Dynamic New Keynesian Model with Government Spending

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1 Linearized Equilibrium Conditions

The following are equilibrium conditions after linearization:

$$c_t = \mathbb{E}_t \, c_{t+1} - \frac{1}{\sigma} \left(i_t - \mathbb{E}_t \, \pi_{t+1} \right) \tag{1}$$

$$\chi n_t = -\sigma c_t + w_t \tag{2}$$

$$w_t = mc_t + a_t \tag{3}$$

$$\pi_t = \zeta mc_t + \beta \, \mathbb{E}_t \, \pi_{t+1} \tag{4}$$

$$y_t = a_t + n_t \tag{5}$$

$$y_t = (1 - \psi) \, c_t + \psi g_t \tag{6}$$

(1) is the linearized Euler equation; σ is the inverse intertemporal elasticity of substitution, i_t is the nominal interest rate, and π_t is inflation. (2) is the linearized labor supply condition; χ is the inverse Frisch elasticity. (3) is the labor demand condition, where mc_t is real marginal cost (the inverse price markup). (4) is the linearized Phillips curve, which comes from price-setting conditions from firms facing staggered price adjustment. The parameter $\zeta = \frac{(1-\phi)(1-\phi\beta)}{\phi}$, where ϕ is the probability of price non-adjustment and β is the subjective discount factor. (5) is the linearized production function (note I am linearizing about a zero inflation steady state, so price dispersion drops out). (6) is the linearized resource constraint. g_t is government spending, and $\psi = \frac{G}{Y}$ is the steady state government spending share of output.

The system would need to be augmented with stochastic processes for g_t and a_t (e.g. AR(1)) as well as some kind of rule for i_t (e.g. a Taylor rule). That would make 9 equations with 9 variables $\left\{ c_t, y_t, n_t, w_t, i_t, mc_t, \pi_t, g_t, a_t \right\}$.

2 System Reduction

We want to reduce this system down to a smaller number of equations. First, sub out consumption from the Euler equation and labor supply curves:

$$\frac{1}{1-\psi}y_t - \frac{\psi}{1-\psi}g_t = \frac{1}{1-\psi}\,\mathbb{E}_t\, y_{t+1} - \frac{\psi}{1-\psi}\,\mathbb{E}_t\, g_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\,\pi_{t+1}) \tag{7}$$

$$\chi n_t = -\frac{\sigma}{1-\psi}y_t + \frac{\psi\sigma}{1-\psi}g_t + w_t \tag{8}$$

Now substitute the production function into (8) to eliminate n_t;

$$\chi(y_t - a_t) = -\frac{\sigma}{1-\psi}y_t + \frac{\psi\sigma}{1-\psi}g_t + w_t \tag{9}$$

Now substitute the labor demand function into (9) to eliminate w_t:

$$\chi(y_t - a_t) = -\frac{\sigma}{1-\psi}y_t + \frac{\psi\sigma}{1-\psi}g_t + mc_t + a_t \tag{10}$$

Collect terms:

$$\left(\frac{\chi(1-\psi)+\sigma}{1-\psi}\right)y_t = (1+\chi)a_t + \frac{\psi\sigma}{1-\psi}g_t + mc_t \tag{11}$$

It is useful to define the flexible price equilibrium level of variables, denoted with a f superscript, as being consistent with $mc_t = 0$ (which would be an implication of price stickiness). y_t^f satisfies:

$$y_t^f = \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}a_t + \frac{\psi\sigma}{\chi(1-\psi)+\sigma}g_t \tag{12}$$

We pretty clearly see that:

$$\left(\frac{\chi(1-\psi)+\sigma}{1-\psi}\right)y_t^f = (1+\chi)a_t + \frac{\psi\sigma}{1-\psi}g_t \tag{13}$$

But this means we can write real marginal cost as:

$$\left(\frac{\chi(1-\psi)+\sigma}{1-\psi}\right)(y_t - y_t^f) = mc_t \tag{14}$$

Hence, we can define $x_t = y_t - y_t^f$ and write the Phillips Curve in terms of the output gap as:

$$\pi_t = \zeta\gamma x_t + \beta\,\mathbb{E}_t\,\pi_{t+1} \tag{15}$$

Where ζ is as before and $\gamma = \left(\frac{\chi(1-\psi)+\sigma}{1-\psi}\right)$.

