Advanced Macro: Iacoviello (2005, *American Economic Review*)

Eric Sims
University of Notre Dame

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1 Overview

This note works through Iacoviello (2005, *AER*): "House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle." In the model, housing is both a source of consumption flows (for households) and also a factor of production (for entrepreneurs/firms). Because of a limited enforcement constraint as in Kiyotaki and Moore (1997), housing serves as collateral for entrepreneurs. In equilibrium, this borrowing constraint being binding results in too little housing being allocated to entrepreneurs.

In addition to these features, the model is a sticky price New Keynesian model. Importantly, debt is denominated in nominal terms. This generates a sort of formal "debt-deflation" mechanism. A contractionary monetary policy shock lowers inflation, which, other things being equal, tightens borrowers' collateral constraints, and further cramps demand.

The paper has two parts. A base model with no adjustment costs and no physical capital, and a more involved model with adjustment costs, capital accumulation, and additional shocks. I work through both parts.

2 Basic Model

The basic model is comprised of the following agents: patient households (who consume housing), entrepreneurs (who use housing as a production input), a competitive final goods producer, a continuum of retailers (they repackage entrepreneurial output and this is where price stickiness is included), and a monetary authority that sets nominal interest rates according to a Taylor rule.

2.1 Patient Household

Choices made by patient households are indicated with $'$ notation. They can choose consumption, c'_t; housing, h'_t, labor, L'_t; and borrowing, B'_t. The gross nominal return on borrowing is R_t, the nominal wage is W_t, the nominal house price is Q_t, and the price of goods is P_t. Money is included in the model of the paper, but money ends up being irrelevant when policy is set via an interest rate rule.

The household problem is:

$$\max_{c'_t, h'_t, L'_t, B'_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln c'_t + j \ln h'_t - \frac{(L'_t)^\eta}{\eta} \right\}$$

$$\text{s.t.}$$

$$P_t c'_t + Q_t h'_t + R_{t-1} B'_{t-1} \leq B'_t + W_t L'_t + P_t F_t - P_t T'_t$$

On the expenditure side of the budget constraint, the household can consume goods ($P_t c'_t$), consume housing ($Q_t h'_t$), and pays interest on its outstanding stock of debt, $R_{t-1} B'_{t-1}$. On the income side, the household earns labor income, $W_t L'_t$, has housing valued at $Q_t h_{t-1}$ that it inherited from the previous period, $P_t F_t$ denotes lump sum profits from firms, $P_t T'_t$ denotes transfers from the government/central bank. It can also issue new debt, B'_t.

Equation (1) in the paper re-writes this in real terms. Define $q_t = Q_t / P_t$ and $w_t = W_t / P_t$. Similarly, let $b_t = B_t / P_t$ denote real debt holdings, and $\pi_t = P_t / P_{t-1}$ as gross inflation. Dividing through by P_t and then using these, we would get:

$$c'_t + q_t h'_t + R_{t-1} b'_{t-1} / \pi_t \leq b'_t + w_t L'_t + q_t h'_{t-1} + F_t - T'_t \tag{1}$$

A Lagrangian where we take the budget constraint written in real terms is:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln c'_t + j \ln h'_t - \frac{(L'_t)^\eta}{\eta} + \lambda'_t \left[b'_t + w_t L'_t + q_t h'_{t-1} + F_t - T'_t - c'_t - q_t h'_t - R_{t-1} b_{t-1} / \pi_t \right] \right\}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial c'_t} = \frac{1}{c'_t} - \lambda'_t$$

$$\frac{\partial \mathbb{L}}{\partial h'_t} = \frac{j}{h'_t} - \lambda'_t q_t + \beta \, \mathbb{E}_t \, \lambda'_{t+1} q_{t+1}$$

$$\frac{\partial \mathbb{L}}{\partial L'_t} = (L'_t)^{\eta - 1} - \lambda'_t w_t$$

$$\frac{\partial \mathbb{L}}{\partial b'_t} = \lambda'_t + \beta \, \mathbb{E}_t \, \lambda'_{t+1} R_t / \pi_{t+1}$$

Setting equal to zero and eliminating λ'_t yields:

$$\frac{q_t}{c'_t} = \frac{j}{h'_t} + \beta \, \mathbb{E}_t \frac{q_{t+1}}{c'_{t+1}} \tag{2}$$

$$(L'_t)^{\eta - 1} = \frac{w_t}{c'_t} \tag{3}$$

$$\frac{1}{c'_t} = \beta \, \mathbb{E}_t \frac{1}{c'_{t+1}} \frac{R_t}{\pi_{t+1}} \tag{4}$$

(3) is a standard labor supply condition, and (4) is the standard Euler equation for bonds. The intuition for (2) is as follows. Purchasing an additional unit of housing costs q_t units of consumption, which is valued at $\frac{1}{c'_t}$ in terms of utility. Hence, the left hand side is the marginal utility cost of purchasing more housing. The first term on the right hand side is the marginal utility, in period t, of having more housing. The second term is the extra utility one gets in period $t+1$ from purchasing more housing in t – purchasing more housing in t generates q_{t+1} additional units of income in $t+1$, which is valued at β/c'_{t+1}. One could alternatively write (2) as:

$$q_t = \frac{jc'_t}{h'_t} + \beta\,\mathbb{E}_t\,\frac{c'_t}{c'_{t+1}}q_{t+1} \tag{5}$$

(5) says that the price of housing, q_t, is equal to the flow benefit of housing, measured in units of consumption, in period t, $\frac{jc'_t}{h'_t}$, plus the expected value of the product of the stochastic discount factor, $\beta c'_t/c'_{t+1}$, with the future price, q_{t+1}.

2.2 Entrepreneurs

Entrepreneurs produce an intermediate good, $Y_{w,t}$, using their stock of real estate and labor hired from the patient household. In particular:

$$Y_{w,t} = Ah^{\nu}_{t-1}L^{1-\nu}_t \tag{6}$$

I'm changing the notation somewhat by adding an w subscript (for "wholesale"); see below. This intermediate output is sold to retailers before being available for consumption. Intermediate output is sold to retailers at P^w_t. Repacked intermediate output is then sold at the aforementioned P_t. $X_t = P_t/P^w_t$ is the markup. Really the w notation stands for "wholesale" rather than retail.

Entrepreneurs do not work. They discount future utility flows at $\gamma < \beta$. They are also subject to a collateral constraint on their housing. Their objective and budget constraints are:

$$\max_{c_t,h_t,L_t,b_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \ln c_t$$

s.t.

$$P^w_t Ah^{\nu}_{t-1}L^{1-\nu}_t - W_tL_t + B_t + Q_th_{t-1} \leq P_tc_t + Q_th_t + R_{t-1}B_{t-1}$$

$$B_t \leq m\,\mathbb{E}\left[\frac{Q_{t+1}h_t}{R_t}\right]$$

The budget constraint says that entrepreneurial resources (left hand side) are the value of output less payments to labor, plus new debt issued, plus the value of the existing housing stock. On the expenditure side, the entrepreneur can consume goods or new housing and pays interest plus principal on its outstanding debt. The borrowing constraint says that borrowing in the present cannot exceed the discounted expected value of future housing. Next period's expected value of housing, $Q_{t+1}h_t$, in effect serves as collateral. m is a parameter between zero and one. If an entrepreneur defaults in $t+1$, the creditor can recover $(1-m)Q_{t+1}h_t$. Hence, the most an entrepreneur can borrow is $m\,\mathbb{E}\left[Q_{t+1}h_t/R_t\right]$.

We can re-write the constraints in real terms by dividing by P_t:

$$\frac{Ah_{t-1}^{\nu}L_t^{1-\nu}}{X_t} - w_tL_t + b_t + q_th_{t-1} \leq c_t + q_th_t + R_{t-1}b_{t-1}/\pi_t \tag{7}$$

$$b_t \leq m\,\mathbb{E}\left[\frac{q_{t+1}h_t\pi_{t+1}}{R_t}\right] \tag{8}$$

Form a Lagrangian:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \left\{ \ln c_t + \mu_t \left[\frac{Ah_{t-1}^{\nu}L_t^{1-\nu}}{X_t} - w_tL_t + b_t + q_th_{t-1} - c_t - q_th_t - R_{t-1}b_{t-1}/\pi_t\right] + \right.$$
$$\left. \lambda_t \left[m\,\mathbb{E}_t\, q_{t+1}h_t\pi_{t+1} - b_tR_t\right] \right\}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial c_t} = \frac{1}{c_t} - \mu_t$$

$$\frac{\partial \mathbb{L}}{\partial L_t} = \mu_t \left[\frac{(1-\nu)Ah_{t-1}^{\nu}L_t^{-\nu}}{X_t} - w_t\right]$$

$$\frac{\partial \mathbb{L}}{\partial h_t} = -\mu_tq_t + m\lambda_t\,\mathbb{E}_t\, q_{t+1}\pi_{t+1} + \gamma\,\mathbb{E}_t\,\mu_{t+1}\left[\frac{\nu Ah_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1}\right]$$

$$\frac{\partial \mathbb{L}}{\partial b_t} = \mu_t - \lambda_tR_t + \gamma\,\mathbb{E}_t\,\mu_{t+1}R_t/\pi_{t+1}$$

Setting these equal to zero and eliminating the multiplier on the budget constraint, we get:

$$(1-\nu)Ah_{t-1}^{\nu}L_t^{-\nu} = X_tw_t \tag{9}$$

$$\frac{q_t}{c_t} = \mathbb{E}_t\left[\frac{\gamma}{c_{t+1}}\left(\frac{\nu Ah_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1}\right) + m\lambda_tq_{t+1}\pi_{t+1}\right] \tag{10}$$

$$\frac{1}{c_t} = \gamma\,\mathbb{E}_t\,\frac{1}{c_{t+1}}\frac{R_t}{\pi_{t+1}} + \lambda_tR_t \tag{11}$$

(9)-(11) are the same as in the paper. If you like, you can re-arrange (10) to be:

$$q_t = \mathbb{E}_t\left[\frac{\gamma c_t}{c_{t+1}}\left(\frac{\nu Ah_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1}\right) + m\lambda_tc_tq_{t+1}\pi_{t+1}\right] \tag{12}$$

(12) says that the price of real estate is the expectation of the stochastic discount factor with

(i) the flow payout, which is the marginal product of housing received in $t+1$, plus (ii) the continuation value, q_{t+1}. The final term is the amount by which having more housing eases the borrowing constraint; λ_t is the shadow value (in utils) of easing the constraint, so $\lambda_t c_t$ puts this into units of consumption. (9) is a standard labor demand condition. (11) is a standard Euler equation for bonds, except for the $\lambda_t R_t$ term at the end. $\lambda_t R_t$ is effectively how much more you could borrow by relaxing the constraint.

2.3 Final Goods and Retailers

There are a continuum of retailers indexed by $z \in [0,1]$. They costlessly transform wholesale output, $Y_{w,t}$, purchased at P_t^w, into retail output, $Y_t(z)$. They then sell this retail output to a competitive final goods firm at $P_t(z)$. The competitive final goods firm produces final output:

$$Y_t = \left(\int_0^1 Y_t(z)^{\frac{\epsilon-1}{\epsilon}} dz \right)^{\frac{\epsilon}{\epsilon-1}} \tag{13}$$

Demand for each retail good is:

$$Y_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t \tag{14}$$

And the price index is:

$$P_t^{1-\epsilon} = \int_0^1 P_t(z)^{1-\epsilon} dz \tag{15}$$

Note that there is bad notation in the paper, and footnote 10 about aggregation is wrong. So I am changing things up a bit.

