Optimal Monetary Policy in the New Keynesian Model

Eric Sims
University of Notre Dame

Spring 2017

1 Introduction

These notes describe optimal monetary policy in the basic New Keynesian model.

2 Re-writing the Basic Model

The basic NK model can be characterized by two main (log-linear) equations: the Phillips Curve and the Euler/IS equation. Here I have taken the liberty of setting the elasticity of intertemporal substitution to unity:

$$\widetilde{\pi}_t = \gamma(\widetilde{Y}_t - \widetilde{Y}_t^f) + \beta E_t \widetilde{\pi}_{t+1}$$
$$\widetilde{Y}_t = E_t \widetilde{Y}_{t+1} - \widetilde{r}_t$$

To economize on notation, let's define $\widetilde{X}_t \equiv \widetilde{Y}_t - \widetilde{Y}_t^f$ as the output gap. Similarly, let's define $\widetilde{r}_t^f$ as the "flexible price real interest rate" (or "natural real interest rate") as that rate that would obtain if prices were fully flexible. We can solve for this by looking at the Euler equation:

$$\widetilde{r}_t^f = E_t \widetilde{Y}_{t+1}^f - \widetilde{Y}_t^f$$

Because of the assumption on preferences, in particular that the coefficient of relative risk aversion is 1, we know that labor hours would be constant if prices were flexible, so we know that the flexible price level of output evolves exogenously in line with the level of technology, with $\widetilde{y}_t^f = \widetilde{a}_t$. If we assume that technology obeys an AR(1), then we can model the flexible price equilibrium level of output as following the same AR(1).

$$\widetilde{Y}_t^f = \rho \widetilde{Y}_{t-1}^f + \varepsilon_t$$

This means that we can solve for the flexible price real interest rate as:

$$\widetilde{r}_t^f = (\rho - 1)\widetilde{Y}_t^f$$

Plugging in the process for $\widetilde{Y}_t^f$ and simplifying we get a process for the natural rate of interest:

$$\widetilde{r}_t^f = (\rho - 1)\rho\widetilde{Y}_{t-1}^f + (\rho - 1)\varepsilon_t$$
$$\widetilde{r}_t^f = \rho\widetilde{r}_{t-1}^f + (\rho - 1)\varepsilon_t$$

We can then summarize the main equations of the model as follows:

$$\widetilde{\pi}_t = \gamma\widetilde{X}_t + \beta E_t\widetilde{\pi}_{t+1} \tag{1}$$
$$\widetilde{X}_t = E_t\widetilde{X}_{t+1} - (\widetilde{r}_t - \widetilde{r}_t^f) \tag{2}$$
$$\widetilde{r}_t^f = \rho\widetilde{r}_{t-1}^f + (\rho - 1)\varepsilon_t \tag{3}$$

In the background there is also (i) a money demand relationship and (ii) a Fisher relationship. For now, we can think about the central bank effectively being able to choose $\widetilde{r}_t$, given a choice of $\widetilde{i}_t$ and an implied path for $E_t\widetilde{\pi}_{t+1}$. Given that, as well as $\widetilde{r}_t^f$ (which is the exogenous driving force), X_t and π_t will be determined.

3 Distortions and Welfare

There are two welfare-reducing distortions in the NK model, one of which is essentially "long run" and the other which is "short run". The "long run" distortion is that the flexible price level of output will be lower than what would obtain in the first best. This is because, in the flexible price version of the model, firms will set price equal to a markup over marginal cost. Hence there will be too little employment. The "short run" distortion is due to price stickiness, and leads to non-optimal fluctuations in relative prices.

We assume that the central bank is concerned with the "short run distortion" and that the "long run distortion" has been taken care of via some kind of Pigouvian tax. This works out to a subsidy for labor, equal to the inverse price markup. This means we can interpret $\widetilde{y}_t^f$ as the optimal equilibrium value of output from the perspective of the central bank. This means that, other things being equal, the central bank would like to eliminate output gaps.

4 Optimal Policy

In addition to disliking output gaps, we also assume that the central bank dislikes inflation. We assume that welfare of the central bank is a present discounted value of a quadratic loss function in inflation and the output gap. This loss function can actually be derived from taking a quadratic approximation to household welfare, while using the linearized equilibrium conditions (see Gali or Woodford's textbooks for a formal derivation. You may wonder why the central bank cares about inflation over and above the output gap (which, via the logic above, the central bank would like to eliminate). If you go back to the CES aggregator over intermediate goods, you will note that it is concave, meaning that households (or the final goods firm, if you like) would like to smooth over intermediate inputs. In a flexible price world, all intermediate producers would choose the same price (e.g. they all desire a relative price of 1). If aggregate inflation is different from zero, with price stickiness relative prices at the intermediate firm level get distorted (e.g. there is price dispersion). This leads to a non-smooth allocation of intermediates, which results in a welfare loss.