Now, let's re-write the Euler equation slightly:

$$y_t - \psi g_t = \mathbb{E}_t\, y_{t+1} - \psi\,\mathbb{E}_t\, g_{t+1} - \frac{1-\psi}{\sigma}(i_t - \mathbb{E}_t\,\pi_{t+1}) \tag{16}$$

Add and subtract y_t^f to both sides of (17):

$$y_t - y_t^f - \psi g_t + y_t^f = \mathbb{E}_t\, y_{t+1} - \mathbb{E}_t\, y_{t+1}^f - \psi\,\mathbb{E}_t\, g_{t+1} + \mathbb{E}_t\, y_{t+1}^f - \frac{1-\psi}{\sigma}(i_t - \mathbb{E}_t\,\pi_{t+1}) \tag{17}$$

Write in terms of the gap, and plug in for y_t^f:

$$x_t - \psi g_t + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} a_t + \frac{\psi\sigma}{\chi(1-\psi)+\sigma} g_t =$$
$$\mathbb{E}_t\, x_{t+1} - \psi\, \mathbb{E}_t\, g_{t+1} + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, a_{t+1} + \frac{\psi\sigma}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, g_{t+1} - \frac{1-\psi}{\sigma}(i_t - \mathbb{E}_t\, \pi_{t+1}) \quad (18)$$

Combining the g_t terms:

$$x_t + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} a_t - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma} g_t =$$
$$\mathbb{E}_t\, x_{t+1} + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, a_{t+1} - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, g_{t+1} - \frac{1-\psi}{\sigma}(i_t - \mathbb{E}_t\, \pi_{t+1}) \quad (19)$$

Now define the natural rate of interest, r_t^f, as the real rate consistent with flexible prices (which would mean $x_t = 0$). We have:

$$\frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} a_t - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma} g_t = \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, a_{t+1} - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, g_{t+1} - \frac{1-\psi}{\sigma} r_t^f \quad (20)$$

Or:

$$r_t^f = \frac{\sigma}{1-\psi} \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} (\mathbb{E}_t\, a_{t+1} - a_t) - \frac{\sigma}{1-\psi} \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma} (\mathbb{E}_t\, g_{t+1} - g_t) \quad (21)$$

Let's further assume that a_t and g_t obey AR(1) processes, with AR(1) coefficients ρ_a and ρ_g, respectively. Then we can write this further as:

$$r_t^f = \frac{\sigma}{1-\psi} \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} (\rho_a - 1) a_t - \frac{\sigma}{1-\psi} \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma} (\rho_g - 1) g_t \quad (22)$$

As long as the AR(1) parameters are both less than 1, increases in a_t lower r_t^f, whereas increases in g_t raise it.

Now go back to (20). Add $\frac{1-\psi}{\sigma} r_t^f$ to both sides:

$$x_t + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma} a_t - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma} g_t + \frac{1-\psi}{\sigma} r_t^f =$$
$$\mathbb{E}_t\, x_{t+1} + \frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, a_{t+1} - \frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma}\, \mathbb{E}_t\, g_{t+1} - \frac{1-\psi}{\sigma}(i_t - \mathbb{E}_t\, \pi_{t+1} - r_t^f) \quad (23)$$

But then the r_t^f on the left hand side cancels with all the other junk involving a_t and g_t, leaving us with:

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-\psi}{\sigma}\left(i_t - \mathbb{E}_t\, \pi_{t+1} - r_t^f\right) \tag{24}$$

Hence, (15) and (24) (Phillips Curve and IS equation) define the non-policy block of the model. We can treat r_t^f as an exogenous process, as given in (24). So we can write the reduced system as:

$$\pi_t = \zeta\gamma x_t + \beta\, \mathbb{E}_t\, \pi_{t+1} \tag{25}$$

$$x_t = \mathbb{E}_t\, x_{t+1} - \frac{1-\psi}{\sigma}\left(i_t - \mathbb{E}_t\, \pi_{t+1} - r_t^f\right) \tag{26}$$

$$r_t^f = \frac{\sigma}{1-\psi}\frac{(1+\chi)(1-\psi)}{\chi(1-\psi)+\sigma}(\rho_a - 1)a_t - \frac{\sigma}{1-\psi}\frac{\psi\chi(1-\psi)}{\chi(1-\psi)+\sigma}(\rho_g - 1)g_t \tag{27}$$

We need to append a rule for i_t. But once we get the dynamics for π_t and x_t, we can recover the dynamics for y_t given the definition of y_t^f, (12).

3 IRFs Under a Taylor Rule

Solving the "full system," (1)-(6), yields identical dynamics to the "reduced" system written in terms of the gap and natural rates, (25)-(28). To close the model, I assume a Taylor rule (and I assume a_t and g_t obey AR(1) processes):

$$i_t = \phi_\pi \pi_t \tag{28}$$

Below are impulse responses to a one unit productivity shock (assuming $\rho_a = 0.9$, $\rho_g = 0.9$, $\beta = 0.99$, $\sigma = \chi = 1$, $\phi = 0.75$, $\psi = 0.2$, and $\phi_\pi = 1.5$). I multiply the output response to the government spending shock by the inverse government spending share of output; this puts the response in "multiplier" terms (the model is solved in log deviations).