Retailers can update their price in each period with probability $1-\theta$. They discount future profits via the stochastic discount factor of patient households, $\Lambda_{t,t+k} = \beta^k \frac{c'_t}{c'_{t+k}}$. Flow nominal profit for each intermediary is:

$$F_t(z)^n = P_t(z) Y_t(z) - P_t^w Y_t(z)$$

They produce $Y_t(z)$, and they use $Y_{w,t}$ as input, but this is transformed costlessly into $Y_t(z)$, so we can eliminate $Y_{w,t}$ and just write this in terms of $Y_t(z)$. Plugging in the demand function:

$$F_t(z)^n = P_t(z)^{1-\epsilon} P_t^{\epsilon} Y_t - P_t^w P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t$$

Write this in real terms by dividing by P_t:

$$F_t(z) = P_t(z)^{1-\epsilon} P_t^{\epsilon-1} Y_t - X_t^{-1} P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t$$

Where again $X_t = P_t / P_t^w$. The problem of a firm getting to reset is therefore to pick $P_t(z)$ to maximize the PDV of $F_t(z)$, where discounting is by the stochastic discount factor, $\Lambda_{t,t+k}$, as well as the probability that a price chosen in t is still in place in period $t+k$, θ^k:

$$\max_{P_t(z)} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \left\{ \Lambda_{t,t+k} \left(P_t(z)^{1-\epsilon} P_{t+k}^{\epsilon-1} Y_{t+k} - X_{t+k}^{-1} P_t(z)^{-\epsilon} P_{t+k}^{\epsilon} Y_{t+k} \right) \right\}$$

The FOC is:

$$(\epsilon - 1) P_t(z)^{-\epsilon} \, \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} Y_{t+k} = \epsilon P_t(z)^{-\epsilon-1} \, \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} X_{t+k}^{-1} P_{t+k}^{\epsilon} Y_{t+k}$$

Or:

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} X_{t+k}^{-1} P_{t+k}^{\epsilon} Y_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} Y_{t+k}}$$

We can write the numerator and denominator recursively as:

$$Z_{1,t} = X_t^{-1} P_t^{\epsilon} Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} Z_{1,t+1}$$

$$Z_{2,t} = P_t^{\epsilon-1} Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} Z_{2,t+1}$$

We will need to re-scale these to get rid of price levels. Define $z_{1,t} = Z_{1,t}/P_t^{\epsilon}$ and $z_{2,t} = Z_{2,t}/P_t^{\epsilon-1}$. We then have:

$$z_{1,t} = X_t^{-1} Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} \pi_{t+1}^{\epsilon} z_{1,t+1} \tag{16}$$

$$z_{2,t} = Y_t + \theta \, \mathbb{E}_t \, \Lambda_{t,t+1} \pi_{t+1}^{\epsilon-1} z_{2,t+1} \tag{17}$$

Since $Z_{1,t}/Z_{2,t} = z_{1,t}/z_{2,t} \times P_t$, we can then define $\pi_t^* = P_t^*/P_t$ as the relative reset price. Then we simply have:

$$\pi_t^* = \frac{\epsilon}{\epsilon - 1} \frac{z_{1,t}}{z_{2,t}} \tag{18}$$

2.4 Monetary Policy

The monetary policy rule is:

$$R_t = (\bar{rr})^{1-r_R} (R_{t-1})^{r_R} \left(\pi_{t-1}^{1+r_\pi} (Y_{t-1}/Y)^{r_Y} \right)^{1-r_R} e_{R,t} \tag{19}$$

Here $\bar{rr}$ is the steady state nominal rate and Y is steady state output. r_R is a smoothing parameter, $1 + r_\pi$, with $r_\pi > 0$, is the coefficient on lagged inflation, and r_Y is the coefficient on the deviation of output from steady state. $e_{R,t}$ is a shock. Note that the Taylor rule is purely backward-looking.

2.5 Aggregation

The aggregate price level evolves according to (15) taking into account properties of Calvo pricing:

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon} \tag{20}$$

Divide both sides by $P_t^{1-\epsilon}$ to write this in terms of inflation rates:

$$1 = \theta \pi_t^{\epsilon-1} + (1-\theta)(\pi_t^*)^{1-\epsilon} \tag{21}$$

Integrate (14) across z, noting that $Y_t(z) = Ah_{t-1}^{\nu} L_t^{1-\nu}$. We get:

$$Ah_{t-1}^{\nu} L_t^{1-\nu} = Y_t v_t^p \tag{22}$$

v_t^p is a measure of price dispersion, which can be written:

$$v_t^p = (1-\theta)(\pi_t^*)^{-\epsilon} + \theta \pi_t^{\epsilon} v_{t-1}^p \tag{23}$$

Sum the budget constraints of the patient household and the entrepreneur together:

$$c_t + c_t' + q_t(h_t + h_t') + \frac{R_{t-1}}{\pi_t}(b_{t-1} + b_{t-1}') = (b_t + b_t') + q_t(h_{t-1} + h_{t-1}') + \frac{Ah_{t-1}^{\nu} L_t^{1-\nu}}{X_t} + F_t - T_t'$$

Here I have imposed labor market-clearing, so that the $w_t L_t'$ and $w_t L_t$ terms cancel (i.e. $L_t = L_t'$). Market-clearing for bonds requires $b_t + b_t' = 0$ (i.e. one lends, the other borrows). The aggregate stock of housing is fixed at H, so $h_t + h_t' = H$. But then these terms on the left and right hand sides cancel, leaving:

$$c_t + c_t' = \frac{Ah_{t-1}^{\nu} L_t^{1-\nu}}{X_t} + F_t - T_t'$$

Because we have omitted money, $T_t' = 0$ (i.e. there is no transfer/tax from the government). What about F_t? Recall from above that we have:

$$F_t(z) = P_t(z)^{1-\epsilon} P_t^{\epsilon-1} Y_t - X_t^{-1} P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t$$

Aggregate profits are just profits integrated across retailers:

$$F_t = \int_0^1 F_t(z) = P_t^{\epsilon-1} Y_t \int_0^1 P_t(z)^{1-\epsilon} dz - X_t^{-1} Y_t \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz$$

Now, note from above that $\int_0^1 P_t(z)^{1-\epsilon} dz = P_t^{1-\epsilon}$, so in the first term the price terms just drop out. In the second term, the term inside the integral, $\int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon} dz = v_t^p$. So we have:

$$F_t = Y_t - \frac{Y_t v_t^p}{X_t}$$

But since $Ah_{t-1}^{\nu} L_t^{1-\nu} = Y_t v_t^p$, the summed budget constraints work out to the typical resource constraint:

$$c_t + c_t' = Y_t \tag{24}$$

2.6 Full Set of Equilibrium Conditions

For completeness, the full set of equilibrium conditions are presented below:

$$\frac{q_t}{c_t'} = \frac{j}{h_t'} + \beta\,\mathbb{E}_t \frac{q_{t+1}}{c_{t+1}'} \tag{25}$$

$$(L_t)^{\eta-1} = \frac{w_t}{c_t'} \tag{26}$$

$$\frac{1}{c_t'} = \beta\,\mathbb{E}_t \frac{1}{c_{t+1}'} \frac{R_t}{\pi_{t+1}} \tag{27}$$

$$(1-\nu) A h_{t-1}^{\nu} L_t^{-\nu} = X_t w_t \tag{28}$$

$$\frac{q_t}{c_t} = \mathbb{E}_t \left[\frac{\gamma}{c_{t+1}} \left(\frac{\nu A h_t^{\nu-1} L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1} \right) + m \lambda_t q_{t+1} \pi_{t+1} \right] \tag{29}$$

$$\frac{1}{c_t} = \gamma\,\mathbb{E}_t \frac{1}{c_{t+1}} \frac{R_t}{\pi_{t+1}} + \lambda_t R_t \tag{30}$$

$$b_t = m\,\mathbb{E}_t (q_{t+1} h_t \pi_{t+1} / R_t) \tag{31}$$

$$z_{1,t} = X_t^{-1} Y_t + \theta\,\mathbb{E}_t\, \Lambda_{t,t+1} \pi_{t+1}^{\epsilon} z_{1,t+1} \tag{32}$$

$$z_{2,t} = Y_t + \theta\,\mathbb{E}_t\, \Lambda_{t,t+1} \pi_{t+1}^{\epsilon-1} z_{2,t+1} \tag{33}$$

$$\pi_t^* = \frac{\epsilon}{\epsilon-1} \frac{z_{1,t}}{z_{2,t}} \tag{34}$$

$$R_t = (\bar{r r})^{1-r_R} (R_{t-1})^{r_R} \left(\pi_{t-1}^{1+r_\pi} (Y_{t-1}/Y)^{r_Y} \right)^{1-r_R} e_{R,t} \tag{35}$$

$$1 = \theta \pi_t^{\epsilon-1} + (1-\theta)(\pi_t^*)^{1-\epsilon} \tag{36}$$

$$A h_{t-1}^{\nu} L_t^{1-\nu} = Y_t v_t^p \tag{37}$$

$$v_t^p = (1-\theta)(\pi_t^*)^{-\epsilon} + \theta \pi_t^{\epsilon} v_{t-1}^p \tag{38}$$

$$c_t + c_t' = Y_t \tag{39}$$

$$h_t + h_t' = H \tag{40}$$

$$b_t = c_t + q_t (h_t - h_{t-1}) + \frac{R_{t-1} b_{t-1}}{\pi_t} + w_t L_t - \frac{Y_t v_t^p}{X_t} \tag{41}$$

This is 17 variables, $\left\{c'_t, h'_t, L_t, c_t, h_t, b_t, Y_t, X_t, v^p_t, q_t, w_t, R_t, \pi_t, \pi^*_t, z_{1,t}, z_{2,t}, \lambda_t\right\}$ and 17 equations.

2.7 Log-Linearization

Let's log-linearize these conditions about the steady state. Denote the steady state with variables without a time subscript. Let variables with "hats" denote log-deviations.

Without showing the work, we know that the price-setting conditions, (32)-(34), plus (36), become the standard Phillips Curve:

$$\widehat{\pi}_t = -\kappa \widehat{X}_t + \beta\, \mathbb{E}_t\, \widehat{\pi}_{t+1} \tag{42}$$

The resource constraint is:

$$\widehat{Y}_t = \frac{c}{Y}\widehat{c}_t + \frac{c'}{Y}\widehat{c}'_t \tag{43}$$

The patient household's linearized Euler equation is:

$$\widehat{c}'_t = \mathbb{E}_t\, \widehat{c}'_{t+1} - \widehat{rr}_t \tag{44}$$

Where $\widehat{rr}_t$ is the real interest rate:

$$\widehat{rr}_t = \widehat{R}_t - \mathbb{E}_t\, \widehat{\pi}_{t+1} \tag{45}$$

The linearized household labor supply condition is:

$$(\eta - 1)\widehat{L}_t = \widehat{w}_t - \widehat{c}'_t \tag{46}$$

The linearized labor demand condition is:

$$\nu \widehat{h}_{t-1} - \nu \widehat{L}_t = \widehat{w}_t + \widehat{X}_t \tag{47}$$

Re-write the Euler equation for housing for the patient household:

$$q_t = j\frac{c'_t}{h'_t} + \beta\, \mathbb{E}_t\, \frac{c'_t}{c'_{t+1}} q_{t+1}$$

Take logs:

$$\ln q_t = \ln\left[j\frac{c'_t}{h'_t} + \beta\, \mathbb{E}_t\, \frac{c'_t}{c'_{t+1}} q_{t+1}\right]$$

Totally differentiate (ignoring the expectation operator):

$$\frac{dq_t}{q} = \frac{1}{q}\left[\frac{j}{h'}dc'_t - j\frac{c'}{(h')^2}dh'_t + \beta q\frac{1}{c'}dc'_t - \beta q\frac{c}{(c')^2}dc'_{t+1} + \beta dq_{t+1}\right]$$

Simplifying:

$$\widehat{q}_t = j\frac{c'}{qh'}\left[\widehat{c}'_t - \widehat{h}'_t\right] + \beta\widehat{c}'_t - \beta\,\mathbb{E}_t\,\widehat{c}'_{t+1} + \beta\,\mathbb{E}_t\,\widehat{q}_{t+1} \tag{48}$$

Now take logs of the bond Euler equation for entrepreneurs:

$$-\ln c_t = \ln\left[\frac{\gamma}{c_{t+1}}\frac{R_t}{\pi_{t+1}} + \lambda_t R_t\right]$$

Now totally differentiate:

$$-\frac{dc_t}{c} = c\left[-\frac{\gamma}{c^2}Rdc_{t+1} + \frac{\gamma}{c}dR_t - \frac{\gamma R}{c}d\pi_{t+1} + Rd\lambda_t + \lambda dR_t\right]$$

Using hat notation and re-arranging a bit:

$$-\widehat{c}_t = -\frac{\gamma}{\beta}\,\mathbb{E}_t\,\widehat{c}_{t+1} + \frac{\gamma}{\beta}\widehat{rr}_t + \frac{\lambda c}{\beta}\left(\widehat{\lambda}_t + \widehat{R}_t\right) \tag{49}$$

Now focus on the Euler equation for housing for the entrepreneur. Re-arrange slightly:

$$q_t = \mathbb{E}_t\left[\frac{\gamma c_t}{c_{t+1}}\left(\frac{\nu A h_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1}\right) + m\lambda_t c_t q_{t+1}\pi_{t+1}\right]$$

Take logs, ignoring the expectations operator:

$$\ln q_t = \ln\left[\frac{\gamma c_t}{c_{t+1}}\left(\frac{\nu A h_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} + q_{t+1}\right) + m\lambda_t c_t q_{t+1}\pi_{t+1}\right]$$

Totally differentiate, noting that $\frac{\nu A h_t^{\nu-1}L_{t+1}^{1-\nu}}{X_{t+1}} = \frac{\nu Y_{t+1}}{h_t X_{t+1}}$:

$$\frac{dq_t}{q} = \frac{1}{q}\Bigg[\left(\frac{\gamma}{c}dc_t - \frac{\gamma}{c}dc_{t+1}\right)\left(\frac{\nu Y}{hX} + q\right) + \gamma\left(\frac{\nu}{hX}dY_{t+1} - \frac{\nu Y}{hX^2}dX_{t+1} - \frac{\nu Y}{h^2X}dh_t + dq_{t+1}\right)$$
$$mcqd\lambda_t + m\lambda q dc_t + m\lambda c dq_{t+1} + m\lambda cqd\pi_{t+1}\Bigg]$$

Using hat notation, this can be simplified to:

$$\widehat{q}_t = \gamma(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1})\left(\frac{\nu Y}{qhX} + 1\right) + \frac{\gamma\nu Y}{qhX}\left(\mathbb{E}_t\,\widehat{Y}_{t+1} - \mathbb{E}_t\,\widehat{X}_{t+1} - \widehat{h}_t\right) + \gamma\widehat{q}_{t+1} + m\lambda c\left(\widehat{\lambda}_t + \widehat{c}_t + \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{\pi}_{t+1}\right) \tag{50}$$

Now, linearize the borrowing constraint:

$$\ln b_t = \ln m + \ln q_{t+1} + \ln h_t + \ln \pi_{t+1} - \ln R_t$$

Or:

$$\widehat{b}_t = \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{h}_t - \widehat{rr}_t \tag{51}$$

The price-setting conditions are already linearized and expressed via the Phillips Curve.