Let ω denote the relative weight that the central bank places on the output gap. In the formal derivation, you can show that this is equal to $\frac{\gamma}{\epsilon}$, where γ is the slope of the Phillips Curve and ϵ is the price elasticity of demand. The central bank would therefore like to minimize the following:

$$\min \quad \frac{1}{2} E_0 \left(\sum_{t=0}^{\infty} \beta^t \left(\widetilde{\pi}_t^2 + \omega \widetilde{X}_t^2 \right) \right)$$

The 1/2 on the outside is just a scaling term that doesn't affect the optimum but simplifies things a bit. As noted above, we can think about the central bank as choosing inflation and the output gap, given its choice of $\widetilde{i}_t$, which then determines r_t given a path of $E_t \pi_{t+1}$. This must be done subject to the constraint of the Phillips Curve, however.

We consider two cases. In the first, called "discretion", the central bank solves the one period problem each period. In the other, called "commitment", the central bank solves the entire problem at the beginning of time and commits to its policy. We start first with the discretion case. The problem can be written:

$$\min_{\widetilde{\pi}_t, \widetilde{x}_t} \quad \frac{1}{2} \left(\widetilde{\pi}_t^2 + \omega \widetilde{X}_t^2 \right)$$

$$\text{s.t.}$$

$$\widetilde{\pi}_t = \gamma \widetilde{X}_t + \beta E_t \widetilde{\pi}_{t+1}$$

Set the problem up as a Lagrangian:

$$\mathcal{L} = -\frac{1}{2} \left(\widetilde{\pi}_t^2 + \omega \widetilde{X}_t^2 \right) + \lambda \left(\widetilde{\pi}_t - \gamma \widetilde{X}_t - \beta E_t \widetilde{\pi}_{t+1} \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\pi}_t} = 0 \Leftrightarrow \widetilde{\pi}_t = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{x}_t} = 0 \Leftrightarrow \omega \widetilde{X}_t = -\lambda \gamma$$

Combing FOCs so as to eliminate the Lagrange multiplier, we get:

$$\widetilde{X}_t = -\frac{\gamma}{\omega} \widetilde{\pi}_t \tag{4}$$

Loosely speaking, this first order condition can be interpreted as a "lean against the wind" policy. If the output gap is positive, the Fed will want to pursue a policy in which it lowers inflation (and vice-versa).

Next, consider the problem under commitment. Here, the objective of the central bank is not just the current objective, but the present discounted value of the flow objective functions. A Lagrangian is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2} \left(\widetilde{\pi}_t^2 + \omega \widetilde{X}_t^2 \right) + \lambda_t \left(\widetilde{\pi}_t - \gamma \widetilde{X}_t - \beta E_t \widetilde{\pi}_{t+1} \right) \right)$$

The time -1 expectation of inflation at time 0, $E_{-1}\widetilde{\pi}_0$, is taken as given. Hence, other that $\widetilde{\pi}_0$, $\widetilde{\pi}_t$ shows up twice in the period t objective, the period t constraint, and the period $t-1$ constraint (i.e. as $E_{t-1}\widetilde{\pi}_t$). Let's work through the FOCs:

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\pi}_0} = 0 \Leftrightarrow \widetilde{\pi}_0 = -\lambda_0$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{\pi}_t} = 0 \Leftrightarrow -\beta^{t-1}\lambda_{t-1}\beta - E_{t-1}\beta^t \widetilde{\pi}_t + E_{t-1}\beta^t \lambda_t = 0 \Rightarrow E_{t-1}\widetilde{\pi}_t = E_{t-1}\lambda_t - \lambda_{t-1} \;\; \forall t > 0$$

$$\frac{\partial \mathcal{L}}{\partial \widetilde{X}_t} = 0 \Leftrightarrow \widetilde{X}_t = -\frac{\gamma}{\omega}\lambda_t$$

We can combine these first order conditions to get:

$$\widetilde{X}_0 = -\frac{\gamma}{\omega}\widetilde{\pi}_0$$

$$E_t X_{t+1} = X_t - \frac{\gamma}{\omega} E_t \widetilde{\pi}_{t+1} \;\; \forall t$$

Start this at the beginning of time and substitute forward:

$$\widetilde{X}_0 = -\frac{\gamma}{\omega}\widetilde{\pi}_0$$

$$E_0 \widetilde{X}_1 = X_0 - \frac{\gamma}{\omega} E_0 \widetilde{\pi}_1 = -\frac{\gamma}{\omega}\widetilde{\pi}_0 - \frac{\gamma}{\omega} E_0 \widetilde{\pi}_1$$

$$E_0 \widetilde{X}_0 = E_0 \widetilde{X}_1 - \frac{\gamma}{\omega} E_0 \widetilde{\pi}_2 = -\frac{\gamma}{\omega}\widetilde{\pi}_2 - \frac{\gamma}{\omega} E_0 \widetilde{\pi}_1 - \frac{\gamma}{\omega} E_0 \widetilde{\pi}_2$$

$$\vdots$$

$$E_0 \widetilde{X}_t = -\frac{\gamma}{\omega} \sum_{j=0}^{t} \widetilde{\pi}_{t-j}$$