Figure 1: IRFs to Productivity Shock

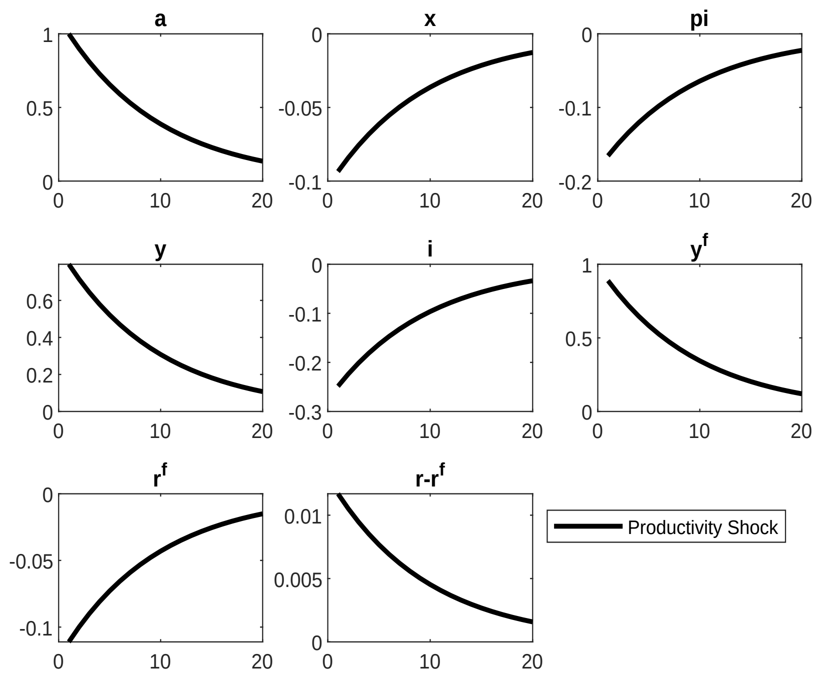
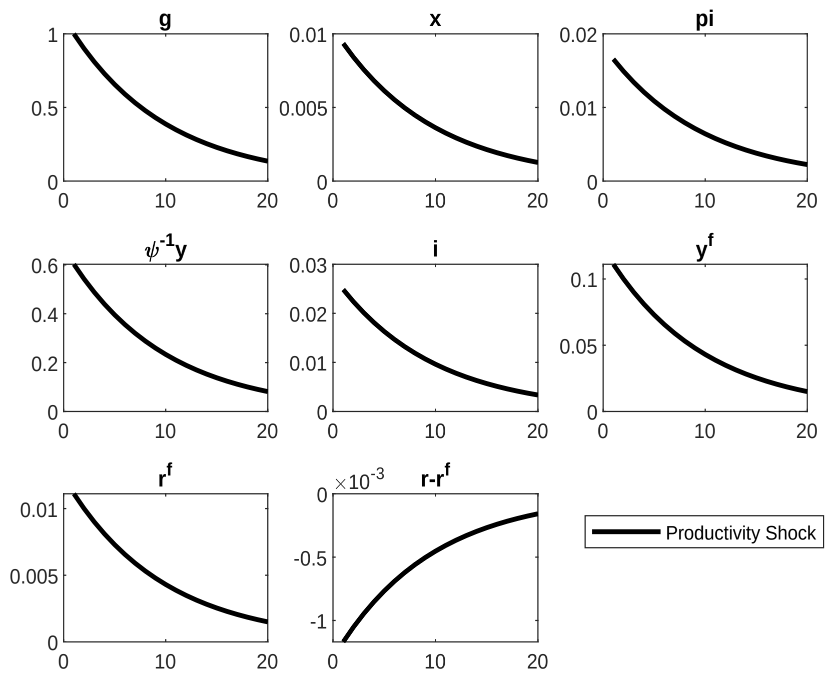


Figure 2: IRFs to Government Spending Shock



4 Approximating the Effects of the ZLB

We can approximate the effects of the ZLB in the model by assuming that $i_t = 0$ (which really means it is pegged at a constant value, not necessarily at zero; what matters is not whether the nominal rate is at zero but rather that it does not move in response to shocks). Within the context of the linearized model, there are two relatively easy ways to implement this: (1) a deterministic interest rate peg and (2) a stochastic interest rate peg. One can, in principle, do both of these "by hand."

