Graduate Macro Theory II:
Notes on Neoclassical Growth Model

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1 Basic Neoclassical Growth Model

The economy is populated by a large number of infinitely lived agents. These agents are identical, and so we can effectively treat them as one. These agents consume, save in physical capital, and supply one unit of labor each period inelastically. Time runs from $t = 0, \dots, \infty$. The population at each point in time is $N_t = (1+n)^t N_0$, with N_0 given. Households get flow utility from consumption, C_t, by an increasing and concave function, $u(C_t)$. They discount the future by $0 < \beta < 1$.

Firms produce output using capital and labor, where these inputs are turned into outputs through a constant returns to scale production function. The firms are owned by the households. There are two other exogenous inputs to production: Z_t, which is called labor augmenting technology; and A_t, which is neutral technological progress. Assume that $Z_t = (1 + z)^t Z_0$, Z_0 given. Suppose that A_t follows some stationary stochastic process. Both Z_t and A_t are measures of productivity in the sense that they govern how efficient firms are at transforming inputs into output. We differentiate between the two along two dimensions: (i) Z_t controls the trend while A_t measures stochastic fluctuations in productivity about that trend, and (ii) Z_t is labor augmenting technology (it multiplies N_t, not K_t) whereas A_t is neutral productivity (it multiplies a function of N_t and K_t). The second assumption is really just one of convenience in terms of stationarizing variables. The production function is:

$$Y_t = A_t F(K_t, Z_t N_t) \tag{1}$$

Output must be either consumed or used as investment in new capital goods. Hence the aggregate accounting identity is:

$$Y_t = C_t + I_t \tag{2}$$

Capital accumulates according to the following equation, with $0 < \delta < 1$ the depreciation rate on capital. There is an assumed one period delay between when new capital is accumulated and when it is productive:

$$K_{t+1} = I_t + (1-\delta)K_t \tag{3}$$

Equations (1)-(3) can be combined:

$$K_{t+1} = A_t F(K_t, Z_t N_t) - C_t + (1-\delta)K_t \tag{4}$$

The equilibrium of the economy can be expressed as the solution to a social planner's problem (we will discuss decentralized competitive equilibria versus social planner's solutions later):

$$\max_{C_t, K_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\text{s.t.}$$

$$K_{t+1} = A_t F(K_t, Z_t N_t) - C_t + (1-\delta)K_t$$

In this problem the social planner wants to pick a sequence of consumption and one period ahead capital from the beginning of time until the end to maximize the expected lifetime utility subject to the sequence of resource constraints. There are no prices in the planner's problem – a planner directly picks allocations, whereas in a decentralized equilibrium prices are determined endogenously so as to coordinate behavior of different models actors in such a way as to make markets clear.

There are two sources of non-stationarity in the model – population growth, n; and technological progress, z. Standard solution methodologies require that the variables of the model be stationary. There are two ways to proceed – either transform the economy and then characterize the equilibrium conditions, or get the equilibrium conditions and then transform them to be consistent with stationarity. I always find the optimality conditions of untransformed variables and then re-write the equilibrium conditions in stationary form.

I'm going to impose a standard functional form specification on preferences, which is that flow utility over consumption is iso-elastic:

$$u(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

For two functions, $f(x)$ and $g(x)$, if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $+/-\infty$, L'Hopital's rule says:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

L'Hopital's rule is applicable in this case because, as $\sigma \to 1$, both numerator and denominator of flow utility go to zero. The derivative of $C_t^{1-\sigma}$ with respect to σ is $-\ln C_t \times C_t^{1-\sigma}$. As $\sigma \to 1$, this is just $-\ln C_t$. The derivative of $1-\sigma$ with respect to σ is -1. Hence the flow utility function simply converges to $\ln C_t$ as $\sigma \to 1$. Because the "-1" is additive in this specification of flow utility, it is common to write this utility function without it: $u(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$. This is fine, but technically one needs the "$-1''$ to invoke L'Hopital's rule.

Finally, we will almost always be working with a Cobb-Douglas production technology, so that:

$$F(K_t, Z_t N_t) = K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha}, \quad 0 < \alpha < 1$$

2 First Order Conditions

We can find the first order conditions necessary for an interior solution a couple of different ways – either the method of Lagrange multipliers or by expressing the problem as a dynamic program. I'll begin with the Lagrangian formulation.

