

Graduate Macro Theory II: Notes on Log-Linearization

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The solutions to many discrete time dynamic economic problems take the form of a system of non-linear difference equations. There generally exists no closed-form solution for such problems. As such, we must resort to numerical and/or approximation techniques.

One particularly easy and very common approximation technique is that of log linearization. We first take natural logs of the system of non-linear difference equations. We then linearize the logged difference equations about a particular point (usually a steady state), and simplify until we have a system of linear difference equations where the variables of interest are percentage deviations about a point (again, usually a steady state). Linearization is nice because we know how to work with linear difference equations. Putting things in percentage terms (that's the "log" part) is nice because it provides natural interpretations of the units (i.e. everything is in percentage terms).

First consider some arbitrary univariate function, $f(x)$. Taylor's theorem tells us that this can be expressed as a power series about a particular point x^* , where x^* belongs to the set of possible x values:

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!} (x - x^*) + \frac{f''(x^*)}{2!} (x - x^*)^2 + \frac{f^{(3)}(x^*)}{3!} (x - x^*)^3 + \dots$$

Here $f'(x^*)$ is the first derivative of f with respect to x evaluated at the point x^* , $f''(x^*)$ is the second derivative evaluated at the same point, $f^{(3)}$ is the third derivative, and so on. $n!$ reads "n factorial" and is equal to $n! = n(n-1)(n-2) \cdot \dots \cdot 1$. In words, the factorial of n is the product of all non-negative integers less than or equal to n . Hence $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, and so on.

For a function that is sufficiently smooth, the higher order derivatives will be small, and the function can be well approximated (at least in the neighborhood of the point of evaluation, x^*) linearly as:

$$f(x) = f(x^*) + f'(x^*) (x - x^*)$$

Taylor's theorem also applies equally well to multivariate functions. As an example, suppose we have $f(x, y)$. The first order approximation about the point (x^*, y^*) is:

$$f(x, y) \approx f(x^*, y^*) + f_x(x^*, y^*) (x - x^*) + f_y(x^*, y^*) (y - y^*)$$

Here f_x denotes the partial derivative of the function with respect to x and similarly for y . Suppose that we have the following (non-linear) function:

$$f(x) = \frac{g(x)}{h(x)}$$

To log-linearize it, first take natural logs of both sides:

$$\ln f(x) = \ln g(x) - \ln h(x)$$

Now use the first order Taylor series expansions:

$$\begin{aligned} \ln f(x) &\approx \ln f(x^*) + \frac{f'(x^*)}{f(x^*)} (x - x^*) \\ \ln g(x) &\approx \ln g(x^*) + \frac{g'(x^*)}{g(x^*)} (x - x^*) \\ \ln h(x) &\approx \ln h(x^*) + \frac{h'(x^*)}{h(x^*)} (x - x^*) \end{aligned}$$

The above follows from the fact that $\frac{d \ln f(x)}{dx} = \frac{f'(x)}{f(x)}$. Now put these all together:

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)} (x - x^*) = \ln g(x^*) + \frac{g'(x^*)}{g(x^*)} (x - x^*) - \ln h(x^*) - \frac{h'(x^*)}{h(x^*)} (x - x^*)$$

Group terms:

$$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)} (x - x^*) = \ln g(x^*) - \ln h(x^*) + \frac{g'(x^*)}{g(x^*)} (x - x^*) - \frac{h'(x^*)}{h(x^*)} (x - x^*)$$

But since $\ln f(x^*) = \ln g(x^*) - \ln h(x^*)$, these terms cancel out, leaving:

$$\frac{f'(x^*)}{f(x^*)} (x - x^*) = \frac{g'(x^*)}{g(x^*)} (x - x^*) - \frac{h'(x^*)}{h(x^*)} (x - x^*)$$

To put everything in percentage terms, multiply and divide each term by x^* :

$$\frac{x^* f'(x^*) (x - x^*)}{f(x^*) x^*} = \frac{x^* g'(x^*) (x - x^*)}{g(x^*) x^*} - \frac{x^* h'(x^*) (x - x^*)}{h(x^*) x^*}$$

For notational ease, define $\tilde{x} = \frac{(x-x^*)}{x^*}$, or the percentage deviation of x about x^* . Then we have:

$$\frac{x^* f'(x^*)}{f(x^*)} \tilde{x} = \frac{x^* g'(x^*)}{g(x^*)} \tilde{x} - \frac{x^* h'(x^*)}{h(x^*)} \tilde{x}$$

The above discussion and general cookbook procedure applies equally well in multivariate contexts. To summarize, the cookbook procedure for log-linearizing is:

1. Take logs
2. Do a first order Taylor series expansion about a point (usually a steady state)
3. Simplify so that everything is expressed in percentage deviations from steady state

A number of examples arise in economics. I will log-linearize the following four examples: (a) Cobb-Douglas production function; (b) accounting identity; (c) capital accumulation equation; and (d) consumption Euler equation.