To make life a bit easier, let's drop the Taylor rule assumption, and instead assume that, so long as i_t is not fixed, we just have $i_t = r_t^f$. This means that, so long as there is no ZLB, we will have $x_t = \pi_t = 0$. The basic idea of either approach is to assume that, once the ZLB episode is over, policy reverts to tracking the natural rate of interest, which means both inflation and the output gap are zero. These provide "terminal conditions" if you will that allow us to solve backwards for the time paths of x_t and π_t during the period in which the interest rate is pegged.

4.1 Deterministic Peg

Assume, at period t, that $i_t = 0$. It will stay there through period i_{t+H-1}, reverting to equaling the natural rate starting in period $t + H$. This means that, starting in period $t + H$, we will have

$x_{t+H} = \pi_{t+H} = 0$.

Let's start with a government spending shock. Suppose that g_t goes up in period t and follows its AR(1) process; assume that a_t is fixed. This in turn generates a time path of r_t^f. Taking that time path of r_t^f as given, plus the terminal conditions that $\pi_{t+H} = x_{t+H} = 0$, we can solve for π_{t+H-1} and x_{t+H-1} from the Phillips Curve and IS equations:

$$x_{t+H-1} = \frac{1-\psi}{\sigma} r^f_{t+H-1} \tag{29}$$

$$\pi_{t+H-1} = \zeta\gamma x_{t+H-1} \tag{30}$$

Then, taking these as given, we can just solve backwards in time in $t + H - 2$ and so on back to t. In principle one can do this analytically, but it gets messy. I'll instead just do it numerically using loops in Matlab.

Below I show impulse responses to a government spending shock for $H = 6$ and $H = 10$. I scale the output response by $1/\psi$, which puts the units of the response in the form of a multiplier.

Figure 3: IRFs to Government Spending Shock, 6 Period Deterministic Peg

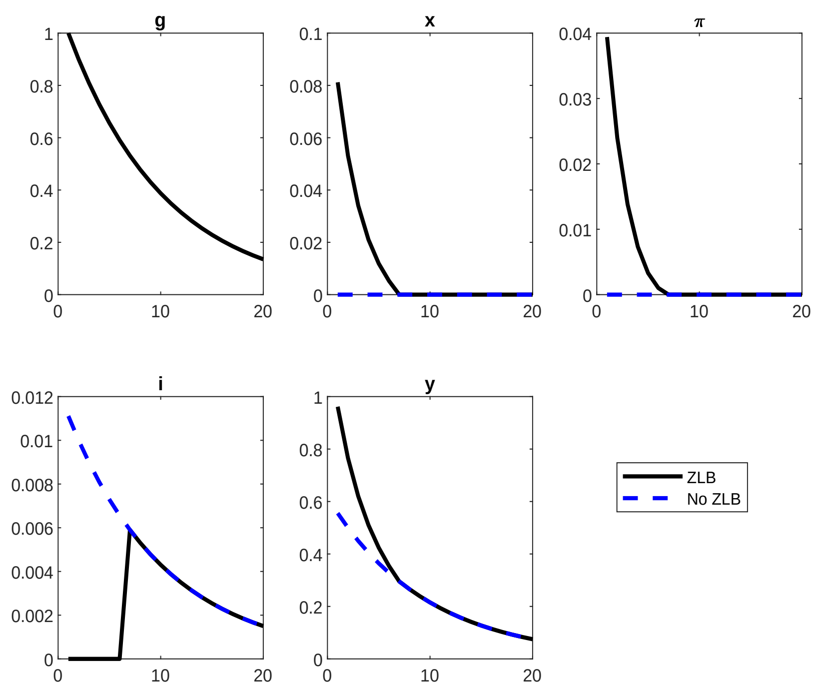
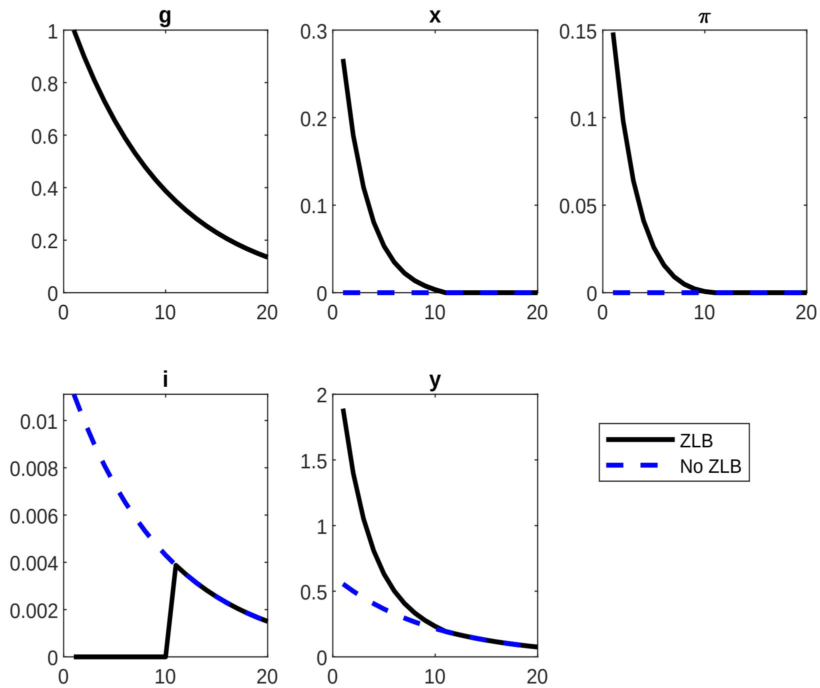