2.1 A Lagrangian Formulation

I write Lagrangians as current value Lagrangians, which means that the discount factor multiplies the constraints. An alternative formulation is to use a present value Lagrangian, in which case the discount factor does not multiply the constraints. These yield identical solutions and only differ in the interpretation of the multipliers. The current value Lagrangian for the planner's problem of the neoclassical growth model is is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t E_0 \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_t \left(A_t K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha} - C_t + (1-\delta)K_t - K_{t+1} \right) \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow C_t^{-\sigma} = \lambda_t \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} \left(Z_{t+1} N_{t+1}\right)^{1-\alpha} + (1-\delta) \right) \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Leftrightarrow K_{t+1} = A_t K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha} - C_t + (1-\delta)K_t \tag{7}$$

This Lagrangian formulation is called "current value" because the Lagrange multiplier is the marginal utility of consumption at time t. A present value Lagrangian would have the multiplier equal to the marginal utility of consumption at time t discounted back to the beginning of time. The present value Lagrangian formulation would be:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t E_0 \frac{C_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} E_0 \mu_t \left(A_t K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha} - C_t + (1-\delta)K_t - K_{t+1} \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \beta^t C_t^{-\sigma} = \mu_t \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \mu_t = E_t \mu_{t+1} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} \left(Z_{t+1} N_{t+1}\right)^{1-\alpha} + (1-\delta) \right) \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = 0 \Leftrightarrow K_{t+1} = A_t K_t^{\alpha} \left(Z_t N_t\right)^{1-\alpha} - C_t + (1-\delta) K_t \tag{10}$$

The only difference here with the earlier setup is that μ_t is the present value multiplier on the constraint – i.e. it is multiplied by β^t to discount it back to period 0. If you eliminate the multipliers you get the same first order conditions in both setups.

In addition to the first order conditions, which are necessary for an optimum, there is also the transversality condition, which combined with the above FOC is necessary and sufficient for an optimum:

$$\lim_{T \to \infty} \beta^T \lambda_T K_{T+1} = 0$$

The transversality condition is most easily understand by supposing that the problem is finite horizon. At the end of time, you would never want to leave any utility on the table, so to speak, since there is no more tomorrow. K_{T+1}, the amount of capital left over at the end of the time (period T is the last period, K_{T+1} is how much would be left over after producing in period T), could always be consumed, yielding utility $\lambda_T K_{T+1}$. If the present value of this is not zero, then the household has "over-saved" and could not have been optimizing.

(5) can be combined with (6) to eliminate the multiplier, yielding:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} \left(Z_{t+1} N_{t+1}\right)^{1-\alpha} + (1-\delta) \right) \tag{11}$$

The variables of the model are growing, inheriting deterministic trends from both Z_t and N_t. To re-scale the first order conditions so that all the variables are the model are stationary, define lower case variables as those divided by $Z_t N_t$:

$$c_t \equiv \frac{C_t}{Z_t N_t} \tag{12}$$

$$y_t \equiv \frac{Y_t}{Z_t N_t} \tag{13}$$

$$k_t \equiv \frac{K_t}{Z_t N_t} \tag{14}$$

Multiply and divide both sides of (8) by $Z_t N_t$ as needed to make it stationary:

$$(Z_tN_t)^{-\sigma}\left(\frac{C_t}{Z_tN_t}\right)^{-\sigma} = \beta\left(Z_{t+1}N_{t+1}\right)^{-\sigma}E_t\left(\frac{C_{t+1}}{Z_{t+1}N_{t+1}}\right)^{-\sigma}\left(\alpha A_{t+1}\left(\frac{K_{t+1}}{Z_{t+1}N_{t+1}}\right)^{\alpha-1} + (1-\delta)\right)$$

Using the transformed variable notation and simplifying somewhat, we have:

$$c_t^{-\sigma} = \beta\left(\frac{Z_{t+1}N_{t+1}}{Z_tN_t}\right)^{-\sigma}E_tc_{t+1}^{-\sigma}\left(\alpha A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta)\right)$$

Noting that $\frac{Z_{t+1}}{Z_t} = 1+z$ and $\frac{N_{t+1}}{N_t} = 1+n$, we can define $\gamma = (1+z)(1+n)$ and write this as:

$$c_t^{-\sigma} = \beta\gamma^{-\sigma}E_tc_{t+1}^{-\sigma}\left(\alpha A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta)\right) \tag{15}$$

Effectively the presence of trend growth changes the discount factor that shows up in the Euler equation: instead of β, it is $\beta\gamma^{-\sigma}$. For plausible parameterizations, there isn't much difference.

