}$$

As we imposed originally, we have $B_{1,t} + B_{2,t} = 0$.

Hence, we observe that consumption is constant across time for both agents (though it isn't the same for both agents – type 1 consumes more, which ends up being by virtue of getting an endowment of 1 in the first period instead of having to wait until period $t+1$). What is important to note here is that consumption of an agent doesn't depend on whether his/her income is high or low – it is the same in both periods. Effectively, the bond serves as an insurance mechanism – if I have lots of income in the first period but little in the second, I save in the first period and vice-versa. In doing so I can smooth out my consumption relative to my income.

While in this setup agents can eliminate their idiosyncratic income variability through the saving instrument, as in the problem of a single type of agent, the agents here cannot smooth out aggregate income changes. Suppose, for example, that I gave type 1 agents 2 units of the endowment in period t instead of just 1. They would like to smooth their consumption across time, but will not be able

to do that. It will still be the case that the relative consumption of type 1 to type 2 agents will be the same in both periods, but the levels of consumption will be different than in the baseline setup. The important point here is that agent's consumption patterns depend on the *aggregate* endowment, not their idiosyncratic endowment.

3.2 Stochastic Heterogeneous Endowments

Now let's change the setup slightly. Rather than the future endowments being deterministic, let's make them stochastic. Assume that current endowments are known, $Y_{i,t}$ for $i = 1, 2$. For the future endowments, suppose that there are two states of the world: states 1 and 2. State 1 occurs with probability p and state 2 occurs with probability $1 - p$. The period $t + 1$ endowment of agent i in state 1 is $Y_{i,t+1}(1)$, and the endowment in state 2 is $Y_{i,t+1}(2)$. For simplicity, let's assume that the aggregate future endowment is 1 regardless of the state: this means $Y_{1,t+1}(1) + Y_{2,t+1}(1) = 1$ and $Y_{1,t+1}(2) + Y_{2,t+1}(2) = 1$. This means that there is no aggregate uncertainty, but the agents face individual uncertainty.

Given the individual uncertainty faced by agents, we need re-state the problem somewhat. First, they will want to maximize expected utility (since future income, and hence future consumption, are uncertain, the future utility flow will be uncertain). Second, I want to augment the description of bonds. In the setup we've worked with so far, there is one bond – you buy the bond today by giving up q_t units of consumption, and it pays out 1 unit of consumption in $t + 1$ (equivalently, you buy the bond today giving up one unit of consumption, and it pays out $1 + r_t$ units of consumption in $t + 1$). Now, I want to introduce *two* bonds which have "state-contingent" payoffs: $B_t(1)$ pays out 1 unit of consumption in period $t + 1$ if state 1 materializes, while $B_t(2)$ pays out one unit of consumption in $t + 1$ should state 2 materialize. The price of these bonds in terms of period t consumption are $q_t(1)$ and $q_t(2)$, respectively. If you buy $B_t(1)$ and state 2 materializes, you get nothing – the payout on these bonds is state-contingent.

The first period flow budget constraint for household of type i is:

$$C_{i,t} + q_t(1)B_{i,t}(1) + q_t(2)B_{i,t}(2) \leq Y_{i,t} \tag{25}$$

In words, the household's consumption plus its purchases of the two state-contingent bonds cannot exceed income. Note that $B_{i,t}(1)$ and $B_{i,t}(2)$ can be negative. The household effectively faces two separate second period budget constraints – this is because budget constraints must hold ex-post, not just in expectation. So we will have one period $t + 1$ constraint for each state of the world (in this case two). For agent $i = 1, 2$, these are:

$$C_{i,t+1}(1) \leq Y_{i,t+1}(1) + B_{i,t}(1) \tag{26}$$

$$C_{i,t+1}(2) \leq Y_{i,t+1}(2) + B_{i,t}(2) \tag{27}$$

Household of type i will want to choose current consumption, purchases of both bonds, and future state-contingent consumption to maximize the expected discounted value of lifetime utility

subject to the three flow budget constraints. The problem can be written:

$$\max_{C_{i,t},B_{i,t}(1),B_{i,t}(2),C_{i,t+1}(1),C_{i,t+1}(2)} \ln C_{i,t} + p\beta \ln C_{i,t+1}(1) + (1-p)\beta \ln C_{i,t+1}(2)$$

$$\text{s.t.}$$

$$C_{i,t} + q_t(1)B_{i,t}(1) + q_t(2)B_{i,t}(2) \leq Y_{i,t}$$

$$C_{i,t+1}(1) \leq Y_{i,t+1}(1) + B_{i,t}(1)$$

$$C_{i,t+1}(2) \leq Y_{i,t+1}(2) + B_{i,t}(2)$$

Form a Lagrangian with three constraints:

$$\mathcal{L} = \ln C_{i,t} + p\beta \ln C_{i,t+1}(1) + (1-p)\beta \ln C_{i,t+1}(2) + \lambda_{i,1}\left[Y_{i,t} - C_{i,t} - q_t(1)B_{i,t}(1) - q_t(2)B_{i,t}(2)\right] + \ldots$$
$$\cdots + \lambda_{i,2}\left[Y_{i,t+1}(1) + B_{i,t}(1) - C_{i,t+1}(1)\right] + \lambda_{i,3}\left[Y_{i,t+1}(2) + B_{i,t}(2) - C_{i,t+1}(2)\right]$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_{i,t}} = 0 \Leftrightarrow \frac{1}{C_{i,t}} = \lambda_{i,1} \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial B_{i,t}(1)} = 0 \Leftrightarrow q_t(1)\lambda_{i,1} = \lambda_{i,2} \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial B_{i,t}(2)} = 0 \Leftrightarrow q_t(2)\lambda_{i,1} = \lambda_{i,3} \tag{30}$$

$$\frac{\partial \mathcal{L}}{\partial C_{i,t+1}(1)} = 0 \Leftrightarrow p\beta \frac{1}{C_{i,t+1}(1)} = \lambda_{i,2} \tag{31}$$

$$\frac{\partial \mathcal{L}}{\partial C_{i,t+1}(2)} = 0 \Leftrightarrow (1-p)\beta \frac{1}{C_{i,t+1}(2)} = \lambda_{i,3} \tag{32}$$

Divide (31) by (32):

$$\frac{p}{1-p}\frac{C_{i,t+1}(2)}{C_{i,t+1}(1)} = \frac{\lambda_{i,2}}{\lambda_{i,3}} \tag{33}$$

Now divide (29) by (30) to get:

$$\frac{\lambda_{i,2}}{\lambda_{i,3}} = \frac{q_t(1)}{q_t(2)} \tag{34}$$

Since agents face the same prices, this means that $\frac{\lambda_{1,2}}{\lambda_{1,3}} = \frac{\lambda_{2,2}}{\lambda_{2,3}}$ – i.e. this ratio of multipliers is the same for both kinds of agents. Combine this with (23) to get:

$$\frac{p}{1-p}\frac{C_{i,t+1}(2)}{C_{i,t+1}(1)} = \frac{q_t(1)}{q_t(2)} \tag{35}$$

12

Since this is the same across agents, we can write this as:

$$\frac{C_{1,t+1}(2)}{C_{1,t+1}(1)} = \frac{C_{2,t+1}(2)}{C_{2,t+1}(1)} \tag{36}$$

We can re-arrange terms to get:

$$\frac{C_{1,t+1}(2)}{C_{2,t+1}(2)} = \frac{C_{1,t+1}(1)}{C_{2,t+1}(1)} \tag{37}$$

This tells us that the *relative* consumption across agents in period $t+1$ is independent of the realization of the state – it is the same in state 1 as in state 2. Since I assumed that aggregate income in states 1 and 2 is the same at 1, and since in equilibrium total consumption will again have to equal total income, it will be the case that the levels of consumption will be the same in across states. I'm not going to go through here and solve for all equilibrium prices and allocations, but it would be straightforward to do so.

What's going on here is that these state-contingent bonds serve as an insurance mechanism. In period t, an agent will "buy" bonds (have positive holdings of bonds) for states in which he expects to have relatively low future income and will "sell' (negative holdings) bonds that pay out in states where his future income is relatively high. By doing so, agents can effectively eliminate the idiosyncratic income risk that they face – as we see above, the relative consumption across agents is identical in each state, which has the implication that individual consumption is perfectly correlated with aggregate consumption. If the aggregate endowment is fixed, the levels of consumption for both types of agents will be the same in both future states (though not necessarily the same across agents). If the aggregate endowment is higher in one state than the other, then the levels of consumption for agents in the higher state will be higher than in the low state, but the relative levels of consumption will still be the same across states.

The setup I have described here is a (simple) description of *complete markets*: there are state-contingent securities (bonds) which span the set of possible states, making it possible for agents to use these securities to eliminate any idiosyncratic income risk. Note that it is not possible to eliminate *aggregate* endowment risks here – the market-clearing condition is always that the aggregate endowment must equal the aggregate consumption, so if the aggregate endowment fluctuates consumption will fluctuate. But, as we've seen above in this simple example, the ratio of consumption across agents will not vary across states, provided these state-contingent bonds exist. This means that individual consumption is perfectly correlated with aggregate consumption. The existence of these complete markets forms the basis for representative agent assumptions in macroeconomics – if markets are complete, idiosyncratic risks are eliminated and we can treat the economy as though there is a single type of agent. Finally, note that our setup here does not imply consumption equalization across agents – it implies that an agent's consumption only varies with the aggregate endowment.

Formally, the setup I've used here are called Arrow securities – there exists securities in each period that have state-contingent payouts in the next period, and can be used to eliminate idiosyn-

cratic risk. Arrow securities are one period securities that pay off in every conceivable idiosyncratic state in the next period. An alternative arrangement that typically yields identical outcomes is for trading in securities to being at the beginning of time for all states in all periods – this is often called the Arrow-Debrue setup. The bottom line here is that the existence of complete markets (either via the one period Arrow setup or the Arrow-Debrue securities at the beginning of time) can eliminate idiosyncractic risk, which makes individual consumption perfectly correlated with aggregate consumption, which helps motivate the use of representative agent economies.