Figure 4: IRFs to Government Spending Shock, 10 Period Deterministic Peg



Regardless of H, the ZLB causes the policy rate to under-react relative to the natural rate. This causes output and inflation to react more. This magnification is larger the bigger is H. For $H = 6$, the government spending multiplier still below 1. For $H = 10$, it goes up to about 2. Note that, because there are no endogenous state variables in the model, once we get to horizon $t + H$, the responses under the peg converge exactly to those without the peg.

Next, I show response to the productivity shock with the same peg lengths. The story is reversed relative to government spending. Ideally, the central bank wants to lower the policy rate to match the lower natural rate. The inability to do so means that policy is too tight. This exerts a contractionary effect, so output increases less and inflation falls more, both more so the bigger is H. When $H = 10$, for example, a positive productivity shock in fact becomes contractionary for output.

Figure 5: IRFs to Productivity Shock, 6 Period Deterministic Peg

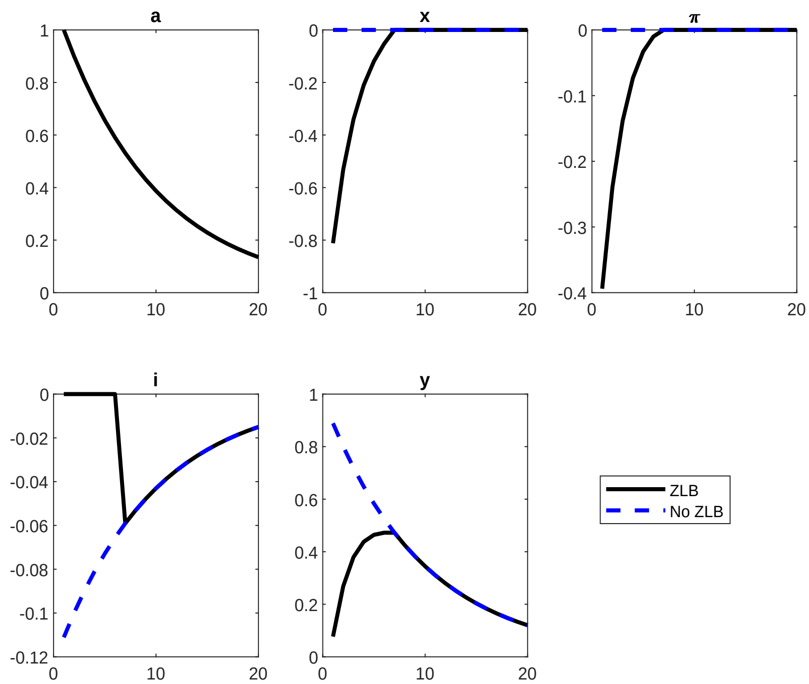
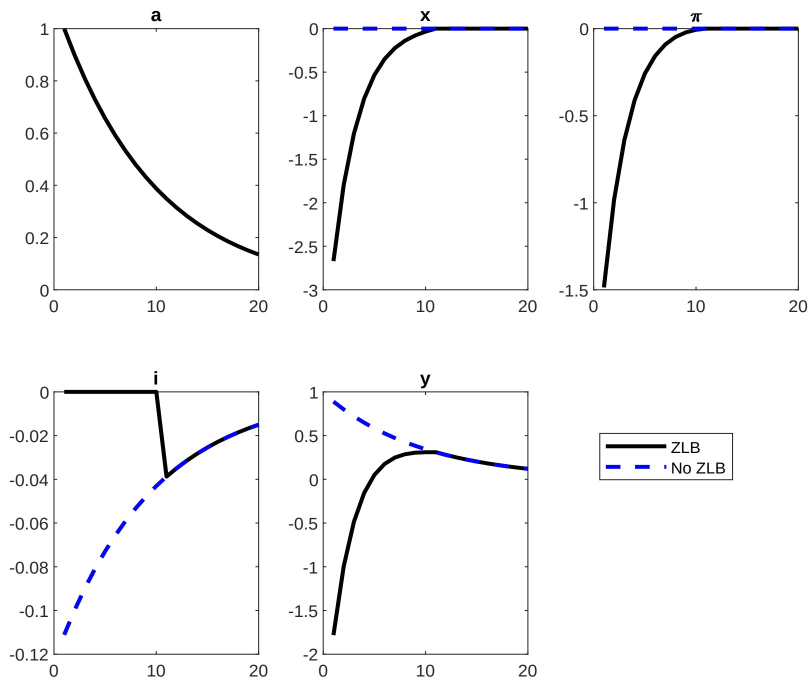