Now transform the capital accumulation equation by multiplying and dividing as necessary:

$$Z_{t+1}N_{t+1}\frac{K_{t+1}}{Z_{t+1}N_{t+1}} = A_t\left(\frac{K_t}{Z_tN_t}\right)^{\alpha}Z_tN_t - Z_tN_t\frac{C_t}{Z_tN_t} + (1-\delta)\frac{K_t}{Z_tN_t}Z_tN_t$$

Simplifying:

$$\gamma k_{t+1} = A_tk_t^{\alpha} - c_t + (1-\delta)k_t \tag{16}$$

Again, this only differs from the regular specification by the factor γ on the left hand side, which is close to one anyway. (12)-(13), along with the transversality condition (which is the same expressed in "efficiency units", divided by Z_tN_t, or regular units), characterize the solution to the planner's problem. Because of the fact that the equilibrium conditions of the transformed economy do not differ much from the efficiency units representation, it is common to just ignore trend growth altogether. It really has very little effect on the numerical solution of the model.

At the end of the day, the solution of the model is a system of two difference equations – (15) and (16). These jointly describe the dynamics of the endogenous state variable, k_t, and the control, or "jump," variable c_t. Since capital is predetermined, you have to have initial initial condition on capital exogenously given. The "solution" of the model is a value of c_t, given k_t, that is consistent with the two difference equations. In general, there are *many* different values of c_t consistent with the two difference equations (at an intuitive level, think about it – there are two equations but only one real "unknown" since k_t is given). To get a unique solution, we use the transversality condititon combined with a feasibility constraint – neither c_t nor k_t can go to zero in the limit, or, colloquially, the economy cannot "explode." If the economy cannot explode, we *must* in the limit approach a steady state, a system in which neither k_t nor c_t are growing. Given the dynamics imposed by the two difference equations, if we weren't going to a steady state, consumption would either be forever decreasing (hitting zero, which would drive the marginal utility of consumption to infinity, violating the transversality condition) or capital would go to zero (which would mean consumption would have to go to zero for the allocation to be feasible, which would also violate the transversality condition because the marginal utility of consumption would go to infinity). So we can think about the solution as picking c_t, given an initial condition on k_t, such that it (i) obeys the dynamics of the two difference equations, and (ii) the system approaches the steady state over time.

2.2 A Value Function Representation

An alternative way to find the first order conditions is by setting the problem up as a dynamic programming problem. Before beginning, there is an alternative way to deal with trend growth. Rather than characterizing the solution and then re-writing it in efficiency units, you can first transform the problem and then get the equilibrium conditions. For practice, let's do it this way now.

The problem is:

$$\max_{C_t, K_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

$$\text{s.t.}$$

$$K_{t+1} = A_t K_t^{\alpha} (Z_t N_t)^{1-\alpha} - C_t + (1-\delta) K_t$$

We can re-write both objective function and constraint in terms of efficiency units by fiddling around. Begin with the objection function, multiplying and dividing by $(Z_t N_t)^{1-\sigma}$. At date t flow utility is (ignore the extra -1 in the objective function, which is just a constant and is there only so that we can technically apply L'Hopital's rule in the case that $\sigma = 1$):

$$\beta^t \frac{C_t^{1-\sigma}}{1-\sigma} = \beta^t \frac{\left(\frac{C_t}{Z_t N_t}\right)^{1-\sigma} (Z_t N_t)^{1-\sigma}}{1-\sigma}$$

Use the fact that $Z_t = (1+z)^t Z_0$ and $N_t = (1+n)^t N_0$, and normalize the initial conditions to $Z_0 = N_0 = 1$ (which is fine because that is effectively going to just normalize flow utility by a constant, which does not matter) and we can write $Z_t N_t = \gamma^t$, where again $\gamma = (1+n)(1+z)$. Then we can re-write period t flow utility as:

$$\left(\beta \gamma^{1-\sigma}\right)^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

Now go to the constraint:

$$Z_{t+1} N_{t+1} \frac{K_{t+1}}{Z_{t+1} N_{t+1}} = A_t \left(\frac{K_t}{Z_t N_t}\right)^{\alpha} Z_t N_t - Z_t N_t \frac{C_t}{Z_t N_t} + (1-\delta) \frac{K_t}{Z_t N_t} Z_t N_t$$

Simplifying:

$$\gamma k_{t+1} = A_t k_t^{\alpha} - c_t + (1-\delta) k_t$$

Hence, written in efficiency units, the planner's problem is:

$$\max_{c_t, k_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \left(\beta\gamma^{1-\sigma}\right)^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

s.t.

$$\gamma k_{t+1} = A_t k_t^{\alpha} - c_t + (1-\delta)k_t$$

For the purposes of the Bellman equation, the state variables are A_t and k_t. The choice variable is c_t. The Bellman equation writes the infinite horizon dynamic programming problem into two periods:

$$V(A_t, k_t) = \max_{c_t} \quad \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta\gamma^{1-\sigma} E_t V(A_{t+1}, k_{t+1}) \right\}$$

s.t.