Now, what is the sum of the inflation rates between period 0 and period t? It is the (log) price level minus the price level in the period before period 0 . Then the first order condition becomes:

$$E_0 \widetilde{X}_t = -\frac{\gamma}{\omega} E_0 \left(\widetilde{P}_t - \widetilde{P}_{-1} \right) \tag{5}$$

Since this must hold in expectation, and there are no disturbances that show up here either, it must also hold ex-post. This means we can get rid of the expectations operator in the FOC. To simplify matters, we can also normalize the initial price level to 0, so the FOC becomes:

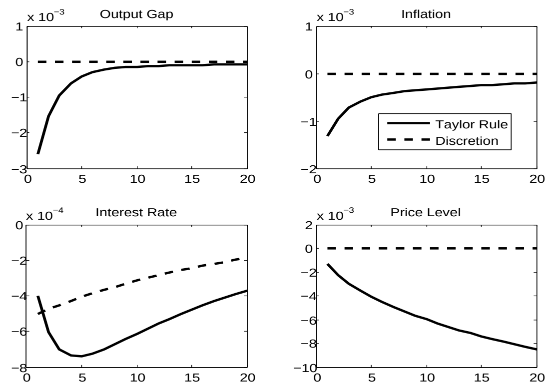
$$\widetilde{X}_t = -\frac{\gamma}{\omega} \widetilde{P}_t \tag{6}$$

This looks similar to the first order condition under commitment, but it features the price *level* as opposed to price inflation. As such, this kind of rule is called a price level targeting rule. As $t \to \infty$, $X_t \to 0$, which means that $\lim_{t \to \infty} E_0 \widetilde{P}_t = \widetilde{P}_{-1}$. This means that the policy under commitment implies that the price level always returns to trend.

Now, let's give a little thought to the solutions under discretion and commitment. From the FOC for discretion, note that a policy of $\widetilde{X}_t = \widetilde{\pi}_t = 0$ is consistent with the FOC holding. But is it consistent with the other equations of the model holding? It turns out that it is. If you plug these into the Phillips Curve, you can see that the Phillips Curve can hold at all times with $\widetilde{\pi}_t = \widetilde{X}_t = 0$. From the IS equation, this implies that $\widetilde{r}_t = \widetilde{r}_t^f$ at all times. With inflation always equal to zero, this implies that the nominal interest rate should track the natural rate of interest at all times, $\widetilde{i}_t = \widetilde{r}_t^f$. Blanchard and Gali (2007) term this result the "Divine Coincidence." What they mean by this term is that, in this model, there is no tradeoff between the two objectives in the Fed's "dual mandate." In the simples NK model, a policy of stabilizing inflation completely will automatically stabilize the output gap, and vice versa.

Interestingly, this also means that there is no gain from commitment over discretion. If the discretion solution always has both inflation and the gap equal to zero, it achieves the global minimum of the objective function. There is also no relevant tradeoff between real (gap) and nominal (inflation) stabilization: the central bank can achieve both. There are a couple of ways to implement this policy in a Dynare code. The easiest is to simply replace a Taylor rule (or money supply equation) with the first order condition in the list of equilibrium conditions. This will spit out a time path for the nominal interest rate and time paths for inflation and the gap.

Below I show impulse responses for a quantitative version of the model. I set $\gamma = 0.3$, $\beta = 0.99$, $\rho = 0.95$, and $\epsilon = 10$, which implies a weight on the output gap of 0.03. For ease of comparison, I compare the optimal discretion/commitment responses (as noted above, these are the same) with the response that would obtain with a Taylor rule of the form $\widetilde{i}_t = \rho_i \widetilde{i}_{t-1} + (1 - \rho_i)\phi_\pi \widetilde{\pi}_t$, where I set $\rho_i = 0.8$ and $\phi_\pi = 1.5$. These are impulse responses to a productivity shock (which manifests itself as a shock to the natural rate of interest):



As predicted via our discussion above, the optimal policy results in no movement of the gap or inflation at any horizon. This also then shows up as a constant price level. In contrast, the Taylor rule results in a negative output gap (output rises by less than the flexible price level), disinflation, and a pretty large fall in the price level. The Taylor rule also results in a larger and more persistent decline in the nominal interest rate than does the optimal policy under discretion.

Would it be possible to implement the optimal discretionary/commitment policy (again, they are the same here) using a rule for the interest rate or money growth? At least in this circumstance, the answer is yes.