Figure 6: IRFs to Productivity Shock, 10 Period Deterministic Peg



4.2 Stochastic Peg

Suppose that, in period t, $i_t = 0$ (i.e. fixed). It will stay fixed going into period $t+1$ with probability α, and will lift and return to r^f_{t+1} with probability $1-\alpha$. Then the same thing happens again going forward. If the interest rate remains fixed in $t+1$, then there is an α probability it remains fixed in $t+2$, and a $1-\alpha$ probability it lifts in period $t+2$. And so on.

We can map the parameter α into an expected duration of the peg/ZLB as follows. The probability that the interest rate is pegged for 1 period is $1-\alpha$ – this is the probability that the peg lifts in $t+1$ conditional on being in a peg in t. The probability that it is pegged for 2 periods is $\alpha(1-\alpha)$ – α is the probability it lasts into $t+1$, and $(1-\alpha)$ is the probability it lifts after $t+1$. The probability of 3 periods is $\alpha^2(1-\alpha)$ – α^2 is the probability of getting to $t+2$ with the interest rate still fixed, and $1-\alpha$ is the probability of lifting going into $t+3$. And so on. So the expected duration of the peg is:

$$\mathbb{E}(\text{Duration}) = (1-\alpha) + \alpha(1-\alpha)2 + \alpha^2(1-\alpha)3 + \dots = (1-\alpha)\left[1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots\right] \quad (31)$$

Focus on the last term in brackets. Define it is as S:

$$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots \tag{32}$$

Multiply both sides by α:

$$S\alpha = \alpha + 2\alpha^2 + 3\alpha^3 + 4\alpha^4 + \dots \tag{33}$$

Subtract the latter from the former:

$$S(1-\alpha) = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \dots \tag{34}$$

We know that the right hand side is $\frac{1}{1-\alpha}$. Therefore $S = \frac{1}{(1-\alpha)^2}$. But then from above, this gives us a simple formula for the expected duration:

$$\mathbb{E}(\text{Duration}) = \frac{1}{1-\alpha} \tag{35}$$

So, for example, if $\alpha = 3/4$, the expected duration of the peg/ZLB is 4 periods (if quarters, then this is one year). This, by the way, is exactly the same way to map a Calvo price stickiness parameter to an expected duration between price changes.

We can analytically solve for paths of output, inflation, and the output gap at the ZLB. Once the ZLB lifts, we know that $\pi_t = x_t = 0$ since we are assuming the nominal interest rate will equal the natural rate. Using the method of undetermined coefficients, guess that, during the ZLB, inflation and the gap relate to the natural rate of interest as follows:

$$x_t = \theta_1 r_t^f \tag{36}$$

$$\pi_t = \theta_2 r_t^f \tag{37}$$

Plug these guesses into the Phillips Curve and IS equations:

$$\theta_1 r_t^f = \alpha\theta_1 \mathbb{E}_t r_{t+1}^f - \frac{1-\psi}{\sigma}\left(-\alpha\theta_2 \mathbb{E}_t r_{t+1}^f - r_t^f\right) \tag{38}$$

$$\theta_2 r_t^f = \zeta\gamma\theta_1 r_t^f + \alpha\beta\theta_2 \mathbb{E}_t r_{t+1}^f \tag{39}$$

In doing all this, we are noting that, with probability α, $\pi_{t+1} = \theta_2 r_{t+1}^f$ and with probability $1-\alpha$ it equals zero, and similarly for the output gap. If we assume that $\rho_a = \rho_g = \rho$, we can replace $\mathbb{E}_t r_{t+1}^f = \rho r_t^f$. So we can write these as:

$$\theta_1 r_t^f = \alpha\theta_1\rho r_t^f - \frac{1-\psi}{\sigma}\left(-\alpha\theta_2\rho r_t^f - r_t^f\right) \tag{40}$$

$$\theta_2 r_t^f = \zeta\gamma\theta_1 r_t^f + \alpha\beta\theta_2\rho r_t^f \tag{41}$$