(a) Cobb-Douglas Production Function: Consider a Cobb-Douglas production function:

$$y_t = a_t k_t^\alpha n_t^{1-\alpha}$$

First take logs:

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1 - \alpha) \ln n_t$$

Now do the Taylor series expansion about the steady state values:

$$\ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln a^* + \frac{1}{a^*}(a_t - a^*) + \alpha \ln k^* + \frac{\alpha}{k^*}(k_t - k^*) + (1 - \alpha) \ln n^* + \frac{(1 - \alpha)}{n^*}(n_t - n^*)$$

As above, note that $\ln y^* = \ln a^* + \alpha \ln k^* + (1 - \alpha) \ln n^*$, so these terms cancel:

$$\frac{1}{y^*}(y_t - y^*) = \frac{1}{a^*}(a_t - a^*) + \frac{\alpha}{k^*}(k_t - k^*) + \frac{(1 - \alpha)}{n^*}(n_t - n^*)$$

Now using our definition of “tilde” variables being percentage deviations from steady state, we have:

$$\tilde{y}_t = \tilde{a}_t + \alpha \tilde{k}_t + (1 - \alpha) \tilde{n}_t$$

(b) Accounting Identity: Consider the closed economy accounting identity:

$$y_t = c_t + i_t$$

Take logs:

$$\ln y_t = \ln (c_t + i_t)$$

Now do the first order Taylor series expansion:

$$\ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln(c^* + i^*) + \frac{1}{(c^* + i^*)}(c_t - c^*) + \frac{1}{(c^* + i^*)}(i_t - i^*)$$

Now we have to fiddle with this a bit more than we did for the production function case. First, note that $\ln(c^* + i^*) = \ln y^*$, so that these terms cancel out:

$$\frac{1}{y^*}(y_t - y^*) = \frac{1}{(c^* + i^*)}(c_t - c^*) + \frac{1}{(c^* + i^*)}(i_t - i^*)$$

Now multiply and divide (so as to leave the expression unchanged) each of the two terms on the right hand side by c^* and i^* , respectively:

$$\frac{1}{y^*}(y_t - y^*) = \frac{c^*}{(c^* + i^*)} \frac{(c_t - c^*)}{c^*} + \frac{i^*}{(c^* + i^*)} \frac{(i_t - i^*)}{i^*}$$

Now simplify and use our “tilde” notation:

$$\tilde{y}_t = \frac{c^*}{y^*} \tilde{c}_t + \frac{i^*}{y^*} \tilde{i}_t$$

(c) Capital Accumulation Equation: Consider the standard capital accumulation equation:

$$k_{t+1} = i_t + (1 - \delta)k_t$$

Take logs:

$$\ln k_{t+1} = \ln(i_t + (1 - \delta)k_t)$$

Do the first order Taylor series expansion:

$$\ln k^* + \frac{1}{k^*}(k_{t+1} - k^*) = \ln(i^* + (1 - \delta)k^*) + \frac{1}{(i^* + (1 - \delta)k^*)}(i_t - i^*) + \frac{(1 - \delta)}{(i^* + (1 - \delta)k^*)}(k_t - k^*)$$

Now simplify terms a bit, noting that $\ln(i^* + (1 - \delta)k^*) = \ln k^*$, so that again terms cancel:

$$\frac{1}{k^*}(k_{t+1} - k^*) = \frac{1}{k^*}(i_t - i^*) + \frac{(1 - \delta)}{k^*}(k_t - k^*)$$

Now multiply and divide the first term on the right hand side by i^* :

$$\frac{1}{k^*}(k_{t+1} - k^*) = \frac{i^*}{k^*} \frac{(i_t - i^*)}{i^*} + \frac{(1 - \delta)}{k^*}(k_t - k^*)$$

Using our “tilde” notation:

$$\tilde{k}_{t+1} = \frac{i^*}{k^*} \tilde{i}_t + (1 - \delta) \tilde{k}_t$$

(d) Consumption Euler equation: Consider the standard consumption Euler equation that emerges from household optimization problems with CRRA utility:

$$\left(\frac{c_{t+1}}{c_t} \right)^\sigma = \beta(1 + r_t)$$

$\sigma > 0$ is the coefficient of relative risk aversion. Take logs:

$$\sigma \ln c_{t+1} - \sigma \ln c_t = \ln \beta + \ln(1 + r_t)$$

Now do the first order Taylor series expansion:

$$\sigma \ln c^* + \frac{\sigma}{c^*} (c_{t+1} - c^*) - \sigma \ln c^* - \frac{\sigma}{c^*} (c_t - c^*) = \ln \beta + \ln(1 + r^*) + \frac{1}{1 + r^*} (r_t - r^*)$$

Some terms on the left hand side obviously cancel:

$$\frac{\sigma}{c^*} (c_{t+1} - c^*) - \frac{\sigma}{c^*} (c_t - c^*) = \ln \beta + \ln(1 + r^*) + \frac{1}{1 + r^*} (r_t - r^*)$$

Note that, in the steady state, $1 + r^* = \frac{1}{\beta}$, hence $\ln(1 + r^*) = -\ln \beta$. Using this, we have:

$$\frac{\sigma}{c^*} (c_{t+1} - c^*) - \frac{\sigma}{c^*} (c_t - c^*) = \frac{1}{1 + r^*} (r_t - r^*)$$

There are two semi-standard things to do with the right hand side. First, since r_t is already a percent, it is common to leave it in absolute (as opposed to percentage) deviations. Hence, we can define $\tilde{r}_t = (r_t - r^*)$, while, for all other variables, like consumption, we use the tilde notation to denote percentage deviations, so $\tilde{c}_t = \frac{(c_t - c^*)}{c^*}$, as before. Secondly, we approximate the term $\frac{1}{1 + r^*} = 1$. If the discount factor is sufficiently high, this will be a good approximation. Then, simplifying, we can write:

$$\tilde{c}_{t+1} - \tilde{c}_t = \frac{1}{\sigma} \tilde{r}_t$$

This says that the growth rate of consumption is approximately proportional to the deviation of the real interest rate from steady state, with $\frac{1}{\sigma}$ interpreted as the elasticity of intertemporal substitution.