First, consider a money rule. Suppose that the demand for real balances is given by:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{Y}_t - \kappa\widetilde{i}_t$$

The solution discussed above means that $\widetilde{Y}_t = \widetilde{Y}_t^f$ and $\widetilde{i}_t = \widetilde{r}_t^f$. Plug this in:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{Y}_t^f - \kappa\widetilde{r}_t^f$$

From above, we have $\widetilde{r}_t^f = (\rho - 1)\widetilde{Y}_t^f$. Plug this in:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{Y}_t^f - \kappa(\rho-1)\widetilde{Y}_t^f = \left(\frac{1}{\nu}\left(1 - \kappa(\rho-1)\right)\right)\widetilde{Y}_t^f$$

Now we know that $\widetilde{Y}_t^f = \rho\widetilde{Y}_{t-1}^f$. Hence:

$$\widetilde{m}_t = \frac{1}{\nu}\widetilde{Y}_t^f - \kappa(\rho-1)\widetilde{Y}_t^f = \left(\frac{1}{\nu}\left(1 + \kappa(1-\rho)\right)\right)\rho\widetilde{Y}_{t-1}^f + \left(\frac{1}{\nu}\left(1 + \kappa(1-\rho)\right)\right)\varepsilon_t$$

Iterate this back one period to eliminate $\widetilde{Y}_t^f$:

$$\widetilde{m}_t = \rho\widetilde{m}_{t-1} + \left(\frac{1}{\nu}\left(1 + \kappa(1-\rho)\right)\right)\varepsilon_t$$

Now let's write this in terms of the nominal money supply, which the central bank controls, by noting that $\widetilde{m}_t = \widetilde{M}_t - \widetilde{P}_t$, with $\widetilde{\pi}_t = \widetilde{P}_t - \widetilde{P}_{t-1}$:

$$\widetilde{M}_t = \rho \widetilde{M}_{t-1} - \rho \widetilde{\pi}_t - (1-\rho)\widetilde{P}_t + \left(\frac{1}{\nu} \left(1 + \kappa(1-\rho)\right) \right) \varepsilon_t \tag{7}$$

In other words, the central bank needs to set the money supply according to an AR(1) in the level, with AR coefficient equal to the AR coefficient on the technology process. As long as $\rho < 1$, it will raise the money supply in response to positive technology shocks. There is also a slight adjustment for lagged inflation and the price level.

The central bank can also accomplish its goal with a modified Taylor rule. In particular, suppose it sets the nominal interest rate according to the following rule.

$$\widetilde{i}_t = \widetilde{r}_t^f + \phi_\pi \widetilde{\pi}_t + \phi_x \widetilde{X}_t \tag{8}$$

This Taylor rule looks similar to the Taylor rules we've looked at, but with two differences: there is no smoothing parameter, and there is a "stochastic intercept" equal to the natural rate of interest. A policy rule of $\widetilde{i}_t = \widetilde{r}_t^f$ would result in equilibrium indeterminacy – for an interest rate rule to work, as we have seen, there needs to be a sufficiently strong reaction to endogenous variables like inflation and/or the output gap. What is kind of interesting here, however, is that in equilibrium inflation and the output gap would always be zero with this interest rate rule (provided ϕ_π and/or ϕ_x are sufficiently large). This means that one would observe $\widetilde{i}_t = \widetilde{r}_t^f$, but if the central bank announced that as the policy rule, it would result in indeterminacy. The central bank in essence has to promise to move interest rates sufficiently in response to inflation and the output gap to prevent those from ever occurring.

5 Cost-Push Shocks and an Output-Inflation Tradeoff

Let's modify the Phillips curve to contain an additional term:

$$\widetilde{\pi}_t = \gamma \widetilde{X}_t + \beta E_t \pi_{t+1} + \widetilde{u}_t \tag{9}$$

Here $\widetilde{u}_t$ is referred to as the "cost-push" term. It is a shock which, in a sense, changes the output-inflation tradeoff. It is exogenous and the central bank takes it as given. The structural interpretation of this term is not always very clear – one specific interpretation is time-variation in ϵ, which means there are time-varying desired markups. More generally, the cost-push term can be thought of as something which drives a wedge between marginal cost and the output gap. In reality it is a convenient shortcut to make the central bank's problem more interesting. Assume that it follows an AR(1):

$$\widetilde{u}_t = \rho_u \widetilde{u}_{t-1} + \varepsilon_{u,t} \tag{10}$$

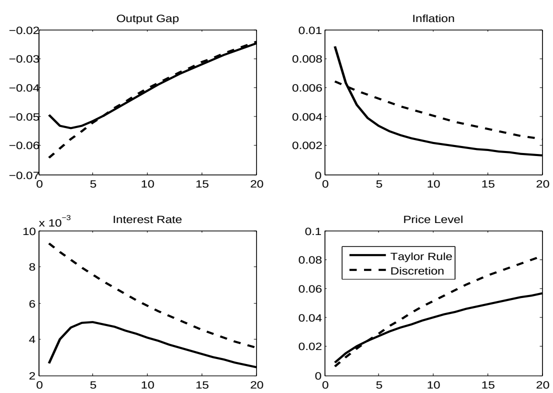
Why does the inclusion of the cost-push term make the central bank's problem more interesting? The optimization problem of the central bank is identical to before, and results in the same first order conditions under discretion and commitment:

$$\widetilde{X}_t = -\frac{\gamma}{\omega}\widetilde{\pi}_t \tag{11}$$

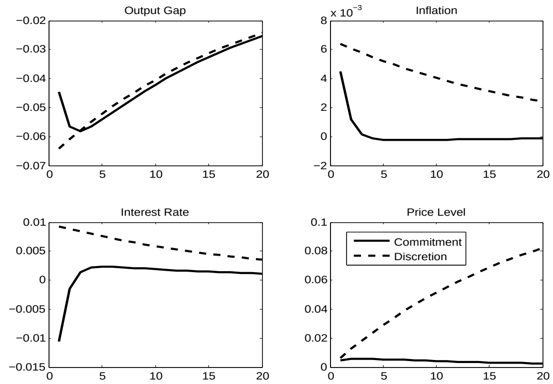
$$\widetilde{X}_t = -\frac{\gamma}{\omega}\widetilde{P}_t \tag{12}$$

The reason the cost-push shock makes things more interesting is that it is in general *not* going to be feasible to implement a no gap / no inflation equilibrium. If $\widetilde{u}_t \neq 0$, then $\widetilde{X}_t = \widetilde{\pi}_t$ is not consistent with the Phillips Curve holding. In other words, the inclusion of the cost-push shock (i) generates a non-trivial tradeoff for the central bank, as it cannot achieve a zero inflation / zero gap outcome, and (ii) opens up the door for welfare gains from commitment. Of course, conditional on a productivity shock (so that $\widetilde{u}_t = 0$), there is no real tradeoff – so it would be optimal for the central bank to completely stabilize both inflation and the output gap in response to productivity shocks, with no resulting welfare gain from commitment.

Below I compute impulse responses to a cost-push shock under various different monetary policy rules. Given the linearity of the model, the responses to the productivity shock are identical to what I showed above, and under either commitment or discretion both the output gap and inflation are completely stabilized in response to the productivity shock. Regardless of the kind of policy (either of the optimal policies or the Taylor rule), the cost-push shock causes the output gap to decline (output falls) and inflation rises. The first set of plots shows the responses under the baseline Taylor rule as well as those under the optimal policy under discretion. Interestingly, the simple Taylor rule appears to do better in the sense of a small decline in the output gap and a smaller increase in the price level (smaller increase in inflation over most horizons). Under either scenario, there is a permanent increase in the price level following the cost-push shock.



Next, I compare the impulse responses from the optimal policy under discretion with the optimal policy under commitment. Here we see some stark differences. First, the output gap response is smaller (on impact) under commitment than under discretion. Second, the inflation response is much smaller under commitment and much less persistent, so much so that the price level returns to its original value under commitment, whereas the price level permanently rises under discretion. Third, there is a very different behavior of the nominal interest rate: under commitment the nominal rate initially falls, whereas under discretion it rises.



What is the source of the gains from commitment? As you will recall from looking at the first order conditions, the difference between commitment and discretion boils down to an implicit price level (commitment) versus inflation (discretion) target. The price level target has the effect of better anchoring expected inflation, $E_t\widetilde{\pi}_{t+1}$, because agents know that the central bank will always enact policy so as to return the price level to its target. Better anchored inflation expectations (e.g. less volatile expected inflation) improves the available tradeoff between current inflation and the output gap that the central bank can achieve. Mechanically from the Phillips Curve, the more $E_t\widetilde{\pi}_{t+1}$ moves around, the more either inflation or the output gap will have to move around (or both), either of which reduce welfare. Hence, a central message that comes out of this exercise is that commitment and expectations are of central importance for good monetary policy.

6 Optimal Policy Numerically Using Second Order Approximations

In many contexts, monetary policy being one, we can't use straight up linearization about a non-stochastic steady state to think about optimal policy.[1] Why is this? In a linearization, expected values of variables are equal to their steady state values, and in many contexts policy doesn't affect the steady state, and hence doesn't affect means. Concretely, in a linearization the expected value of utility (or welfare, taken to mean the present discounted value of flow utility) is independent of policy parameters like what show up in a Taylor rule. Therefore, the natural objective function for a policy-maker (expected welfare) isn't impacted by the policy parameters, and the problem is degenerate.