$$\gamma k_{t+1} = A_t k_t^{\alpha} - c_t + (1-\delta)k_t$$

It is easiest to eliminate the choice variable and instead write the problem as choosing the future state. We can do this by imposing that the constraint holds and eliminating c_t:

$$V(A_t, k_t) = \max_{k_{t+1}} \quad \left\{ \frac{\left(A_t k_t^{\alpha} + (1-\delta)k_t - \gamma k_{t+1}\right)^{1-\sigma}}{1-\sigma} + \beta\gamma^{1-\sigma} E_t V(A_{t+1}, k_{t+1}) \right\}$$

The function $V(\cdot)$ is unknown, but assume it is differentiable. The max operator outside means that k_{t+1} needs to be chosen optimally. This implies taking a derivative with respect to k_{t+1}. The first order condition is:

$$\gamma c_t^{-\sigma} = \beta\gamma^{1-\sigma} E_t \frac{\partial V(A_{t+1}, k_{t+1})}{\partial k_{t+1}}$$

Re-arranging slightly yields:

$$c_t^{-\sigma} = \beta\gamma^{-\sigma} E_t \frac{\partial V(A_{t+1}, k_{t+1})}{\partial k_{t+1}} \tag{17}$$

Well, it isn't so immediately obvious why this helpful, because we have a derivative of an unknown function on the right hand side. Here we make use of the Benveniste-Scheinkman envelop condition. Suppose that the sequence of future capital stocks has been chosen optimally (i.e. we have taken care of the max operator):

$$V(A_t,k_t) = \frac{(A_t k_t^{\alpha} + (1-\delta)k_t - \gamma k_{t+1})^{1-\sigma}}{1-\sigma} + \beta\gamma^{1-\sigma} E_t V(A_{t+1},k_{t+1})$$

Differentiate $V(\cdot)$ with respect to k_t:

$$\frac{\partial V(A_t,k_t)}{\partial k_t} = c_t^{-\sigma}\left(\alpha A_t k_t^{\alpha-1} + (1-\delta) - \gamma\frac{dk_{t+1}}{dk_t}\right) + \beta\gamma^{1-\sigma}E_t\frac{\partial V(A_{t+1},k_{t+1})}{\partial k_{t+1}}\frac{dk_{t+1}}{dk_t}$$

This can be re-arranged:

$$\frac{\partial V(A_t,k_t)}{\partial k_t} = c_t^{-\sigma}\left(\alpha A_t k_t^{\alpha-1} + (1-\delta)\right) - \left(\gamma c_t^{-\sigma} - \beta\gamma^{1-\sigma}E_t\frac{\partial V(A_{t+1},k_{t+1})}{\partial k_{t+1}}\right)\frac{dk_{t+1}}{dk_t}$$

This can be further re-written as:

$$\frac{\partial V(A_t,k_t)}{\partial k_t} = c_t^{-\sigma}\left(\alpha A_t k_t^{\alpha-1} + (1-\delta)\right) - \gamma\left(c_t^{-\sigma} - \beta\gamma^{-\sigma}E_t\frac{\partial V(A_{t+1},k_{t+1})}{\partial k_{t+1}}\right)\frac{dk_{t+1}}{dk_t}$$

But the FOC as written above tells us that $c_t^{-\sigma} = \beta\gamma^{-\sigma}E_t\frac{\partial V(A_{t+1},k_{t+1})}{\partial k_{t+1}}$. This means that the second term drops out, leaving:

$$\frac{\partial V(A_t,k_t)}{\partial k_t} = c_t^{-\sigma}\left(\alpha A_t k_t^{\alpha-1} + (1-\delta)\right)$$

This is the derivative of the value function with respect to its argument, the capital stock. The envelope condition essentially tells us that we can treat k_{t+1} as fixed when evaluating the derivative of the value function with respect to k_t. To use this in the first order condition (14) we have evaluate the argument at k_{t+1} and A_{t+1}. Plugging in to (14), we get:

$$c_t^{-\sigma} = \beta\gamma^{-\sigma}E_t c_{t+1}^{-\sigma}\left(\alpha A_{t+1}k_{t+1}^{\alpha-1} + (1-\delta)\right) \tag{18}$$

Note this is exactly the same as (15), the first order condition we got using the Lagrangian. The value function/Bellman equation approach is a little more widely applicable because it can better deal with non-linearities and corner solutions, but most of the time they yield the same solutions.