The Taylor rule is log-linear:

$$\widehat{R}_t = r_R \widehat{R}_{t-1} + (1 - r_R) \left[(1 + r_\pi) \widehat{\pi}_{t-1} + r_Y \widehat{Y}_{t-1} \right] + \widehat{e}_{R,t} \tag{52}$$

The production function is log-linear, noting that price dispersion is constant to first order (so we can drop it):

$$\widehat{Y}_t = \nu \widehat{h}_{t-1} + (1 - \nu) \widehat{L}_t \tag{53}$$

The resource constraint and house market-clearing condition are fairly straightforward:

$$\widehat{Y}_t = \frac{c}{Y} \widehat{c}_t + \frac{c'}{Y'} \widehat{c}'_t \tag{54}$$

$$\widehat{h}'_t = -\frac{h'}{h} \widehat{h}_t \tag{55}$$

Now, we need to log-linearize the budget constraint for the entrepreneur. First, re-arrange with c_t isolated on the LHS:

$$c_t = b_t - q_t(h_t - h_{t-1}) - \frac{R_{t-1} b_{t-1}}{\pi_t} - w_t L_t + \frac{Y_t v_t^p}{X_t}$$

Now take logs:

$$\ln c_t = \ln \left[b_t - q_t(h_t - h_{t-1}) - \frac{R_{t-1} b_{t-1}}{\pi_t} - w_t L_t + \frac{Y_t v_t^p}{X_t} \right]$$

Now totally differentiate:

$$\frac{dc_t}{c} = \frac{1}{c} \left[db_t - dq_t(0) - q dh_t + q dh_{t-1} - b dR_{t-1} - R db_{t-1} + Rb d\pi_t - w dL_t - L dw_t + \frac{1}{X} dY_t - \frac{Y}{X^2} dX_t \right]$$

Which is:

$$\widehat{c}_t = \frac{1}{c} \left[b \widehat{b}_t - qh \widehat{h}_t + qh \widehat{h}_{t-1} - \frac{b}{\beta} \left(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \pi_t \right) - wL(\widehat{w}_t + \widehat{L}_t) + \frac{Y}{X} (\widehat{Y}_t - \widehat{X}_t) \right]$$

Or:

$$c \widehat{c}_t = b \widehat{b}_t - qh \widehat{h}_t + qh \widehat{h}_{t-1} - \frac{b}{\beta} \left(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \pi_t \right) - wL(\widehat{w}_t + \widehat{L}_t) + \frac{Y}{X} (\widehat{Y}_t - \widehat{X}_t) \tag{56}$$

2.7.1 Full Set of Linearized Conditions

$$\widehat{q}_t = j \frac{c'}{qh'} \left[\widehat{c}'_t - \widehat{h}'_t \right] + \beta \widehat{c}'_t - \beta \, \mathbb{E}_t \, \widehat{c}'_{t+1} + \beta \, \mathbb{E}_t \, \widehat{q}_{t+1} \tag{57}$$

$$(\eta-1)\widehat{L}_t = \widehat{w}_t - \widehat{c}'_t \tag{58}$$

$$\widehat{c}'_t = \mathbb{E}_t\,\widehat{c}'_{t+1} - \widehat{rr}_t \tag{59}$$

$$\nu\widehat{h}_{t-1} - \nu\widehat{L}_t = \widehat{w}_t + \widehat{X}_t \tag{60}$$

$$\widehat{q}_t = \gamma(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1})\left(\frac{\nu Y}{qhX}+1\right) + \frac{\gamma\nu Y}{qhX}\left(\mathbb{E}_t\,\widehat{Y}_{t+1} - \mathbb{E}_t\,\widehat{X}_{t+1} - \widehat{h}_t\right) + \gamma\widehat{q}_{t+1} + m\lambda c\left(\widehat{\lambda}_t + \widehat{c}_t + \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{\pi}_{t+1}\right) \tag{61}$$

$$-\widehat{c}_t = -\frac{\gamma}{\beta}\,\mathbb{E}_t\,\widehat{c}_{t+1} + \frac{\gamma}{\beta}\widehat{rr}_t + \frac{\lambda c}{\beta}\left(\widehat{\lambda}_t + \widehat{R}_t\right) \tag{62}$$

$$\widehat{b}_t = \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{h}_t - \widehat{rr}_t \tag{63}$$

$$\widehat{\pi}_t = -\kappa\widehat{X}_t + \beta\,\mathbb{E}_t\,\widehat{\pi}_{t+1} \tag{64}$$

$$\widehat{R}_t = r_R\widehat{R}_{t-1} + (1-r_R)\left[(1+r_\pi)\widehat{\pi}_{t-1} + r_Y\widehat{Y}_{t-1}\right] + \widehat{e}_{R,t} \tag{65}$$

$$\widehat{Y}_t = \nu\widehat{h}_{t-1} + (1-\nu)\widehat{L}_t \tag{66}$$

$$\widehat{Y}_t = \frac{c}{Y}\widehat{c}_t + \frac{c'}{Y'}\widehat{c}'_t \tag{67}$$

$$\widehat{h}'_t = -\frac{h}{h'}\widehat{h}_t \tag{68}$$

$$c\widehat{c}_t = b\widehat{b}_t - qh\widehat{h}_t + qh\widehat{h}_{t-1} - \frac{b}{\beta}\left(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \pi_t\right) - wL(\widehat{w}_t + \widehat{L}_t) + \frac{Y}{X}(\widehat{Y}_t - \widehat{X}_t) \tag{69}$$

$$\widehat{rr}_t = \widehat{R}_t - \mathbb{E}_t\,\widehat{\pi}_{t+1} \tag{70}$$

This is 14 variables $\left\{\widehat{c}'_t, \widehat{L}_t, \widehat{h}'_t, \widehat{c}_t, \widehat{h}_t, \widehat{b}_t, \widehat{Y}_t, \widehat{X}_t, \widehat{R}_t, \widehat{q}_t, \widehat{\pi}_t, \widehat{w}_t, \widehat{rr}_t, \widehat{\lambda}_t\right\}$ and 14 equations. In the paper, he lists nine equations (really ten, because the real interest rate is a separate in-text equation), but focuses only $\left\{\widehat{c}'_t, \widehat{c}_t, \widehat{h}_t, \widehat{Y}_t, \widehat{X}_t, \widehat{rr}_t, \widehat{R}_t, \widehat{b}_t, \widehat{q}_t, \widehat{\pi}_t\right\}$ (i.e. ten variables). The variables $\widehat{L}_t$, $\widehat{h}'_t$, $\widehat{\lambda}_t$, and $\widehat{w}_t$ have been eliminated. Let's eliminate these and see if we can recover what Iacoviello has in the paper (equations L.1-L.9).

First, combine labor supply and demand to eliminate $\widehat{w}_t$:

$$(\eta-1)\widehat{L}_t + \widehat{c}'_t = \nu\widehat{h}_{t-1} - \nu\widehat{L}_t - \widehat{X}_t$$

But then we can solve for $\widehat{L}_t$ as:

$$\widehat{L}_t = \frac{\nu}{\eta-(1-\nu)}\widehat{h}_{t-1} - \frac{1}{\eta-(1-\nu)}\left(\widehat{X}_t + \widehat{c}'_t\right)$$

Now plug this into the linearized production function. That gives:

$$\widehat{Y}_t = \nu\widehat{h}_{t-1} + \frac{\nu(1-\nu)}{\eta-(1-\nu)}h_{t-1} - \frac{1-\nu}{\eta-(1-\nu)}\left(\widehat{X}_t + \widehat{c}'_t\right)$$

Which simplifies to:

$$\widehat{Y}_t = \frac{\eta\nu}{\eta-(1-\nu)}\widehat{h}_{t-1} - \frac{1-\nu}{\eta-(1-\nu)}\left(\widehat{X}_t + \widehat{c}'_t\right) \tag{71}$$

(71) is exactly the same as (L7) in the paper.

Now, if you look at (25), we can solve for something about the steady state. Which is:

$$\frac{j}{h'} = \frac{q}{c'}(1-\beta)$$

But this means that:

$$\frac{jc'}{qh'} = 1-\beta$$

Now, use this, along with the fact that $\widehat{h}'_t = -\frac{h'}{h}\widehat{h}_t$, and we can write (57) as:

$$\widehat{q}_t = \widehat{c}'_t + (1-\beta)\frac{h}{h'}\widehat{h}_t - \beta\,\mathbb{E}_t\,\widehat{c}'_{t+1} + \beta\,\mathbb{E}_t\,\widehat{q}_{t+1} \tag{72}$$

(72) is the same as (L1) in the paper with $\iota = (1-\beta)h/h'$.

Now, let's re-write the budget constraint for the entrepreneur, subbing out $\widehat{w}_t$ and $\widehat{L}_t$. First, note that in steady state, we have $wL = \frac{(1-\nu)Y}{X}$. Second, from (60), we can write:

$$\widehat{w}_t + \widehat{L}_t = \widehat{Y}_t - \widehat{X}_t$$

Hence, we have the term:

$$-wL(\widehat{w}_t + \widehat{L}_t) + \frac{Y}{X}(\widehat{Y}_t - \widehat{X}_t) = -\frac{Y}{X}\left((1-\nu)(\widehat{Y}_t - \widehat{X}_t) - (\widehat{Y}_t - \widehat{X}_t)\right) = \frac{\nu Y}{X}(\widehat{Y}_t - \widehat{X}_t)$$

But then we can write the entrepreneur budget constraint as:

$$c\widehat{c}_t = b\widehat{b}_t - qh\widehat{h}_t + qh\widehat{h}_{t-1} - \frac{b}{\beta}\left(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \pi_t\right) + \frac{\nu Y}{X}(\widehat{Y}_t - \widehat{X}_t) \tag{73}$$

Now, we need to deal with the Euler equation for housing for the entrepreneur. It is first helpful to start with the steady state. In steady state, we can write:

$$q = \gamma q\left(\frac{\nu Y}{qhX} + 1\right) + m\lambda q$$

But then the q drop out, leaving:

$$\gamma\left(\frac{\nu Y}{qhX} + 1\right) = 1 - m\lambda c$$

And consequently:

$$\frac{\gamma\nu Y}{qhX} = 1 - m\lambda c - \gamma \tag{74}$$

This means we can write the Euler equation for housing for the entrepreneur as:

$$\widehat{q}_t = (1-m\lambda c)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) + \gamma\widehat{q}_{t+1} + m\lambda c(\widehat{\lambda}_t + \widehat{c}_t + \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{\pi}_{t+1})$$

Solve for $\widehat{\lambda}_t$ from the Euler equation for bonds:

$$\lambda c\widehat{\lambda}_t = -\beta\widehat{c}_t + \gamma\widehat{c}_{t+1} - \gamma\widehat{rr}_t - \lambda c(\widehat{rr}_t + \widehat{\pi}_{t+1})$$