Define welfare as the present discounted value of the flow utility of a representative agent. This can be written recursively as:

[1] Note that in different contexts where policy choices affect the steady state this is not the case. For example, the level of trend inflation will affect the non-stochastic steady state in the basic New Keynesian model. Similarly, tax instruments will affect the steady state of a model with fiscal policy.

$$V_t = U(C_t, N_t) + \beta E_t V_{t+1} \tag{13}$$

Note that this looks like a Bellman equation, but there is no max operator, because I assume that C_t and N_t have been chosen optimally. The objective of a policy maker will be to pick policies to maximize the unconditional expectation of welfare, e.g. $E(V_t)$. In a first order approximation, $E(V_t) = V^* = \frac{1}{1-\beta}U(C^*, N^*)$, which in this case is independent of the stance of monetary policy.

To think about policy, we therefore need to use a higher order approximation to welfare. There are two approaches one can take here. In the one described above, you linearize the equilibrium conditions of the model, but take a second order approximation to the recursive representation of welfare. Doing that and simplifying yields the quadratic loss function I showed above. In the other approach, you can take a second order approximation to *all* the equilibrium conditions, including the recursive representation of welfare. This isn't as neat in the sense that it's not possible to derive simple quadratic loss functions that are clean, but one can numerically calculate the expected values of welfare for different policy specifications and compare those to one another.

I'll do a concrete example. I take the basic NK model where policy is characterized by an exogenous money supply rule. I solve the model in Dynare using a second order approximation (rather than first), which is Dynare's default anyway. Below is the text of my .mod file:

```
var Y C int infl inflr N w mc A vp x1 x2 m dm r Yf gap V;
varexo ea em;

parameters psi beta phi sigma eta epsi theta rhoa rhom sigea sigem pistar;

load parameter_nk_money;

set_param_value('psi',psi);
set_param_value('phi',phi);
set_param_value('eta',eta);
set_param_value('sigma',sigma);
set_param_value('theta',theta);
set_param_value('rhoa',rhoa);
set_param_value('sigea',sigea);
set_param_value('sigem',sigem);
set_param_value('pistar',pistar);
set_param_value('rhom',rhom);
set_param_value('beta',beta);
set_param_value('epsi',epsi);

model;

% (1) Euler equation
exp(C)^(-sigma) = beta*exp(C(+1))^(-sigma)*(1+int)*(1+infl(+1))^(-1);

% (2) Labor supply
```

```
psi*exp(N)^(eta) = exp(C)^(-sigma)*exp(w);

% (3) Money demand
exp(m) = theta*((1+int)/int)*exp(C)^(sigma);

% (4) Marginal cost
exp(mc) = exp(w)/exp(A);

% (5) Resource constraint
exp(C) = exp(Y);

% (6) Production function
exp(Y) = exp(A)*exp(N)/exp(vp);

% (7) Price dispersion
exp(vp) = (1-phi)*(1+inflr)^(-epsi)*(1+infl)^(epsi) + (1+infl)^(epsi)*phi*exp(vp(-1));

% (8) Price evolution
(1+infl)^(1-epsi) = (1-phi)*(1+inflr)^(1-epsi) + phi;

% (9) Reset price
1 + inflr = (epsi/(epsi - 1))*(1+infl)*exp(x1)/exp(x2);

% (10) x1
exp(x1) = exp(C)^(-sigma)*exp(mc)*exp(Y) + phi*beta*(1+infl(+1))^(epsi)*exp(x1(+1));

% (11) x2
exp(x2) = exp(C)^(-sigma)*exp(Y) + phi*beta*(1+infl(+1))^(epsi-1)*exp(x2(+1));

% (12) Productivity
A = rhoa*A(-1) + ea;

% (13) Real balance growth
dm = (1-rhom)*pistar - infl + rhom*infl(-1) + rhom*dm(-1) + em;

% (14) Real balance growth definition
dm = m - m(-1);

% (15) Real interest rate (Fisher relationship)
r = int - infl(+1);

% (16) Flexible price output
exp(Yf) = ((1/psi)*(epsi-1)/epsi)^(1/(sigma + eta))*exp(A)^((1+eta)/(sigma + eta));

% (17) Output gap
gap = Y - Yf;

% (18) Welfare
V = C - psi*exp(N)^(1+eta)/(1+eta) + beta*V;
```

```

end;

steady;

shocks;
var ea = sigea^2;
var em = sigem^2;
end;

stoch_simul(order=2,irf=20,ar=0,nocorr,nograph);
```

The above is the same as my earlier Dynare file, except for the fact that I used a second order approximation and that I included a recursive representation of welfare as an equilibrium condition. In writing that, since I assume that $\sigma = 1$, I write flow utility as log over consumption; since the model is log-linearized and Dynare interprets "C" as the natural log of consumption, that's why consumption appears linear in writing down V_t but note that it's actually the log.