3 The Steady State

The solution of the model is a policy function – given the states, what is the optimal choice of the control. Except for very special cases (in particular $\delta = 1$), there is no analytical solution for this model. We can analytically characterize the solution for a special case in which the variables of the model are constant, however. This is called the steady state.

The non-stochastic steady state is defined as a situation in which all variables are constant and where the only source of uncertainty (which in this case is A_t) is held constant at its unconditional mean. In particular, this requires that $k_{t+1} = k_t$ and $c_{t+1} = c_t$. Denote these values k^* and c^*, respectively. Let A^* denote the steady state value of A_t, which is equal to its unconditional mean. We can analytically solve for the steady state capital stock from equation (8):

$$
\begin{aligned}
c^{*-\sigma} &= \beta\gamma^{-\sigma}c^{*-\sigma}\left(\alpha A^* k^{*\alpha-1} + (1-\delta)\right) \\
\frac{\gamma^\sigma}{\beta} - (1-\delta) &= \alpha A^* k^{*\alpha-1}
\end{aligned}
$$

$$
k^* = \left(\frac{\alpha A^*}{\frac{\gamma^\sigma}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{19}
$$

From the accumulation equation, we know that:

$$
A^* k^{*\alpha} = c^* + (\gamma - (1-\delta))k^*
$$

Hence we can solve for steady state consumption as:

$$
c^* = A^*\left(\frac{\alpha A^*}{\gamma^\sigma/\beta - (1-\delta)}\right)^{\frac{\alpha}{1-\alpha}} - (\gamma - (1-\delta))\left(\frac{\alpha A^*}{\gamma^\sigma/\beta - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{20}
$$

We can consider a couple of simple comparative statics. If A^* increases, then k^* and c^* also increase. If β goes up (households become more patient), then k^* and c^* both go up as well.

4 The Phase Diagram

As I discussed above, the solution of the model is a value of c_t, given an initial condition on k_t, such that (i) the system evolves according to the dynamics of the two difference equations and (ii) the system approaches the steady state. A phase diagram allows us to think about the model solution graphically.

We can qualitatively characterize the full solution to the model through a phase diagram. In a two dimensional world, the phase diagram typically puts the endogenous state variable on the horizontal axis (in this case k_t) and the jump variable on the vertical axis (in this case c_t). Phase diagrams are more natural in continuous time; we will proceed in discrete time with one slight abuse of notation.

We want to find the sets of points where the state (capital) and jump variables are not changing in (k_t, c_t) space. Call these two sets of points the $\frac{k_{t+1}}{k_t} = 1$ isocline and the $\frac{c_{t+1}}{c_t} = 1$ isocline. There is only one value of k_{t+1} consistent with $\frac{c_{t+1}}{c_t} = 1$, which is the steady state capital stock if A is at its mean. We can see this from the first order condition.

$$\frac{c_{t+1}}{c_t} = 1: \quad k_{t+1} = \left(\frac{\alpha A_{t+1}}{\gamma^\sigma/\beta - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{21}$$

The $\frac{k_{t+1}}{k_t} = 1$ isocline is found by looking at the capital accumulation equation:

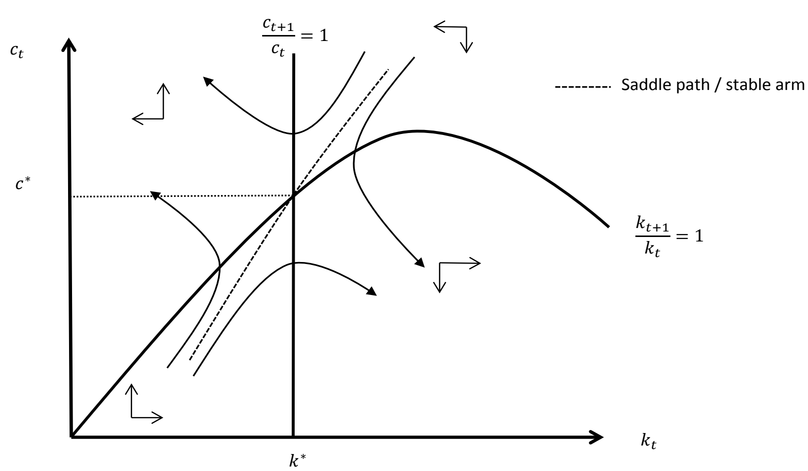
$$\gamma k_{t+1} = A_t k_t^\alpha - c_t + (1-\delta)k_t$$
$$k_{t+1} = k_t \Leftrightarrow c_t = A_t k_t^\alpha + (1-\delta-\gamma)k_t$$