Now combine these. We get:

$$\begin{aligned}\widehat{q}_t = (1-m\lambda c)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) + \gamma\widehat{q}_{t+1} - \\ m\beta\widehat{c}_t + m\gamma\widehat{c}_{t+1} - m\gamma\widehat{rr}_t - m\lambda c(\widehat{rr}_t + \widehat{\pi}_{t+1}) + m\lambda c\widehat{c}_t + m\lambda c\widehat{q}_{t+1} + m\lambda c\widehat{\pi}_{t+1}\end{aligned}$$

Now group terms:

$$\begin{aligned}\widehat{q}_t = (1-m\lambda c)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) + \gamma\widehat{q}_{t+1} + \\ m(\lambda c-\beta)\widehat{c}_t + m\gamma\widehat{c}_{t+1} - m\gamma\widehat{rr}_t - m\lambda c(\widehat{rr}_t + \widehat{\pi}_{t+1}) + m\lambda c\widehat{q}_{t+1} + m\lambda c\widehat{\pi}_{t+1}\end{aligned}$$

Now, note that in steady state, $\lambda c = \beta - \gamma$. So we may write this as:

$$\begin{aligned}\widehat{q}_t = (1-m\lambda c)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) + \gamma\widehat{q}_{t+1} - \\ m(\lambda c-\beta)\widehat{c}_t - m(\lambda c-\beta)\widehat{c}_{t+1} - m\gamma\widehat{rr}_t - m(\beta-\gamma)(\widehat{rr}_t + \widehat{\pi}_{t+1}) + m(\beta-\gamma)\widehat{q}_{t+1} + m(\beta-\gamma)\widehat{\pi}_{t+1}\end{aligned}$$

Which can be reduced further to:

$$\begin{aligned}\widehat{q}_t = (1-m\lambda c)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) + \gamma\widehat{q}_{t+1} - \\ (m\lambda c - m\beta)(\widehat{c}_t - \widehat{c}_{t+1}) + m(\beta-\gamma)\widehat{q}_{t+1}\end{aligned}$$

We can go further:

$$\widehat{q}_t = (1-m\beta)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\lambda c-\gamma)(\mathbb{E}_t\,Y_{t+1} - E_t X_{t+1} - \widehat{h}_t) - m\beta\widehat{rr}_t + (\gamma(1-m)+m\beta)\widehat{q}_{t+1}$$

But since $\lambda c = \beta - \gamma$, we can write $(1-m\lambda c-\gamma) = 1 - m(\beta-\gamma) - \gamma = 1 - m\beta + m\gamma - \gamma = 1 - m\beta - \gamma(1-m)$. So we have:

$$\widehat{q}_t = (1-m\beta)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) + (1-m\beta-\gamma(1-m))(\mathbb{E}_t\,Y_{t+1} - E_tX_{t+1} - \widehat{h}_t) - m\beta\widehat{rr}_t + (\gamma(1-m)+m\beta)\widehat{q}_{t+1} \tag{75}$$

This is identical to (L4) in the paper, where they define $\gamma_e = m\beta + (1-m)\gamma$, since $1-\gamma_e = 1 - m\beta - \gamma(1-m)$.

So then the reduced linear system is:

$$\widehat{c}'_t = \mathbb{E}_t\,\widehat{c}'_{t+1} - \widehat{rr}_t \tag{76}$$

$$\widehat{b}_t = \mathbb{E}_t\,\widehat{q}_{t+1} + \widehat{h}_t - \widehat{rr}_t \tag{77}$$

$$\widehat{\pi}_t = -\kappa\widehat{X}_t + \beta\,\mathbb{E}_t\,\widehat{\pi}_{t+1} \tag{78}$$

$$\widehat{R}_t = r_R\widehat{R}_{t-1} + (1-r_R)\left[(1+r_\pi)\widehat{\pi}_{t-1} + r_Y\widehat{Y}_{t-1}\right] + \widehat{e}_{R,t} \tag{79}$$

$$\widehat{Y}_t = \frac{c}{Y}\widehat{c}_t + \frac{c'}{Y'}\widehat{c}'_t \tag{80}$$

$$\widehat{q}_t = \gamma_e\,\mathbb{E}_t\,\widehat{q}_{t+1} + (1-\gamma_e)\,\mathbb{E}_t(\mathbb{E}_t\,Y_{t+1} - E_tX_{t+1} - \widehat{h}_t) - m\beta\widehat{rr}_t + (1-m\beta)(\widehat{c}_t - \mathbb{E}_t\,\widehat{c}_{t+1}) \tag{81}$$

$$\widehat{Y}_t = \frac{\eta\nu}{\eta-(1-\nu)}\widehat{h}_{t-1} - \frac{1-\nu}{\eta-(1-\nu)}\left(\widehat{X}_t + \widehat{c}'_t\right) \tag{82}$$

$$c\widehat{c}_t = b\widehat{b}_t - qh\widehat{h}_t + qh\widehat{h}_{t-1} - \frac{b}{\beta}\left(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \widehat{\pi}_t\right) + \frac{\nu Y}{X}(\widehat{Y}_t - \widehat{X}_t) \tag{83}$$

$$\widehat{q}_t = \widehat{c}'_t + \iota\widehat{h}_t - \beta\,\mathbb{E}_t\,\widehat{c}'_{t+1} + \beta\,\mathbb{E}_t\,\widehat{q}_{t+1} \tag{84}$$

$$\widehat{rr}_t = \widehat{R}_t - \mathbb{E}_t\,\widehat{\pi}_{t+1} \tag{85}$$

Where $\gamma_e = m\beta + (1-m)\gamma$ and $\iota = (1-\beta)h/h'$, and $\kappa = (1-\theta)(1-\theta\beta)/\theta$. (76)-(85) are identical to (L1)-(L9) in the paper (augmented to include the real interest rate expression).

2.8 Steady State

Let variables without time subscripts denote steady states. Further, assume zero steady state inflation, so $\pi = 1$ (recall this is gross inflation). This means that $\pi^* = 1$ and $v^p = 1$. This implies that $X = \frac{\epsilon}{\epsilon-1}$.

From (27), we get that $R = \beta^{-1} = \bar{rr}$. Since $\gamma < \beta$, this insures that the $\lambda > 0$ (i.e. the borrowing constraint binds in the steady state). Let's start evaluating the other relationships in steady state. We have:

$$\frac{j}{h'} = (1-\beta)\frac{q}{c'} \tag{86}$$

$$L^{\eta-1} = \frac{w}{c'} \tag{87}$$

$$w = \frac{(1-\nu)Y}{XL} \tag{88}$$

$$q = \gamma\left(\frac{\nu Y}{Xh} + q\right) + m\lambda c q \tag{89}$$

$$\lambda c = \beta - \gamma \tag{90}$$

$$b = \beta m q h \tag{91}$$

$$Y = A h^{\nu} L^{1-\nu} \tag{92}$$

$$\frac{c'}{Y} = 1 - \frac{c}{Y} \tag{93}$$

$$\frac{h}{H} = 1 - \frac{h'}{H} \tag{94}$$

$$b = c + \frac{b}{\beta} + wL - \frac{Y}{X} \tag{95}$$

Let's try to write these things out as ratios relative to output or the aggregate housing stock, as this is what appears in the appendix.

We can eliminate wL from (88). This allows us to write (95) as:

$$\frac{\beta - 1}{\beta}\frac{b}{Y} = \frac{c}{Y} - \frac{\nu}{X} \tag{96}$$

Now fiddle with (89)-(90). We have:

$$1 = \gamma\left(\frac{\nu Y}{qXh} + 1\right) + m(\beta - \gamma)$$

I can use this to solve for qh/Y. In particular:

$$\frac{1}{\gamma} - \frac{m(\beta-\gamma)}{\gamma} - 1 = \frac{\nu Y}{qhX}$$

Which is:

$$\frac{1 - m(\beta-\gamma) - \gamma}{\gamma} = \frac{\nu}{X}\frac{Y}{qh}$$

Or:

$$\frac{qh}{Y} = \frac{\nu}{X}\frac{\gamma}{1 - m(\beta-\gamma) - \gamma} = \frac{\gamma\nu}{1-\gamma_e}\frac{1}{X} \tag{97}$$

(97) is identical to the second expression for the steady state in the Appendix. But then we can trivially get $\frac{b}{Y}$ from (91):

$$\frac{b}{Y} = \frac{\beta m \gamma \nu}{1-\gamma_e}\frac{1}{X} \tag{98}$$

(98) is identical to the third expression in the steady state appendix. But now that we know

this, we can solve for $\frac{c}{Y}$:

$$\frac{\beta-1}{\beta}\frac{\beta m\gamma\nu}{1-\gamma_e}\frac{1}{X}+\frac{\nu}{X}=\frac{c}{Y}$$

Which can be simplified further to:

$$\frac{c}{Y}=\frac{1}{X}\left[\nu-\frac{(1-\beta)m\nu\gamma}{1-\gamma_e}\right]=\frac{1}{X}\left[\frac{\nu(1-\gamma_e)-(1-\beta)m\nu\gamma}{1-\gamma_e}\right]$$

The numerator inside the brackets can be written: $\nu\left(1-m\beta-(1-m\gamma)-(1-\beta)m\gamma\right)$. But this equals $1-m\beta-\gamma(1-m\beta)=(1-m\beta)(1-\gamma)$. Hence, we have:

$$\frac{c}{Y}=\frac{\nu}{X}\frac{(1-m\beta)(1-\gamma)}{1-\gamma_e} \tag{99}$$

(99) is identical to the expression in the appendix. But then we can get $\frac{c'}{Y}$ as simply one minus this.

We are left with getting $\frac{h}{H}$, the lone remaining condition in the steady state appendix. Fiddle with (86). We have:

$$jc'=(1-\beta)qh'$$

Now plug in that $h'=H-h$:

$$jc'=(1-\beta)q(H-h)$$

Which can be written:

$$j\frac{c'}{h}=(1-\beta)q\left(\frac{H}{h}-1\right)$$

Or:

$$\frac{jc'}{(1-\beta)qh}=\frac{H}{h}-1$$

So:

$$\frac{H}{h}=1+\frac{jc'}{(1-\beta)qh}$$

Now multiply and divide the fraction on the RHS by Y:

$$\frac{H}{h}=1+\frac{jc'Y}{(1-\beta)qhY}=1+\frac{j}{1-\beta}\frac{c'}{Y}\frac{Y}{qh}$$

But we now know everything on the right hand side. So we have:

$$\frac{h}{H}=\left[1+\frac{j}{1-\beta}\frac{c'}{Y}\frac{Y}{qh}\right]^{-1} \tag{100}$$

At this point we have everything we need.

Note that we need to re-write the entrepreneur's budget constraint to be in terms of ratios. We have:

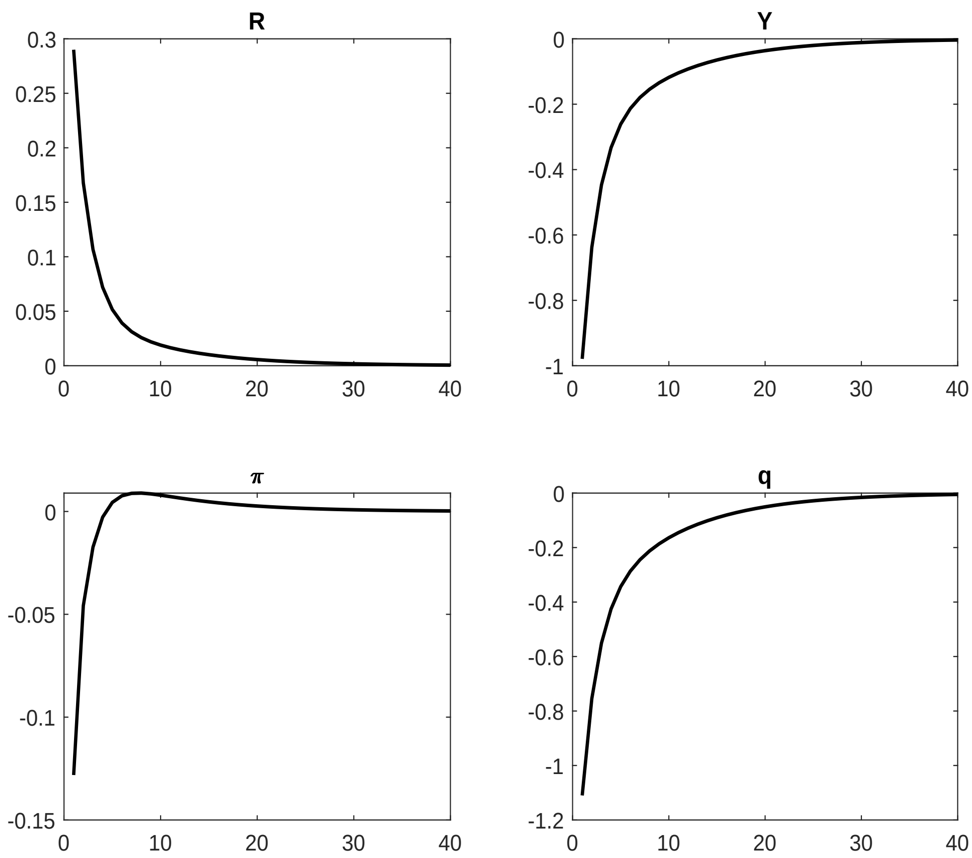
$$\frac{c}{Y}\widehat{c}_t = \frac{b}{Y}\widehat{b}_t - \frac{qh}{Y}\widehat{h}_t + \frac{qh}{Y}\widehat{h}_{t-1} - \frac{b}{\beta Y}(\widehat{R}_{t-1} + \widehat{b}_{t-1} - \widehat{\pi}_t) + \frac{\nu}{X}(\widehat{Y}_t - \widehat{X}_t) \tag{101}$$

2.9 Calibration

He sets $\beta = 0.99$ and $\gamma = 0.98$. He sets $\nu = 0.03$ and $j = 0.1$. He sets $m = 0.89$. Further, we have $\eta = 1.01$. From this, we can solve for ι and γ_e and all the steady state ratios that we need.