Below is the non-stochastic steady state of the model:

```
STEADY-STATE RESULTS:

Y       -0.0526803
C       -0.0526803
int     0.010101
infl    0
inflr   0
N       -0.0526803
w       -0.105361
mc      -0.105361
A       0
vp      0
x1      2.11105
x2      2.21641
m       4.55249
dm      0
r       0.010101
Yf      -0.0526793
gap     0
V       -50.268
```

Below are the moments, including the expected values of the variables:

```
APROXIMATED THEORETICAL MOMENTS
```

VARIABLE	MEAN	STD. DEV.	VARIANCE
Y	-0.0549	0.0290	0.0008
C	-0.0549	0.0290	0.0008
int	0.0100	0.0000	0.0000
infl	0.0000	0.0060	0.0000
inflr	0.0022	0.0240	0.0006
N	-0.0527	0.0129	0.0002
w	-0.1077	0.0314	0.0010
mc	-0.1077	0.0257	0.0007
A	-0.0000	0.0320	0.0010
vp	0.0022	0.0000	0.0000
x1	1.2631	0.0693	0.0048
x2	1.3666	0.0525	0.0028
m	4.5571	0.0290	0.0008
dm	0.0000	0.0082	0.0001
r	0.0100	0.0039	0.0000
Yf	-0.0527	0.0320	0.0010
gap	-0.0022	0.0129	0.0002
V	-50.5002	3.1501	9.9233

We observe here that the mean / expected value of welfare is lower (-50.5002) than steady state welfare (-50.268). The reason why average welfare is lower than steady state welfare is because agents don't like volatility because of concavity in preferences. In a first order approximation this wouldn't show up, but in the second order approximation it does.

We can then solve the model under an alternative monetary policy rule. For example, suppose that we have a money growth rule, but we set the standard deviations of monetary shocks to 0. This in effect means that the money supply would be constant. Below are the steady state values and expected values:

STEADY-STATE RESULTS:

Y	-0.0526803
C	-0.0526803
int	0.010101
infl	0
inflr	0
N	-0.0526803
w	-0.105361
mc	-0.105361
A	0

vp	0
x1	2.11105
x2	2.21641
m	4.55249
dm	0
r	0.010101
Yf	-0.0526793
gap	0
V	-50.268

APROXIMATED THEORETICAL MOMENTS

VARIABLE	MEAN	STD. DEV.	VARIANCE
Y	-0.0537	0.0276	0.0008
C	-0.0537	0.0276	0.0008
int	0.0101	0.0000	0.0000
infl	0.0000	0.0039	0.0000
inflr	0.0009	0.0158	0.0002
N	-0.0527	0.0092	0.0001
w	-0.1064	0.0258	0.0007
mc	-0.1064	0.0184	0.0003
A	0.0000	0.0320	0.0010
vp	0.0009	0.0000	0.0000
x1	1.2560	0.0443	0.0020
x2	1.3606	0.0342	0.0012
m	4.5523	0.0276	0.0008
dm	0.0000	0.0039	0.0000
r	0.0101	0.0025	0.0000
Yf	-0.0527	0.0320	0.0010
gap	-0.0010	0.0092	0.0001
V	-50.3696	3.1488	9.9152

Getting rid of the shock to the money growth rate doesn't affect the steady state value of welfare, but it does increase the mean value of welfare (-50.3696 vs. -50.5002). The units of welfare are not particularly interpretable, so we often want to express the differences in "consumption equivalent" units. The thought experiment is to say "How much consumption would one be willing to give up (each period) under one policy to have the same welfare as under a different policy." It is arbitrary which policy one takes to be the "baseline" – here I'm going to assume it's the policy with no monetary shock.

Let λ denote the fraction of consumption one would be willing to give up in each period in one economy, call this economy 0. With log utility over consumption, we have:

$$E(V_t^0(\lambda)) = E\left[\ln(C_t(1+\lambda)) - \theta\frac{N_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1}^0(\lambda)\right]$$

Since the λ shows up every period and is deterministic, this reduces to:

$$E(V_t^0(\lambda)) = \frac{1}{1-\beta}\ln(1+\lambda) + E\left[\ln C_t - \theta\frac{N_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1}^0\right]$$

The term in brackets is just:

$$E(V_t^0(\lambda)) = \frac{1}{1-\beta}\ln(1+\lambda) + E(V_t^0)$$

Then we want to find the λ that equates this with expected welfare in the alternative economy, call it economy 1. We have:

$$E(V_t^0(\lambda)) = E(V_t^1)$$

Or:

$$\frac{1}{1-\beta}\ln(1+\lambda) + E(V_t^0) = E(V_t^1)$$

Solving for λ:

$$\lambda = \exp\left((1-\beta)(E(V_t^1) - E(V_t^0))\right) - 1$$

If I take the baseline economy 0 to have higher welfare, then the term inside the exp is negative, which means $\lambda < 0$. This makes sense – you would be willing to give up consumption in the high welfare economy to have the same welfare as another economy governed by a policy resulting in lower welfare. Using the differences in expected welfare for the two economies in question here, and a value of $\beta = 0.99$, I get a value of $\lambda = -0.0013$. This means that an agent would be willing to forfeit about 0.1 percent of consumption each period in the economy with no monetary shocks to avoid going to the economy with monetary shocks. This number may not seem large but we typically find pretty small welfare differences under different macro policies, so it's not abnormally low.