Hence the isocline is defined by:

$$\frac{k_{t+1}}{k_t} = 1: c_t = A_t k_t^\alpha + (1-\delta-\gamma)k_t \tag{22}$$

A complication arises because k_t shows up in the $\frac{k_{t+1}}{k_t} = 1$ isocline, whereas the $\frac{c_{t+1}}{c_t} = 1$ isocline depends on k_{t+1}. This problem would not be present in continuous time, which is why continuous time is more natural for phase diagram. I'm going to simply circumvent this issue by assuming that $k_{t+1} \approx k_t$, and will treat the k_{t+1} in the $\frac{c_{t+1}}{c_t} = 1$ isocline as k_t.

Given this simplifying assumption, we can graph each of these lines in a plane with c_t on the vertical axis and k_t on the horizontal axis. The $\frac{c_{t+1}}{c_t} = 1$ isocline is a vertical line at $k_t = k^*$ (assuming that A is at its steady state). The $\frac{k_{t+1}}{k_t} = 1$ isocline is a bit more complicated. We can see that its slope is $\frac{dc_t}{dk_t} = \alpha A_t k_t^{\alpha-1} + (1-\delta-\gamma)$. When k_t is small (i.e. near the origin), then this slope is positive because $\alpha A_t k_t^{\alpha-1}$ will be large. When k_t is large (far away from the origin), the slope will be negative and will approach $(1-\delta-\gamma) < 0$ (because $\alpha A_t k_t^{\alpha-1}$ will go to zero, and $\gamma \geq 1$). The peak occurs where $k_t = \left(\frac{\alpha A_t}{\delta+\gamma-1}\right)^{\frac{1}{1-\alpha}}$, which is greater than k^* when evaluated at the steady state value of A^*. The actual steady state is where the two isoclines cross.



The above picture shows the isoclines and the steady state. It also shows the (i) saddle path and

(ii) some unstable dynamic lines. These dynamics can be derived as follows. "Below" the $\frac{k_{t+1}}{k_t} = 1$ isocline, c_t is "too small", and hence $k_{t+1} > k_t$, and we draw arrows pointing the "right", denoting the direction in which k_t will be expected to travel. "Above" the $\frac{k_{t+1}}{k_t} = 1$ isocline, c_t is "too big", and $k_{t+1} < k_t$, and hence we draw arrows pointing left. To get the dynamic arrows relative to the $\frac{c_{t+1}}{c_t} = 1$ isocline, we have to engage in a slight abuse of terminology. Technically what governs the evolution of c_t is where k_{t+1} is relative to k^*, but what shows up in the diagram is k_t. Let's ignore this distinction and treat $k_t \approx k_{t+1}$. To the "right" of the $\frac{c_{t+1}}{c_t} = 1$ isocline, k_t is "too big"; this means that $\alpha A_{t+1} k_{t+1}^{\alpha-1}$ will be "small", and consumption will be expected to decline. Hence, to the right of the $\frac{c_{t+1}}{c_t} = 1$ isocline we draw arrows pointing down, showing the expected direction of consumption in that region. To the left of the $\frac{c_{t+1}}{c_t} = 1$ the opposite is true; k_t is "too small", and consumption will be expected to grow, so we draw arrows pointing up.

Visually inspecting the picture, we see that the arrows point toward the steady state when the system sits to "northeast" and "southwest" of the steady state. In the regions of the picture that are to the "northwest" or "southeast" of the steady state, the arrows point away from the steady state. The idea of the policy function is to pick c_t given k_t such that (i) the first order conditions hold; (ii) the transversality condition holds; and (iii) the solution is feasible with the constraints. Drawing in the dynamics as we have done presumes (i). The transversality condition rules out picking any value of c_t in the "southeast" region – those regions would eventually lead to 0 consumption (so $\lambda_t \to \infty$) and infinite capital (so $k_{t+1} \to \infty$), which leads to a violation of the transversality condition. Picking any value of c_t in the "northwest" region would violate (iii) – we would move towards infinite consumption with zero capital, which is infeasible. Hence, for any given k_t (the state), consumption must start either in the "southwest" or "northeast" regions. Any old value of c_t will not do – there will be a unique value of c_t for each k_t such that we travel towards the steady state. Any other value of c_t (shown by the "explosive" dynamic arrows), would eventually lead to a violation of (ii) or (iii). The unique set of values of c_t consistent with (i) - (iii) being satisfied and holding A_t fixed is given by dashed curve crossing through the steady state – this is the "saddle path" or "stable arm." This can be interpreted as the policy function when A_t is at its unconditional mean – for any given current k_t, it tells you the value of c_t consistent with optimization. In other words, the saddle path is the "solution" of the model – it tells us what c_t should be for every k_t such that the dynamics of the two difference equations are obeyed and such that we end up approaching the steady state.