Now we can drop the r_t^f. Focus on the second expression (the Phillips Curve). We can write:

$$\theta_2 = \frac{\zeta\gamma}{1-\alpha\beta\rho}\theta_1 \tag{42}$$

Then from the IS equation, we have:

$$\theta_1\sigma(1-\alpha\rho) = (1-\psi) + \alpha\rho(1-\psi)\theta_2 \tag{43}$$

Plug in for θ_2:

$$\theta_1\sigma(1-\alpha\rho) = (1-\psi) + \alpha\rho(1-\psi)\frac{\zeta\gamma}{1-\alpha\beta\rho}\theta_1 \tag{44}$$

Now this is one equation in θ_1. Get rid of the fraction:

$$\theta_1\sigma(1-\alpha\beta\rho)(1-\alpha\rho) = (1-\alpha\beta\rho)(1-\psi) + \alpha\rho\zeta\gamma(1-\psi)\theta_1 \tag{45}$$

Now collect terms on the LHS:

$$\left[\sigma(1-\alpha\beta\rho)(1-\alpha\rho) - \alpha\rho\zeta\gamma(1-\psi)\right]\theta_1 = (1-\alpha\beta\rho)(1-\psi) \tag{46}$$

So:

$$\theta_1 = \frac{(1-\alpha\beta\rho)(1-\psi)}{\sigma(1-\alpha\beta\rho)(1-\alpha\rho) - \alpha\rho\zeta\gamma(1-\psi)} \tag{47}$$

Which in turn implies:

$$\theta_2 = \frac{\zeta\gamma(1-\psi)}{\sigma(1-\alpha\beta\rho)(1-\alpha\rho) - \alpha\rho\zeta\gamma(1-\psi)} \tag{48}$$

Given these coefficients, we can compute impulse responses to shocks. Recall that an impulse response is a displacement of conditional forecasts. The expected value of, say, x_{t+j} is $\alpha^j\theta_1\,\mathbb{E}_t\,r^f_{t+j}$. This is because α^j is the probability that we are in the ZLB regime; otherwise $x_{t+j} = 0$.

Impulse responses to a government spending and productivity shock are shown below for the case of $\alpha = 3/4$, which corresponds to an expected one year duration of the ZLB. These are similar to above, although more magnified, with the exception of the nominal rate response. In this setup, the impulse response of the nominal rate turns positive at a one period forecast horizon – this is because there is a probability that the ZLB lifts after just one period. But the basic story is the same as above – the nominal rate doesn't rise enough (relative to the natural rate) after a government spending shock, so output expands more; the nominal rate doesn't fall enough (relative to the natural rate) after a productivity shock, so output doesn't rise as much (or, as is the case here, actually falls).