I solved the model assuming zero trend inflation. Suppose I solve the model where steady state inflation is instead set to 0.005 (approximately 2 percent at an annualized frequency). Below are the steady states and the means:

```
STEADY-STATE RESULTS:


Y        -0.0541634
C        -0.0541634
int      0.0151515
```

infl	0.005
inflr	0.0216847
N	-0.0522648
w	-0.106428
mc	-0.106428
A	0
vp	0.00189857
x1	1.40985
x2	1.49874
m	4.15053
dm	0
r	0.0101515
Yf	-0.0526793
gap	-0.00148402
V	-50.4537

APROXIMATED THEORETICAL MOMENTS

VARIABLE	MEAN	STD. DEV.	VARIANCE
Y	-0.0574	0.0283	0.0008
C	-0.0574	0.0283	0.0008
int	0.0151	0.0000	0.0000
infl	0.0050	0.0057	0.0000
inflr	0.0245	0.0269	0.0007
N	-0.0524	0.0160	0.0003
w	-0.1097	0.0319	0.0010
mc	-0.1097	0.0297	0.0009
A	0.0000	0.0320	0.0010
vp	0.0050	0.0030	0.0000
x1	1.4241	0.0790	0.0062
x2	1.5106	0.0598	0.0036
m	4.1517	0.0283	0.0008
dm	-0.0000	0.0084	0.0001
r	0.0101	0.0039	0.0000
Yf	-0.0527	0.0320	0.0010
gap	-0.0047	0.0138	0.0002
V	-50.7913	3.2273	10.4153

Unlike the standard deviation of the monetary policy shock, the level of trend inflation does affect the steady state – we can see that steady state welfare is lower with higher trend inflation (-50.4537 vs. -50.268). We also see that expected welfare is lower, at -50.7913 (compared to -50.5002).

We could calculate consumption equivalent welfare differences based both on steady state welfare or expected welfare. Based on steady states, we would get $\lambda^{ss} = -0.0019$; based on the means, we would get $\lambda^{ss} = -0.0029$. Hence, we observe that the welfare loss based on means is higher than the welfare loss based on the steady state. This is because trend inflation does two things: first, it distorts the steady state by effectively increasing the steady state markup and increasing steady state price dispersion. But it also interacts with the shocks to have a bigger effect on means: with positive trend inflation, price dispersion is first order, which makes stochastic shocks more costly.

Now, let's compare the welfare of the money growth rule (baseline case with no trend inflation and the monetary shock turned off) to a Taylor rule with coefficients $\rho_i = 0.8$, $\phi_\pi = 1.5$, and $\phi_x = 0.5$ (again, with no monetary shock). Below are the steady states and means:

```
STEADY-STATE RESULTS:

Y       -0.0526802
C       -0.0526802
int     0.0101019
infl    8.93975e-07
inflr   3.57599e-06
N       -0.0526802
w       -0.10536
mc      -0.10536
A       0
vp      -2.31901e-09
x1      1.2514
x2      1.35676
m       4.5524
r       0.010101
Yf      -0.0526793
gap     -8.75914e-07
V       -50.268
```

```
APROXIMATED THEORETICAL MOMENTS

VARIABLE      MEAN    STD. DEV.   VARIANCE
Y            -0.0528     0.0307     0.0009
C            -0.0528     0.0307     0.0009
int           0.0102     0.0027     0.0000
infl          0.0001     0.0017     0.0000
inflr         0.0006     0.0069     0.0000
N            -0.0527     0.0019     0.0000
w            -0.1055     0.0295     0.0009
```

mc	-0.1055	0.0039	0.0000
A	-0.0000	0.0320	0.0010
vp	0.0002	0.0000	0.0000
x1	1.2561	0.0363	0.0013
x2	1.3610	0.0312	0.0010
m	4.5783	0.2922	0.0854
r	0.0101	0.0014	0.0000
Yf	-0.0527	0.0320	0.0010
gap	-0.0002	0.0019	0.0000
V	-50.2862	3.1894	10.1721

Here, we observe that expected welfare in the Taylor rule economy is -50.2862 (versus -50.3696 in the economy with a money growth rule and no shock to the money growth rate). Calculating the consumption equivalent difference yields -0.0008365, which means that agents would be willing to give up roughly 0.08 percent of consumption each period in the Taylor rule economy to avoid being subjected to the money growth rate economy.

We could alternatively see what welfare is in an economy with inflation targeting. That is, I replace the Taylor rule in my code with a condition that inflation always equals its steady state value. This doesn't affect steady states but affects means. The mean value of welfare is -50.2680. This is slightly preferred to the Taylor rule, which is again slightly preferred to the money growth rule, though again the welfare differences tend to be quite small.