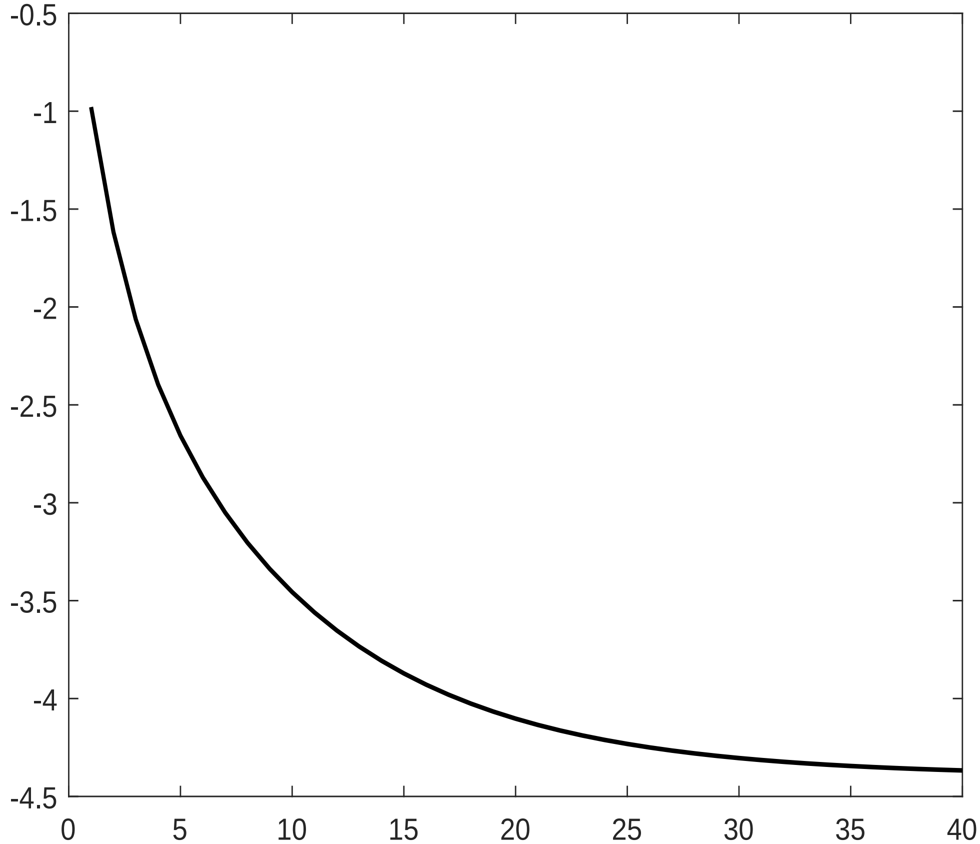
The figure below plots the impulse responses to a policy shock. The interest rate exogenously increases. This results in output and inflation falling. Furthermore, the price of housing falls.

Figure 1: IRFs to Policy Shock



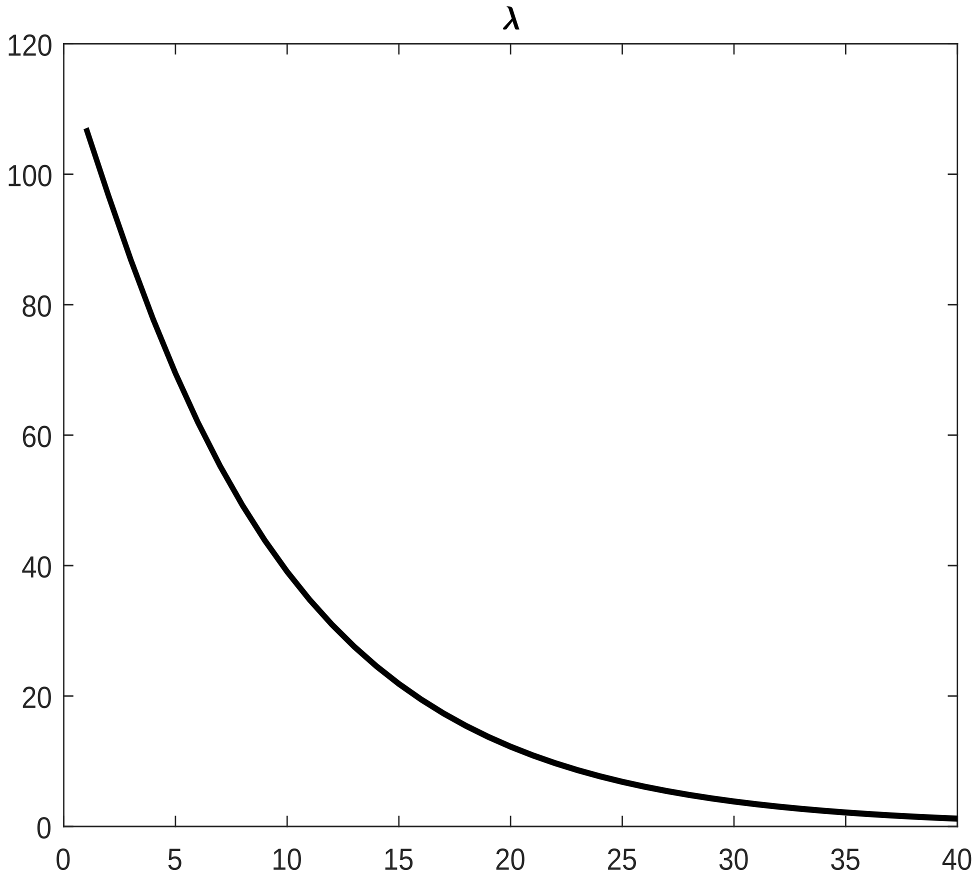
The next figure recreates Figure 2 in the paper, which plots the *cumulative* response of output to a policy shock. This is identical to what he reports in the paper.

Figure 2: Cumulative Output Response



The important insight here is that the borrowing constraint *amplifies* the response to the monetary policy shock. I'm not going to show that explicitly as it is more difficult than just re-parameterizing something (i.e. you need to have households and entrepreneurs to have the same discount factor, which makes the borrowing constraint non-binding). But we can think about the logic for the why constraint exacerbates the output effects of the policy shock by focusing on the response of the multiplier facing the entrepreneur.

Figure 3: Response of Multiplier, λ_t



We observe that λ_t goes up quite markedly – i.e. the borrowing constraint gets tighter. There are several reinforcing effects driving this. Recall that the borrowing constraint is given by:

$$b_t = m\,\mathbb{E}_t\left[\frac{q_{t+1}h_t\pi_{t+1}}{R_t}\right]$$

First, here is a direct effect at play – the increase in R_t causes the right hand side to get smaller, other factors being equal. This tightens the constraint. Second, there is the effect on house prices. With declining aggregate demand q_{t+1} will decline, also tightening the constraint. And then there is a "debt deflation" channel that occurs because nominal debt is not indexed to inflation (in the paper, Iacoviello talks a decent amount about this). In particular, the decline in inflation also itself makes the constraint tighter. All three of these things work in the same direction – λ_t goes up, which *exacerbates* the steady state misallocation wherein the entrepreneur has too little housing relative to what would be efficient. The shock makes it even harder for the entrepreneur to get land, which moves the economy even further from the efficient allocation. And of course, through general equilibrium all these effects on the tightness of the constraint are exacerbated in a "multiplier" or "accelerator" type mechanism – a tighter borrowing constraint for entrepreneurs further reduces the price of housing and inflation, which further tightens the constraint, and so on.

2.10 Solving the Non-Linear Model

Instead of linearizing by hand (which is often useful for intuition but is frankly a pain), we can also simply put in the non-linear equations and let Dynare just solve the model for us (via first order or higher approximation). That is, use (25)-(41) without linearization by hand and without eliminating static variables.

We have to think a bit about the steady state. What matters for the linearization are steady state *ratios* relative to output – the absolute size of steady state output is irrelevant. To solve the steady state of the non-linearized model, however, we do have to worry about absolute sizes. There is typically a "free" normalization at play. Most often, we normalize $A = 1$. Iacoviello instead normalizes $Y = 1$. This is absolutely fine, and maps in nicely to the steady state ratio work when constructing the linearized model. But we have to pick A to be consistent with that normalization, instead of the typical approach of setting $A = 1$. The choice of steady state H (the total available fixed stock of housing) will matter for the requisite normalization of A in this setup but is otherwise not directly relevant.

(97)-(100) give us qh/Y, b/Y, c/Y, and h/H. Normalizing $Y = 1$, this then gives us steady state values of qh, b, and c (and hence c'). Let's just set $H = 1$. But then we can use (100) to give us h:

$$h = \left[1 + \frac{j}{1-\beta}\frac{c'}{qh}\right]^{-1} \tag{102}$$

The FOC for labor supply can be written:

$$L^{\eta} = \frac{wL}{Y}\frac{Y}{c'}$$

But from the labor demand condition we know:

$$\frac{wL}{Y} = \frac{1-\nu}{X}$$

Using this, along with the normalization of $Y = 1$ and the above-found value of c', gives us:

$$L = \left(\frac{1-\nu}{Xc'}\right)^{1/\eta} \tag{103}$$

But since we know L and h now, we can determine the A that is consistent with $Y = 1$ from the production function:

$$A = \frac{1}{h^{\nu}L^{1-\nu}} \tag{104}$$

I can put the non-linear equations into Dynare and let it do the linearization, and I get virtually identical impulse response functions and moments as when I do the linearization by hand.

3 The Extended Model

The extended model is basically the same as the base model, with a couple of modifications. First, entrepreneurs can accumulate physical capital. Second, an additional impatient household is added that is also subject to a borrowing constraint. Third, all households face convex housing stock adjustment costs. Fourth, there is now a stochastic productivity shock and a preference shock to housing.