Figure 7: IRFs to Government Spending Shock, $\alpha = 3/4$

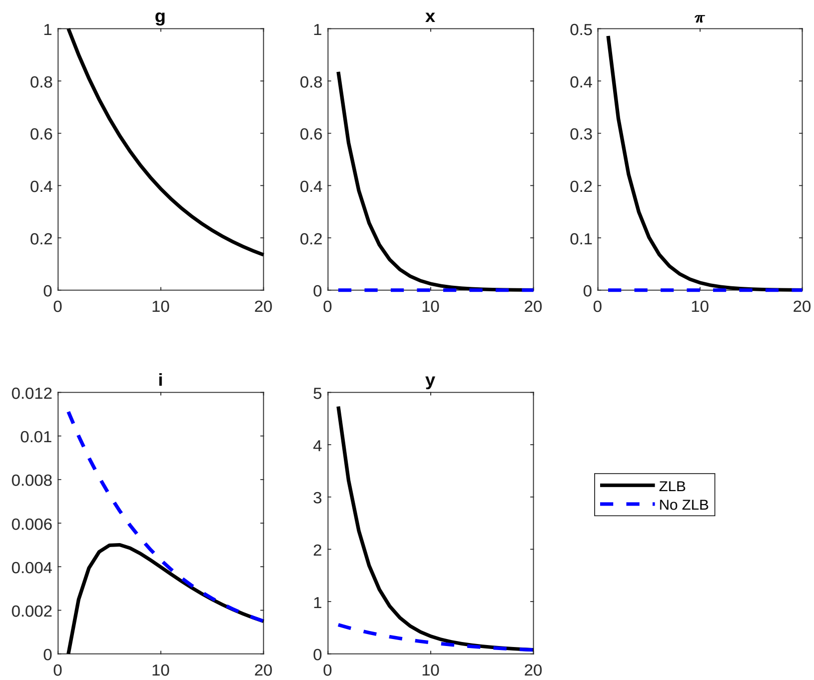
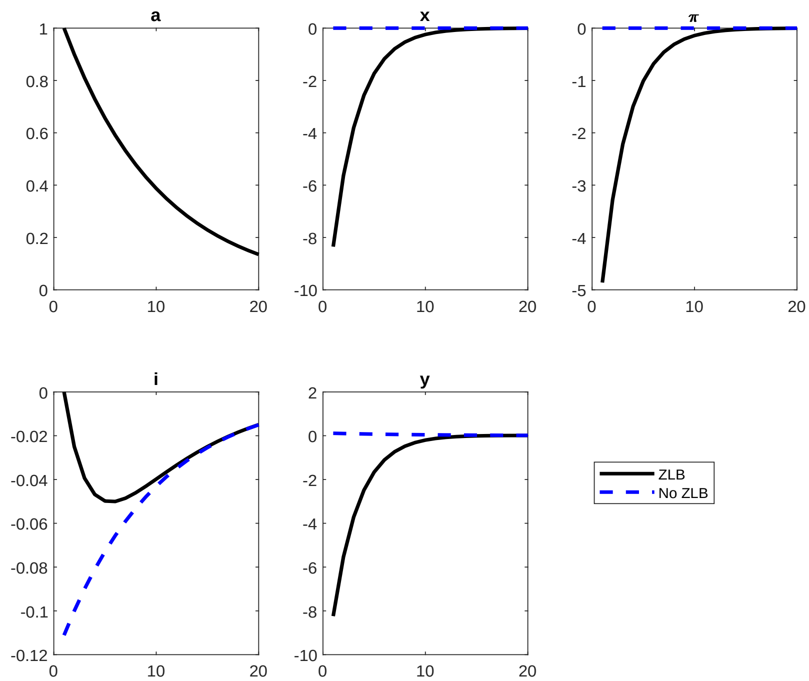


Figure 8: IRFs to Productivity Shock, $\alpha = 3/4$



A weird thing happens with the stochastic peg case. One can see this by looking at the solutions for θ_1 and θ_2. If α gets sufficiently big, the signs of the denominators are going to flip, causing the signs of θ_1 and θ_2 to both also flip. For the parameterization I'm using, this happens between $\alpha = 3/4$ and $\alpha = 4/5$. So when I solve the model with $\alpha = 4/5$, the government spending shock becomes *less* expansionary relative to the flexible price equilibrium, while the productivity shock becomes *more* expansionary. This issue of "sign reversals" with stochastic pegs is discussed in Carlstrom, Fuerst, and Paustian (2015, *JME*).

Figure 9: IRFs to Government Spending Shock, $\alpha = 4/5$

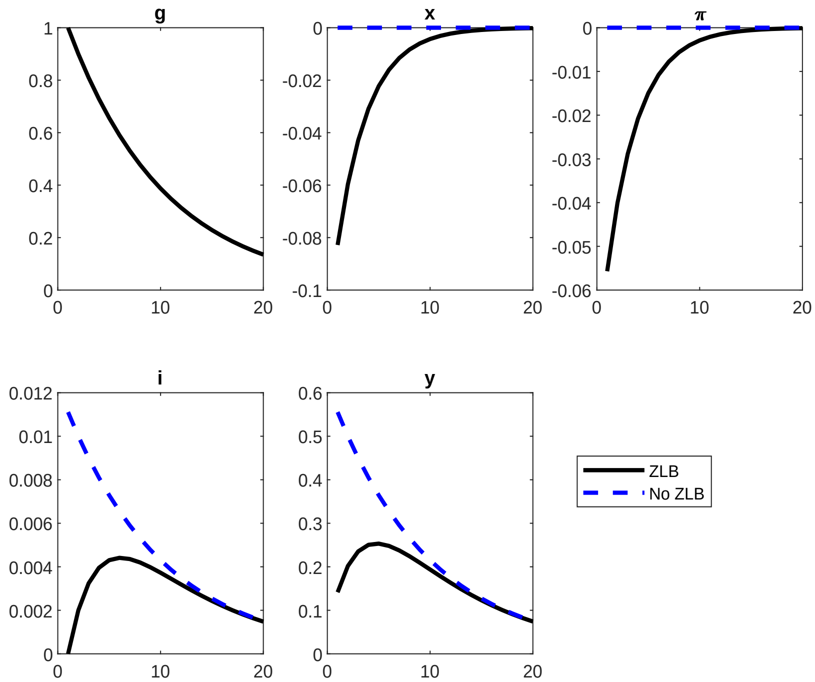


Figure 10: IRFs to Productivity Shock, $\alpha = 4/5$