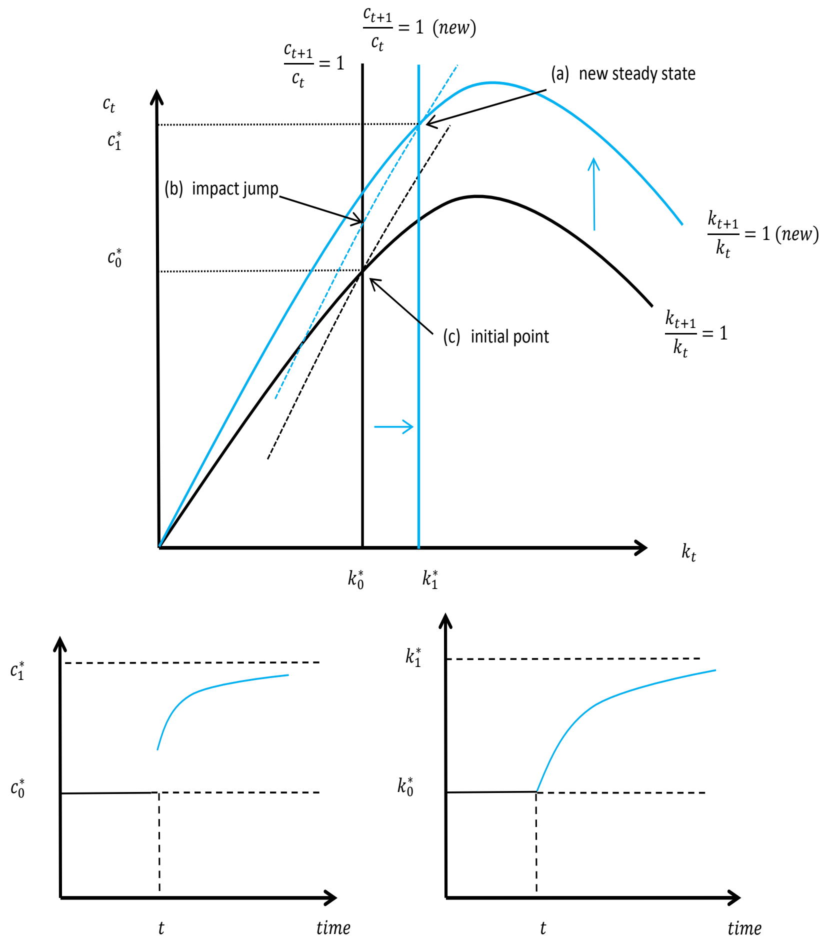
5 Dynamic Effects of Shocks

In this section I work through two different exercises: (1) an unexpected permanent change in A^*; and (2) an unexpected but temporary change in A_t. For (2), suppose that A_t increases immediately and is expected to remain at that level until time $t + T$, at which point is goes back to its initial starting value. For these shifts, always assume that the economy begins in its steady state.