3.1 Entrepreneurs

Entrepreneurs produce wholesale output according to:

$$Y_{w,t} = A_t K_{t-1}^{\mu} h_{t-1}^{\nu} (L_t')^{\alpha(1-\mu-\nu)} (L_t'')^{(1-\alpha)(1-\mu-\nu)} \tag{105}$$

The "prime" denotes the labor supply of the patient household and the double-prime the impatient household. The budget constraint of the entrepreneur is:

$$\frac{Y_{w,t}}{X_t} + b_t + q_t h_{t-1} - w_t' L_t' - w_t'' L_t'' = c_t + q_t h_t + \frac{R_{t-1} b_{t-1}}{\pi_t} + I_t + \xi_{e,t} + \xi_{k,t} \tag{106}$$

On the income side, the entrepreneur earns income from selling to retailers, issues new debt, earns income from its existing stock of housing, and pays labor (to both types of households). On the expenditure side, it consumes, buys new housing, pays interest on its debt, invests in new capital, and pays adjustment costs on housing and capital. These adjustment costs are given by:

$$\xi_{k,t} = \psi \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 \frac{K_{t-1}}{2\delta} \tag{107}$$

$$\xi_{e,t} = \phi_e \left(\frac{h_t - h_{t-1}}{h_{t-1}} \right)^2 \frac{q_t h_{t-1}}{2} \tag{108}$$

Preferences are the same as before, as is the borrowing constraint. A Lagrangian is:

$$\begin{aligned} \mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \Bigg\{ & \ln c_t + \lambda_t \left(m\, \mathbb{E}_t\, q_{t+1} h_t \pi_{t+1} - b_t R_t \right) + \\ & \mu_{1,t} \left[\frac{A_t K_{t-1}^{\mu} h_{t-1}^{\nu} (L_t')^{\alpha(1-\mu-\nu)} (L_t'')^{(1-\alpha)(1-\mu-\nu)}}{X_t} + b_t + q_t h_{t-1} - w_t' L_t' - w_t'' L_t'' \right. \\ & \left. - c_t - q_t h_t - \frac{R_{t-1} b_{t-1}}{\pi_t} - I_t - \psi \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 \frac{K_{t-1}}{2\delta} - \phi_e \left(\frac{h_t - h_{t-1}}{h_{t-1}} \right)^2 \frac{q_t h_{t-1}}{2} \right] + \mu_{2,t} \left(I_t + (1-\delta) K_{t-1} - K_t \right) \Bigg\} \end{aligned}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial c_t} = \frac{1}{c_t} - \mu_{1,t}$$

$$\frac{\partial \mathbb{L}}{\partial L'_t} = \frac{\alpha(1-\mu-\nu)Y_t}{L'_t X_t} - w'_t$$

$$\frac{\partial \mathbb{L}}{\partial L''_t} = \frac{(1-\alpha)(1-\mu-\nu)Y_t}{L''_t X_t} - w''_t$$

$$\frac{\partial \mathbb{L}}{\partial I_t} = -\mu_{1,t}\left(1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right) + \mu_{2,t}$$

$$\frac{\partial \mathbb{L}}{\partial K_t} = -\mu_{2,t} + \gamma\,\mathbb{E}_t\,\mu_{1,t+1}\left[\frac{\mu Y_{t+1}}{K_t X_{t+1}} - \frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)^2 + \frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)\frac{I_{t+1}}{K_t}\right] + \gamma(1-\delta)\,\mathbb{E}_t\,\mu_{2,t+1}$$

$$\frac{\partial \mathbb{L}}{\partial b_t} = -\lambda_t R_t + \mu_{1,t} - \gamma\,\mathbb{E}_t\,\mu_{1,t+1} R_t/\pi_{t+1}$$

$$\frac{\partial \mathbb{L}}{\partial h_t} = \lambda_t m\,\mathbb{E}_t\,q_{t+1}\pi_{t+1} - \mu_{1,t} q_t\left(1 + \phi_e\left(\frac{h_t - h_{t-1}}{h_{t-1}}\right)\right) + \gamma\,\mathbb{E}_t\,\mu_{1,t+1}\left(\frac{\nu Y_{t+1}}{h_t X_{t+1}} + q_{t+1} - \frac{\phi_e}{2}\left(\frac{h_{t+1} - h_t}{h_t}\right)^2 q_t + \phi_e\left(\frac{h_{t+1} - h_t}{h_t}\right)\frac{q_{t+1} h_{t+1}}{h_t}\right)$$

The labor demand schedules are straightforward:

$$w'_t = \frac{\alpha(1-\mu-\nu)Y_t}{L'_t X_t} \tag{109}$$

$$w''_t = \frac{(1-\alpha)(1-\mu-\nu)Y_t}{L''_t X_t} \tag{110}$$

Now let's deal with the multipliers. From the FOC for investment, we have:

$$\mu_{2,t} = \frac{1}{c_t}\left(1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right)$$

This is what he calls υ_t in the appendix. I will use this notation. Then if you go to the capital FOC, we have:

$$\upsilon_t = \gamma\,\mathbb{E}_t\,\frac{1}{c_{t+1}}\left[\frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)\frac{I_{t+1}}{K_t} - \frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)^2\right] + \gamma\,\mathbb{E}_t\left[\frac{\mu Y_{t+1}}{c_{t+1} K_t X_{t+1}} + (1-\delta)\upsilon_{t+1}\right] \tag{111}$$

Now, (111) looks almost just like what is in the appendix, except for the timing terms on the adjustment cost part. This has to be a mistake – K_{t-1} is predetermined, you are choosing K_t, and that only affects adjustment costs in $t+1$, so (i) there should be discounting, (ii) it should be weighted by $1/c_{t+1}$, not $1/c_t$, and (iii) the terms should be I_{t+1} and K_t, not I_t and K_{t-1}. Note if you go to the technical appendix on Iacoviello's homepage, he has the right timing consistent with (111).

The FOC for housing can be re-written:

$$\frac{q_t}{c_t}\left(1+\phi_e\left(\frac{h_t-h_{t-1}}{h_{t-1}}\right)\right)=$$
$$\gamma\,\mathbb{E}_t\,\frac{1}{c_{t+1}}\left[\frac{\nu Y_{t+1}}{h_t X_{t+1}}+q_{t+1}-\frac{\phi_e q_{t+1}}{2}\left(\frac{h_{t+1}-h_t}{h_t}\right)^2+\phi_e\left(\frac{h_{t+1}-h_t}{h_t}\right)\frac{q_{t+1}h_{t+1}}{h_t}\right]+m\lambda_t\,\mathbb{E}_t\,q_{t+1}\pi_{t+1} \tag{112}$$

(112) is the same as in the paper, just modified to include the terms related to the adjustment cost. The first order condition for bonds is the same as in the standard model:

$$\frac{1}{c_t}=\gamma\,\mathbb{E}_t\,\frac{1}{c_{t+1}}\frac{R_t}{\pi_{t+1}}+\lambda_t R_t \tag{113}$$

3.2 Impatient Households

The new agents are impatient households, denoted with $''$. They discount via $\beta''<\beta$. Their problem looks just like the patient household, modified to include a housing adjustment cost, with the exception that, like the entrepreneurs, they face a borrowing constraint. Their budget constraint is:

$$c_t''+q_t h_t''+R_{t-1}b_{t-1}''/\pi_t=b_t''+q_t''h_{t-1}''+w_t''L_t''+T_t''-\frac{\phi_h}{2}\left(\frac{h_t''-h_{t-1}''}{h_{t-1}''}\right)^2 q_t h_{t-1}''$$

Preferences are the same as in the base model, though modified to include a preference shock for housing, j_t (i.e. j is now stochastic, and will apply to both patient and impatient households).A Lagrangian is:

$$\mathbb{L}=\mathbb{E}_0\sum_{t=0}^{\infty}\beta''\Bigg\{\ln c_t''+j_t\ln h_t''-\frac{(L_t'')^\eta}{\eta}+\lambda_t''\left[m''\,\mathbb{E}_t\,q_{t+1}h_t''\pi_{t+1}-b_t''R_t\right]$$
$$+\mu_t''\left[b_t''+q_t h_{t-1}''+w_t''L_t''+T_t''-\frac{\phi_h}{2}\left(\frac{h_t''-h_{t-1}''}{h_{t-1}''}\right)^2 q_t h_{t-1}''-c_t''-q_t h_t''-R_{t-1}b_{t-1}''/\pi_t\right]\Bigg\}$$

The FOC are:

$$\frac{\partial\mathbb{L}}{\partial c_t''}=\frac{1}{c_t''}-\mu_t''$$
$$\frac{\partial\mathbb{L}}{\partial L_t''}=-(L_t'')^{\eta-1}+\mu_t''w_t''$$

$$\frac{\partial \mathbb{L}}{\partial b''_t} = -R_t \lambda''_t + \mu''_t - \beta'' \, \mathbb{E}_t \, \mu''_t R_t / \pi_{t+1}$$

$$\frac{\partial \mathbb{L}}{\partial h''_t} = \frac{j_t}{h''_t} + m'' \lambda_t \, \mathbb{E}_t \, q_{t+1} \pi_{t+1} - \mu''_t \left[\phi_h \left(\frac{h''_t - h''_{t-1}}{h''_{t-1}} \right) q_t - q_t \right] + \\ \beta'' \, \mathbb{E}_t \, \mu''_t \left[q_{t+1} - \frac{\phi_h q_{t+1}}{2} \left(\frac{h''_{t+1} - h''_t}{h''_t} \right)^2 + \phi_h \left(\frac{h''_{t+1} - h''_t}{h''_t} \right) q_{t+1} \frac{h''_{t+1}}{h''_t} \right]$$

We can eliminate the multiplier and write these as:

$$\frac{w''_t}{c''_t} = (L''_t)^{\eta - 1} \tag{114}$$

$$\frac{1}{c''_t} = \beta'' \, \mathbb{E}_t \, \frac{1}{c''_{t+1}} \frac{R_t}{\pi_{t+1}} + R_t \lambda''_t \tag{115}$$

$$\frac{q_t}{c''_t} \left[1 + \phi_h \left(\frac{h''_t - h''_{t-1}}{h''_{t-1}} \right) \right] = \frac{j_t}{h''_t} + \\ \beta'' \, \mathbb{E}_t \left[\frac{q_{t+1}}{c''_{t+1}} \left(1 - \frac{\phi_h}{2} \left(\frac{h''_{t+1} - h''_t}{h''_t} \right)^2 + \phi_h \left(\frac{h''_{t+1} - h''_t}{h''_t} \right) \frac{h''_{t+1}}{h''_t} \right) + m'' \lambda''_t q_{t+1} \pi_{t+1} \right] \tag{116}$$

(116) is close to what he has in the appendix, but he seems to be missing the term involving the square of the difference.

3.3 Patient Households

The patient households look exactly like the impatient households, except they are not subject to a borrowing constraint. They are subject to the same adjustment cost and same preference shock, j_t. Hence, their FOC are:

$$\frac{w'_t}{c'_t} = (L'_t)^{\eta - 1} \tag{117}$$

$$\frac{1}{c'_t} = \beta \, \mathbb{E}_t \, \frac{1}{c'_{t+1}} \frac{R_t}{\pi_{t+1}} \tag{118}$$

$$\frac{q_t}{c'_t} \left[1 + \phi_h \left(\frac{h'_t - h'_{t-1}}{h'_{t-1}} \right) \right] = \frac{j_t}{h'_t} + \\ \beta \, \mathbb{E}_t \left[\frac{q_{t+1}}{c'_{t+1}} \left(1 - \frac{\phi_h}{2} \left(\frac{h'_{t+1} - h'_t}{h'_t} \right)^2 + \phi_h \left(\frac{h'_{t+1} - h'_t}{h'_t} \right) \frac{h'_{t+1}}{h'_t} \right) \right] \tag{119}$$

3.4 Other Parts

The retailer problem is identical to before. Aggregation related to price-setting and aggregate production is the same. Aggregation on the demand side is a bit trickier. We need to sum the budget constraints of the three agents. We have:

$$c'_t + q_t h'_t + \frac{R_{t-1} b'_{t-1}}{\pi_t} + \xi'_{h,t} + c''_t + q_t h''_t + \frac{R_{t-1} b''_{t-1}}{\pi_t} + \xi''_{h,t} + c_t + q_t h_t + \frac{R_{t-1} b_{t-1}}{\pi_t} + I_t + \xi_{e,t} + \xi_{k,t} =$$
$$b'_t + w'_t L'_t + q_t h'_{t-1} + F_t + \frac{Y_{w,t}}{X_t} + b_t + q_t h_{t-1} - w'_t L'_t - w''_t L''_t + b''_t + q''_t h''_{t-1} + w''_t L''_t$$

Bond market-clearing requires $b'_t + b''_t + b_t = 0$. House market-clearing requires $h'_t + h''_t + h_t = H$. Imposing these things plus labor market-clearing gives (using the same fact about remitted profits) gives the resource constraint:

$$c'_t + c''_t + c_t + I_t + \xi_{e,t} + \xi_{k,t} + \xi'_{h,t} + \xi''_{h,t} = Y_t \tag{120}$$

Where the ξ terms are shorthands for the adjustment costs:

3.5 Full Set of Conditions Extended Model

The full set of equilibrium conditions are:

$$\frac{w'_t}{c'_t} = (L'_t)^{\eta-1} \tag{121}$$

$$\frac{1}{c'_t} = \beta\, \mathbb{E}_t \frac{1}{c'_{t+1}} \frac{R_t}{\pi_{t+1}} \tag{122}$$

$$\frac{q_t}{c'_t}\left[1 + \phi_h \left(\frac{h'_t - h'_{t-1}}{h'_{t-1}}\right)\right] = \frac{j_t}{h'_t} +$$
$$\beta\, \mathbb{E}_t \left[\frac{q_{t+1}}{c'_{t+1}} \left(1 - \frac{\phi_h}{2}\left(\frac{h'_{t+1} - h'_t}{h'_t}\right)^2 + \phi_h \left(\frac{h'_{t+1} - h'_t}{h'_t}\right) \frac{h'_{t+1}}{h'_t}\right)\right] \tag{123}$$

$$\frac{w''_t}{c''_t} = (L''_t)^{\eta-1} \tag{124}$$

$$\frac{1}{c''_t} = \beta''\, \mathbb{E}_t \frac{1}{c''_{t+1}} \frac{R_t}{\pi_{t+1}} + R_t \lambda''_t \tag{125}$$

$$\frac{q_t}{c''_t}\left[1+\phi_h\left(\frac{h''_t-h''_{t-1}}{h''_{t-1}}\right)\right]=\frac{j_t}{h''_t}+$$
$$\beta''\,\mathbb{E}_t\left[\frac{q_{t+1}}{c''_{t+1}}\left(1-\frac{\phi_h}{2}\left(\frac{h''_{t+1}-h''_t}{h''_t}\right)^2+\phi_h\left(\frac{h''_{t+1}-h''_t}{h''_t}\right)\frac{h''_{t+1}}{h''_t}\right)\right]+m''\lambda''_t q_{t+1}\pi_{t+1} \quad (126)$$

$$b''_t=m''\,\mathbb{E}_t\left[q_{t+1}h''_t\pi_{t+1}/R_t\right] \quad (127)$$

$$w'_t=\frac{\alpha(1-\mu-\nu)Y_t}{L'_tX_t} \quad (128)$$

$$w''_t=\frac{(1-\alpha)(1-\mu-\nu)Y_t}{L''_tX_t} \quad (129)$$

$$\upsilon_t=\frac{1}{c_t}\left(1+\frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}}-\delta\right)\right) \quad (130)$$

$$\upsilon_t=\gamma\,\mathbb{E}_t\frac{1}{c_{t+1}}\left[\frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t}-\delta\right)\frac{I_{t+1}}{K_t}-\frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t}-\delta\right)^2\right]+\gamma\,\mathbb{E}_t\left[\frac{\mu Y_{t+1}}{c_{t+1}K_tX_{t+1}}+(1-\delta)\upsilon_{t+1}\right] \quad (131)$$

$$\frac{q_t}{c_t}\left(1+\phi_e\left(\frac{h_t-h_{t-1}}{h_{t-1}}\right)\right)=$$
$$\gamma\,\mathbb{E}_t\frac{1}{c_{t+1}}\left[\frac{\nu Y_{t+1}}{h_tX_{t+1}}+q_{t+1}-\frac{\phi_eq_{t+1}}{2}\left(\frac{h_{t+1}-h_t}{h_t}\right)^2+\phi_e\left(\frac{h_{t+1}-h_t}{h_t}\right)\frac{q_{t+1}h_{t+1}}{h_t}\right]+m\lambda_t\,\mathbb{E}_t\,q_{t+1}\pi_{t+1}$$
$$(132)$$

$$\frac{1}{c_t}=\gamma\,\mathbb{E}_t\frac{1}{c_{t+1}}\frac{R_t}{\pi_{t+1}}+\lambda_tR_t \quad (133)$$

$$b_t=m\,\mathbb{E}_t\left[q_{t+1}h_t\pi_{t+1}/R_t\right] \quad (134)$$

$$z_{1,t}=X_t^{-1}Y_t+\theta\,\mathbb{E}_t\,\Lambda_{t,t+1}\pi_{t+1}^{\epsilon}z_{1,t+1} \quad (135)$$

$$z_{2,t}=Y_t+\theta\,\mathbb{E}_t\,\Lambda_{t,t+1}\pi_{t+1}^{\epsilon-1}z_{2,t+1} \quad (136)$$

$$\pi^*_t=\frac{\epsilon}{\epsilon-1}\frac{z_{1,t}}{z_{2,t}} \quad (137)$$

$$R_t=(\bar{rr})^{1-r_R}(R_{t-1})^{r_R}\left(\pi_{t-1}^{1+r_\pi}(Y_{t-1}/Y)^{r_Y}\right)^{1-r_R}e_{R,t} \quad (138)$$

$$1=\theta\pi_t^{\epsilon-1}+(1-\theta)(\pi^*_t)^{1-\epsilon} \quad (139)$$

$$A_tK_{t-1}^{\mu}h_{t-1}^{\nu}(L'_t)^{\alpha(1-\mu-\nu)}(L''_t)^{(1-\alpha)(1-\mu-\nu)}=Y_tv_t^p \quad (140)$$

$$v_t^p=(1-\theta)(\pi^*_t)^{-\epsilon}+\theta\pi_t^{\epsilon}v_{t-1}^p \quad (141)$$

$$c_t+c'_t+c''_t+I_t+\xi_{e,t}+\xi_{k,t}+\xi'_{h,t}+\xi''_{h,t}=Y_t \quad (142)$$

$$h_t+h'_t+h''_t=H \quad (143)$$

$$K_t=I_t+(1-\delta)K_{t-1} \quad (144)$$

$$\xi_{e,t} = \frac{\phi_e}{2}\left(\frac{h_t - h_{t-1}}{h_{t-1}}\right)^2 q_t h_{t-1} \tag{145}$$

$$\xi'_{h,t} = \frac{\phi_h}{2}\left(\frac{h'_t - h'_{t-1}}{h'_{t-1}}\right)^2 q_t h'_{t-1} \tag{146}$$

$$\xi''_{h,t} = \frac{\phi_h}{2}\left(\frac{h''_t - h''_{t-1}}{h''_{t-1}}\right)^2 q_t h''_{t-1} \tag{147}$$

$$\xi_{k,t} = \frac{\psi}{2\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)^2 K_{t-1} \tag{148}$$

$$b_t = c_t + q_t(h_t - h_{t-1}) + \frac{R_{t-1}b_{t-1}}{\pi_t} + w'_t L'_t + w''_t L''_t + I_t + \xi_{e,t} + \xi_{k,t} - \frac{Y_t v^p_t}{X_t} \tag{149}$$

$$b''_t = c''_t + q_t(h''_t - h''_{t-1}) + \frac{R_{t-1}b''_{t-1}}{\pi_t} - w''_t L''_t + \xi''_{h,t} \tag{150}$$

$$j_t = (1-\rho_j)j + \rho_j j_{t-1} + s_j \varepsilon_{j,t} \tag{151}$$

$$A_t = (1-\rho_A)A + \rho_A A_{t-1} + s_A \varepsilon_{A,t} \tag{152}$$

This is 32 equations. There are 32 variables:
$\left\{c_t, c'_t, c''_t, h_t, h'_t, h''_t, L'_t, L''_t, I_t, K_t, Y_t, X_t, b_t, b''_t, q_t, R_t, w'_t, w''_t, \pi_t, \pi^*_t, z_{1,t}, z_{2,t}, v^p_t, \upsilon_t, \lambda_t, \lambda''_t, j_t, A_t, \xi_{e,t}, \xi_{k,t}, \xi'_{h,t}, \xi''_{h,t}\right\}$

3.6 Steady State

I'm going to solve for the steady state by hand. First of all, the steady interest rate is standard:

$$R = \beta^{-1} \tag{153}$$

We know that $\upsilon = \frac{1}{c}$. But this means we can write the Euler equation for capital as:

$$1 = \left[\frac{\mu Y}{KX} + (1-\delta)\right]$$

Since we are targeting $Y = 1$ and have a target value of X, this allows us to solve for the steady state capital stock:

$$K = \frac{\mu}{X\left[\frac{1}{\gamma} - (1-\delta)\right]} \tag{154}$$

Given K, we then have $I = \delta K$. None of the adjustment cost terms will be different than zero outside of steady state, and with zero trend inflation the pricing conditions are straightforward. Let's write out a bunch of other equations in steady state and work from there, given what we know and given the normalization of $Y = 1$.

$$\frac{w'L'}{c'} = (L')^\eta$$

$$\frac{q}{c'} = \frac{j}{h'} + \beta\frac{q}{c'}$$
$$\frac{w''L''}{c''} = (L'')^\eta$$
$$R\lambda'' = \frac{1}{c''}\left(1 - \beta''R\right)$$
$$\frac{q}{c''} = \frac{j}{h''} + \beta''\frac{q}{c''} + m''\lambda''q$$
$$b''R = m''qh''$$
$$w'L' = \frac{\alpha(1-\mu-\nu)}{X}$$
$$w''L'' = \frac{(1-\alpha)(1-\mu-\nu)}{X}$$
$$\frac{q}{c} = \frac{\gamma}{c}\left[\frac{\nu}{hX} + q\right] + m\lambda q$$
$$R\lambda = \frac{1}{c}\left[1 - \gamma R\right]$$
$$bR = mqh$$
$$AK^\mu h^\nu (L')^{\alpha(1-\mu-\nu)}(L'')^{(1-\alpha)(1-\mu-\nu)} = 1$$
$$c + c' + c'' + \delta K = 1$$
$$h + h' + h'' = H$$
$$b = c + Rb + w'L' + w''L'' + \delta K - Y/X$$
$$b'' = c'' + Rb'' - w''L''$$

Let's start eliminating things. First, we can eliminate the wage terms. We get:

$$\frac{\alpha(1-\mu-\nu)}{X} = c'(L')^\eta$$
$$\frac{(1-\alpha)(1-\mu-\nu)}{X} = c''(L'')^\eta$$
$$b(1-R) = c + \frac{1-\mu-\nu}{X} + \delta K - \frac{1}{X}$$
$$b''(1-R) = c'' - \frac{(1-\alpha)(1-\mu-\nu)}{X}$$

Furthermore, note that we can write:

$$\lambda = \frac{1}{c}(\beta - \gamma)$$
$$\lambda'' = \frac{1}{c''}(\beta - \beta'')$$

This means that we can write the housing Euler equations as:

$$\frac{q}{c''} = \frac{j}{h''} + \beta'' \frac{q}{c''} + m'' q \frac{1}{c''}(\beta - \beta'')$$
$$\frac{q}{c} = \frac{\gamma}{c}\left[\frac{\nu}{hX} + q\right] + mq\frac{1}{c}(\beta - \gamma)$$
$$\frac{q}{c'} = \frac{j}{h'} + \beta\frac{q}{c'}$$

Focus on the second expression. The c cancel, and we can multiply both sides by h:

$$qh = \gamma\left[\frac{\nu}{X} + qh\right] + mqh(\beta - \gamma)$$

Or, simplifying:

$$qh = \frac{\gamma\nu}{1 - \gamma - (\beta - \gamma)m}\frac{1}{X}$$

Similarly:

$$qh'' = jc'' + \beta'' qh'' + m'' qh''(\beta - \beta'')$$

Or:

$$qh'' = \frac{jc''}{1 - \beta'' - m''(\beta - \beta'')}$$

And, finally:

$$qh' = jc' + \beta qh'$$

Or:

$$qh' = \frac{jc'}{1 - \beta}$$

Now we can solve for c. How? Because we know:

$$c = b\left(\frac{\beta - 1}{\beta}\right) + \frac{\mu + \nu}{X} - \delta K$$

But we know that:

$$b = \beta mqh$$

So we get:

$$c = mqh(\beta - 1) + \frac{\mu + \nu}{X} - \delta K$$

Knowing qh, we now have c. We similarly now can solve for c'':

$$c'' = b''\left(\frac{\beta-1}{\beta}\right) + \frac{(1-\alpha)(1-\mu-\nu)}{X}$$

But $b'' = \beta m'' q h''$:

$$c'' = m'' q h''(\beta - 1) + \frac{(1-\alpha)(1-\mu-\nu)}{X}$$

But we know qh'' from above. So:

$$c'' = m''(\beta-1)\frac{jc''}{1-\beta''-m''(\beta-\beta'')} + \frac{(1-\alpha)(1-\mu-\nu)}{X}$$

So:

$$\left[1 - \frac{jm''(\beta-1)}{1-\beta''-m''(\beta-\beta'')}\right]c'' = \frac{(1-\alpha)(1-\mu-\nu)}{X}$$

Or:

$$c'' = \left[1 - \frac{jm''(\beta-1)}{1-\beta''-m''(\beta-\beta'')}\right]^{-1}\frac{(1-\alpha)(1-\mu-\nu)}{X}$$

But then we can solve for c' from the resource constraint:

$$c' = 1 - c - c'' - \delta K$$

But then we can solve for L' and L'':

$$L' = \left[\frac{\alpha(1-\mu-\nu)}{c'X}\right]^{\frac{1}{\eta}}$$

$$L'' = \left[\frac{(1-\alpha)(1-\mu-\nu)}{c''X}\right]^{\frac{1}{\eta}}$$

But now we can solve for q, but noting that $qh + qh' + qh'' = q$ (since $h + h' + h'' = H = 1$).

$$q = \frac{\gamma\nu}{1-\gamma-(\beta-\gamma)m}\frac{1}{X} + \frac{jc''}{1-\beta''-m''(\beta-\beta'')} + \frac{jc'}{1-\beta}$$

But then we can recover h and h', knowing q, and then $h'' = 1 - h - h'$. But then we also have the b an b' from the borrowing constraints:

$$b'' = \beta m'' q h''$$

$$b = \beta m q h$$

We can also solve for the A consistent with our normalization:

$$A = \left[K^{\mu}h^{\nu}(L')^{\alpha(1-\mu-\nu)}(L'')^{(1-\alpha)(1-\mu-\nu)}\right]^{-1}$$

3.7 Parameterization and IRFs

For the extended model, Iacoviello calibrates some parameters and estimates others. More on the estimation below. The calibrated parameters are $\beta = 0.99$, $\gamma = 0.98$, and $\beta'' = 0.95$. The parameter on labor in the utility function is $\eta = 1.01$ and the steady state weight on housing in the utility function is $j = 0.1$. Two of the three production function parameters are $\mu = 0.3$ and $\nu = 0.03$. The depreciation rate is $\delta = 0.03$ and the capital adjustment cost parameter is $\psi = 2$. Even though we went to the trouble of writing down the model with the housing adjustment cost, this is set to zero – $\phi_e = \phi_h = 0$. The steady state markup is $X = 1.05$ and the probability of non-price adjustment is $\theta = 0.75$.

A subset of other parameters are estimated. The estimation is via impulse response matching. The exercise is a bit weird so I will describe it in words here. First, he is estimating a four variable VAR in the nominal interest rate, inflation, the housing price, and output. Then he is identifying impulse responses via a recursive Choleski ordering, with the variables following this ordering (i.e. the interest rate is ordered first, etc). To be able to estimate a four variable VAR, he is adding a fourth shock – an "inflation" shock which appears as a residual in the linearized Phillips Curve (I have not included that in my exposition of the model here).

The important point here is this – *this VAR ordering only makes sense from the perspective of the model for the interest rate.* Why does the ordering only make sense for the interest rate? Well, the interest rate specification has monetary policy reacting *only with a lag* to inflation and output. This means that ordering monetary policy "first" means that the reduced-form innovation in the interest rate equation can be interpreted as the monetary shock – this affects all other variables immediately, but the interest rate only reacts to other shocks with a lag of one period.[1] The other orthogonalized shocks from the VAR don't map into model. In the recursive VAR, inflation does not react within period to the last two orthogonalized shocks. But in the model, inflation will react immediately to all shocks. So, we cannot interpret the VAR impulse responses other than the response to the identified monetary shock in a structural way in terms of the model. It is a well-defined exercise to take the DSGE model, form a reduced-form VAR and compute IRFs to the Choleski-identified orthogonal "shocks" – these are, if you will, interesting moments one might hope the model can match. But the IRFs to these shocks do not map one-to-one into the IRFs to the actual shocks in the model (again, other than for the interest rate).

Let $\widehat{\mathbf{\Psi}}$ be a vector collecting of impulse responses of variables from the empirical VAR. Let $\widehat{\mathbf{\Psi}}(\zeta)$ be a vector collecting the model impulse responses, where ζ is the vector of parameters to estimate (see below). But what impulse responses in the model? These are *not* the impulse responses to different shocks in the model. Rather, they are impulse responses to the Choleski orthogonalization of the reduced form VAR representation of these four variables in the model. As I said above, only the impulse response to the "R shock" will be interpretable as the response to a shock in the model.

[1] Interestingly, it is worth noting that this is *exactly the opposite* from how monetary policy shocks are identified in most empirical VARs. In those VARs (see, e.g., Christiano, Eichenbaum, and Rebelo 2005), monetary policy is ordered "late" in the VAR wherein monetary policy is assumed to react *instantaneously* to other shocks but only affects other variables with a *lag*.

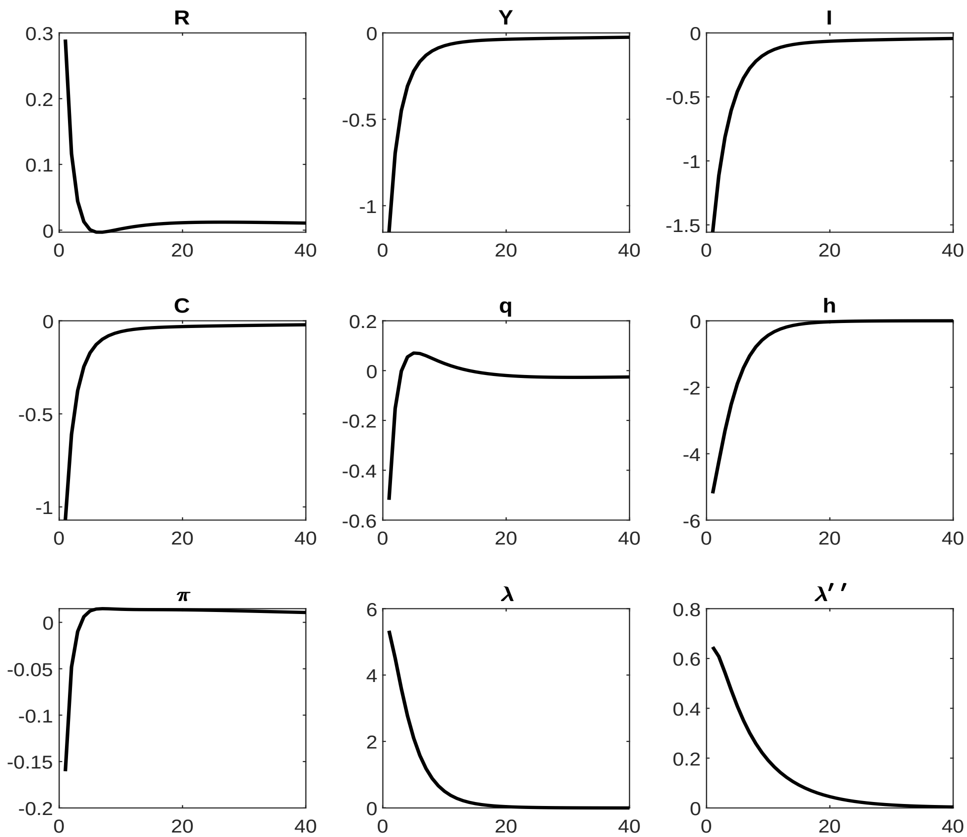
Put differently, the responses shown in Figure 5 are *not* responses to the four shocks in the model; they are responses to the four Choleski-orthogonalized innovations from the reduced-form VAR representation of the model.

The parameters he estimates are α (the share on the two kinds of labor), m and m'' (the down payment requirements for the entrepreneur and the impatient household), and the parameters of the shock processes (but *not* the monetary policy rule, which is based on estimation of a single equation Taylor rule). The objective function is to pick ζ to minimize the distance between the VAR and model impulse responses:

$$\min_{\delta} \left[\mathbf{\Psi}(\zeta) - \widehat{\mathbf{\Psi}}\right]' \mathbf{\Phi} \left[\mathbf{\Psi}(\zeta) - \widehat{\mathbf{\Psi}}\right]$$

Here $\mathbf{\Phi}$ is the weighting matrix. Typically in these kinds of exercises, the weighting matrix is the inverse variance-covariance matrix of empirical moments that you are targeting (in this case, the IRFs to the Choleski shocks in the VAR). Iacoviello does something slightly different – see the discussion in the paper. The estimates are in Table 2. He essentially estimates the productivity shock to be iid. For what I'm going to show below, I use $\rho_A = 0.803$ instead of $\rho_A = 0.03$. Otherwise I use what they report in the paper. The impulse responses to the three shocks I have in the model are shown below:

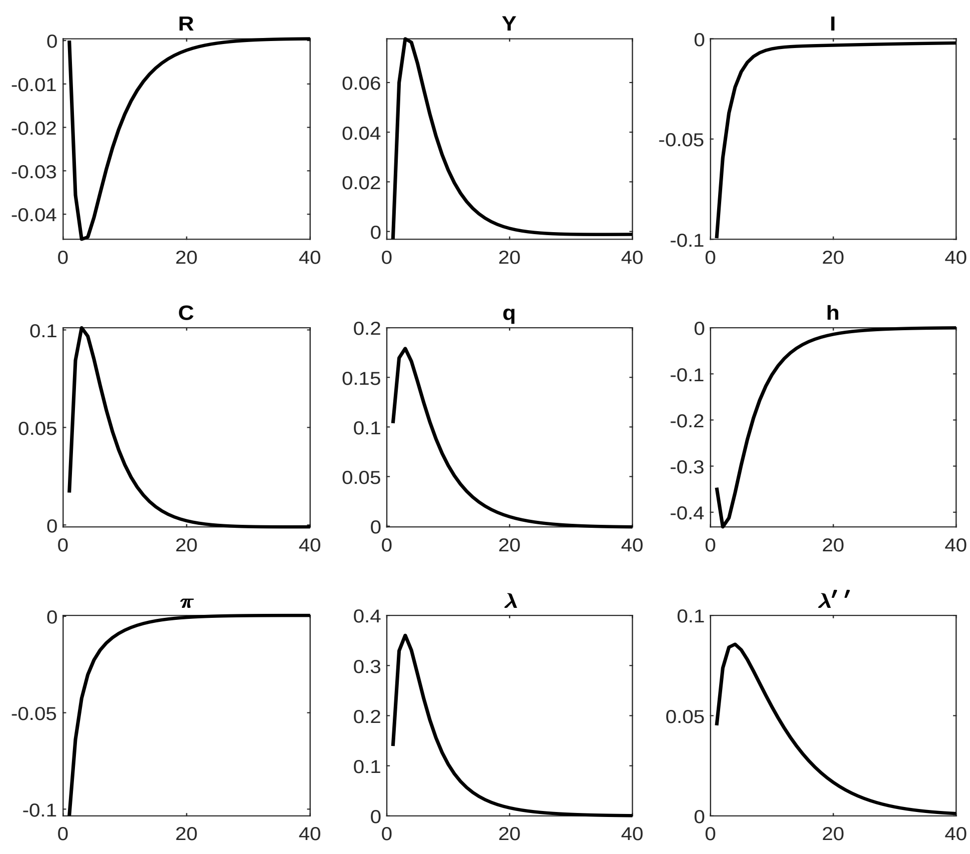
Figure 4: IRFs to Policy Shock



The policy shock impulse responses look similar to what shows up in the simple model. They are also basically exactly the same as what he reports in Figure 5 of the paper (solid lines for the model); based on the discussion above, this makes sense, as we actually can interpret the first orthogonalized response as the response to the structural monetary policy shock. The C response I show is *aggregate* consumption (the sum of consumption of the three types of agents). Investment goes down (by a bit more than output); that's really the only new response relative to the simple model. As shown before, the Lagrange multipliers on the now two borrowing constraints go up – i.e. these constraints become tighter. This amplifies the negative output response to the shock. The decline in the house price, increase in the nominal rate, and decrease in inflation tightens the constraints on patient households and entrepreneurs and *amplifies* the effects of the exogenous monetary policy disturbance.

Next, I show IRFs to the productivity shock in the model. As noted above, these are not directly comparable to the IRFs he shows in Figure 5; furthermore, I have changed the AR parameter on the productivity process to something more reasonable. These look a little weird relative to most standard models – the increase in productivity is *contractionary* on impact for output (before going up) and contractionary for investment, whereas in most standard models it would be expansionary for both over all horizons. What is going on? The productivity improvement leads to an increase in the price of housing – patient households end up wanting more of it because their consumption goes up. This price increases causes entrepreneurs to shed their stock of housing, which keeps output from rising much. Note that the entrepreneurs' constraint becomes tighter (i.e. λ_t goes up), even though q_t also goes up – part of what is driving this is that the productivity shock is deflationary. This is kind of a general result in these collateral constraint models. When constraints apply to nominal asset holdings, the constraint tends to amplify the effects of demand shocks (which move inflation in the same direction as output, therefore loosening constraints in periods where demand is high) but does the opposite for supply shocks (because inflation falls, which works to tighten the constraint).

Figure 5: IRFs to Productivity Shock



The responses to the housing preference shock are shown below. Households and patient households decide they like housing more. The immediate impact of this is pushing up the price of housing. On its own, this would cause the entrepreneur to want less housing. But the increase in q_t eases the entrepreneur's borrowing constraint by quite a lot, as evidenced by the decline in λ_t. In spite of the higher price of capital, this actually increases the amount of housing that entrepreneur's have, h_t, at least immediately. This triggers an increase in investment because the marginal product of capital is higher. Because of the higher housing initially held by the entrepreneur and the higher investment, the wage goes up initially and labor increases, so we get a temporary output boom. But eventually, the higher price of housing dissuades entrepreneurs from holding housing – basically, the constraint on entrepreneurs is only eased for a while. After that time, housing ends up being consumed by patient and impatient households, which actually results in output eventually falling.

Figure 6: IRFs to Housing Preference Shock