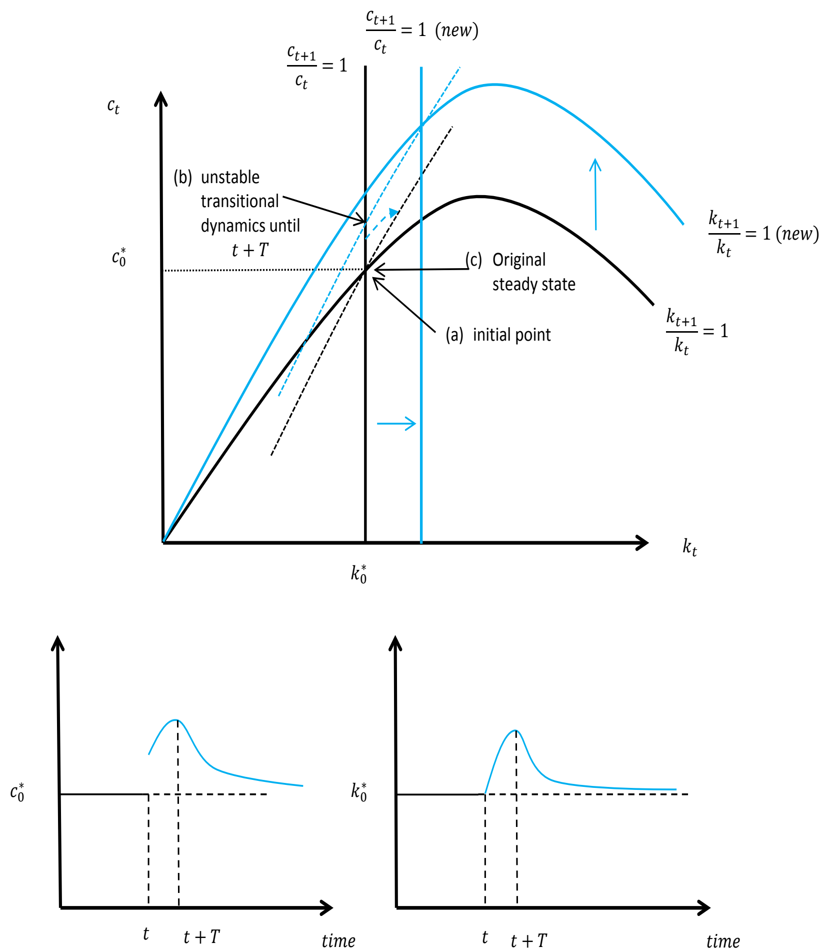
The dynamics of these systems always work as follows. Whenever something exogenous changes, the jump variable (in this case consumption) must jump in such a way that it "rides" the new system dynamics for as long as the change in the exogenous variable is in effect, and in expectation the system must hit a steady state eventually (either returning to the original steady state or going to a new steady state, depending on whether the change in the exogenous variable is permanent or not). The state variable cannot jump immediately, but will follow (in expectation) the dynamics of the system thereafter.

We begin with the permanent increase in A_t. This clearly shifts both isoclines. In particular, the $\frac{c_{t+1}}{c_t} = 1$ isocline shifts to the right, and the $\frac{k_{t+1}}{k_t} = 1$ isocline shifts "up and to the right". The new isoclines are shown as blue; the original isoclines are black. It is clear that the steady state values of both consumption and the capital stock are higher.

There's also a new saddle path/policy function. Depending on the slope of the saddle path, this could cross the original value of k_t either above, below, or exactly at the original value of c_t. This means that the initial jump in consumption is ambiguous. I have shown this system with consumption initially jumping up, which is what will happen under "plausible" parameterizations of σ. Thereafter the system must ride the new dynamics and approach the new steady state. This is labeled point (b) in the diagram – consumption jumps up immediately to the new saddle path, and then thereafter c_t and k_t "ride" the saddle path into the new steady state, labeled (c) in the diagram. Since $k_{t+1} > k_t$, investment increases on impact, so that consumption does not increase as much as output. Below the phase diagram I show the impulse responses, which trace out the dynamic responses of consumption and the capital stock to the shock. Again, note that consumption jumps on impact, whereas the capital stock does not, and from thereafter they "ride" the dynamics to a steady state.



Now consider an unexpected but temporary increase in A. In particular, A_t shifts up in period t, remains at a higher value until $t+T$, and thereafter returns to its original value. Here the new isoclines shift in the same way they do in the case of a permanent shift in A. I show the new isoclines as blue lines, and also show the new saddle path as a dashed blue line. Here, however, consumption cannot jump all the way to the new saddle path as it did in the case of the permanent change in A. Consumption must jump and ride the unstable "new" system dynamics until time $t+T$, when A goes back to its original value. At this point in time it must be back on the original saddle path. From there on it must follow the original saddle path back into the original steady state.



In the figure above we begin at point (a). Consumption jumps up immediately on impact, but it cannot jump all the way to the new saddle path – if it did that, when A goes back down in $t+T$, we would be in an unstable region of the diagram. Rather, consumption must jump in the same direction as it would in response to a permanent increase in A, but less. This is labeled (b) in the diagram. Once it jumps, the system must "ride" the dynamics associate with the blue isoclines until $t+T$. This means that consumption and capital must both be increasing during this period. At period $t+T$, the system must sit on the original saddle path. After that time, the system simply rides the saddle path back in to the original steady state. Since c_t jumps by less on impact but the change in output is the same as if the shock were permanent, investment must jump by more. These results are consistent with the intuition from the permanent income hypothesis.

This exercise turns out to provide important insight into more complicated problems. In these dynamic systems you can almost never find the analytical solution for a general case, though you can do so for the (i) steady state and (ii) what the solution would look like if the jump variable didn't jump at all. For the case of temporary but persistent exogenous shocks, the full solution is somewhere in between those two extreme cases (between (i) and (ii)). If the shock is not very persistent, a good approximation to the solution would have the jump variable not changing at all.