Graduate Macro Theory II: Extensions of Basic RBC Framework

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1 Introduction

The basic RBC model – which is just a stochastic neoclassical growth model with variable labor – is the building block of almost all modern DSGE models. It fits the data well on some dimensions, but less well on others. In this set of notes we consider several extensions and modifications of the basic framework. I call this set of notes "RBC Extensions" because all the extensions I assume here are "real" – I do not yet deal with nominal rigidities, which one loosely think of as "New Keynesian." We will see those later.

2 Common Extensions

This section works through a number of extensions designed to make the RBC model (i) more realistic and (ii) a better fit with the data.

2.1 Indivisible Labor

One failure of the RBC model is that it fails to generate sufficient volatility in hours of work. It also models hours in a rather unrealistic way that is at odds with reality – all fluctuations in hours come from the *intensive* margin (e.g. average hours worked) as opposed to the *extensive* margin (the binary choice of whether to work or not). In the real world most people have a more or less fixed number of hours worked; it is fluctuations in bodies that drive most of the fluctuation in total hours worked.

In reality, households face two decisions: (1) work or not and (2) conditional on working, how much to work. This is difficult to model because it introduces discontinuity into the decisions household make. Hansen (1985) and Rogerson (1988) came up with a convenient technical fix. Suppose that within period preferences of any household are:

$$u(C_t, 1 - N_t) = \ln C_t + \theta \frac{(1 - N_t)^{1-\xi} - 1}{1 - \xi} \tag{1}$$

These preferences nest the basic specification that we've used already if $\xi = 1$ (which means we'd have $\ln(1 - N_t)$. The Frisch labor supply elasticity here would be given by $(\xi\gamma)^{-1}$, where $\gamma = \frac{N^*}{1-N^*}$ (so if $\xi = 1$ we'd be in the standard case we started with). But playing with the value of ξ would allow you to control the Frisch elasticity and have it not simply depend on N^* as it does in the log case.

Suppose that the structure of the world is as follows. There are a large number of identical households. Households either work or they do not. If they work, they work $\bar{N}$ hours, with $0 < \bar{N} < 1$. The amount of work, $\bar{N}$, can be interpreted as a technological constraint and is exogenous to the model. Each period, there is a probability of working, τ_t, with $0 < \tau_t < 1$. This probability is indexed by t because it is a choice variable – essentially the household can choose its probability of working, but not how much it does work if it does work. There is a lottery such that the household has a τ_t chance of being selected to work; the rule of the game is there is perfect insurance in that every household gets paid whether they work or not. Hence, in expectation households will work $N_t = \tau_t \bar{N}$ and they will have identical consumption (because of the implicit assumption of perfect insurance combined with separability across consumption and leisure).

We can write out the households expected flow utility function as:

$$u(C_t, 1 - N_t) = \ln C_t + \tau_t \theta \frac{(1 - \bar{N})^{1-\xi} - 1}{1 - \xi} + (1 - \tau_t)\theta \frac{(1)^{1-\xi} - 1}{1 - \xi} \tag{2}$$

We observe that preferences are linear in τ_t. Collecting terms we get:

$$u(C_t, 1 - N_t) = \ln C_t + \tau_t \theta \left(\frac{(1 - \bar{N})^{1-\xi} - 1}{1 - \xi} - \frac{(1)^{1-\xi} - 1}{1 - \xi} \right) + \theta \frac{(1)^{1-\xi} - 1}{1 - \xi} \tag{3}$$

Now, from above we know that $\tau_t = \frac{N_t}{\bar{N}}$. Make this substitution:

$$u(C_t, 1 - N_t) = \ln C_t + \frac{N_t}{\bar{N}} \theta \left(\frac{(1 - \bar{N})^{1-\xi} - 1}{1 - \xi} - \frac{(1)^{1-\xi} - 1}{1 - \xi} \right) + \theta \frac{(1)^{1-\xi} - 1}{1 - \xi} \tag{4}$$

As long as $\xi > 0$, then $\frac{1^{1-\xi}-1}{1-\xi} > \frac{(1-\bar{N})^{1-\xi}-1}{1-\xi}$. Hence, re-write this again as:

$$u(C_t, 1 - N_t) = \ln C_t - \frac{N_t}{\bar{N}} \theta \left(\frac{(1)^{1-\xi} - 1}{1 - \xi} - \frac{(1 - \bar{n})^{1-\xi} - 1}{1 - \xi} \right) + \theta \frac{(1)^{1-\xi} - 1}{1 - \xi}$$

Let's define two constants as:

$$B = \frac{\theta}{\bar{N}} \left(\frac{(1)^{1-\xi} - 1}{1 - \xi} - \frac{(1 - \bar{N})^{1-\xi} - 1}{1 - \xi} \right)$$

$$D = \theta \frac{(1)^{1-\xi} - 1}{1 - \xi}$$

We can actually just drop D altogether from the analysis – adding a constant to the utility function won't change the household's optimal choices. Then we can write the within period utility function as:

$$u(C_t, 1 - N_t) = \ln C_t - BN_t \tag{5}$$

In other words, utility effectively becomes linear in labor under this indivisible labor with lotteries framework. This holds for *any* value of ξ. But indeed, it is *as if* $\xi = 0$. In other words, the aggregate labor supply elasticity is *infinite* even if the micro labor supply is very small (i.e. ξ very large). This is potentially very helpful – one can generate more hours volatility with a higher Frisch elasticity, and is (potentially) not subject to the criticisms that the labor supply elasticity is inconsistent with micro evidence.

These preferences are isomorphic to the ones where there is disutility from labor if the parameter $\chi = 0$:

$$u(C_t) - v(N_t) = \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi}$$

The full model can then be written as follows. I assume that households own the capital stock and lease it to firms. Households also have access to one period bonds. I abstract from the presence of these bonds in the firm first order condition because the quantity of bonds ends up being indeterminate anyway. As such, the firm problem becomes static.

Households:

$$\max_{C_t, N_t, B_{t+1}, K_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - BN_t \right)$$

$$\text{s.t.}$$

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t \leq w_t N_t + R_t K_t + r_t B_t$$

Firms:

$$\max_{N_t, K_t} \quad A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t K_t$$

In a competitive equilibrium the first order conditions hold and all budget constraints hold. This gives rise to the following characterization of equilibrium:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left(R_{t+1} + (1-\delta) \right) \right) \tag{6}$$

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (1 + r_{t+1}) \right) \tag{7}$$

$$B = \frac{1}{C_t} w_t \tag{8}$$

$$w_t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{9}$$

$$R_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{10}$$

$$K_{t+1} = I_t + (1-\delta) K_t \tag{11}$$

$$Y_t = C_t + I_t \tag{12}$$

$$Y_t = A_t A_t^{\alpha} N_t^{1-\alpha} \tag{13}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{14}$$

We would like to come up with a calibration of this model that is consistent with our previous calibrations. We don't actually need to calibrate anything that goes into B, just the value of B. But how do we do that?

From (6), combined with (10), we can solve for the steady state capital to labor ratio

$$\left(\frac{K^*}{N^*} \right) = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)} \right)^{\frac{1}{1-\alpha}} \tag{15}$$

Now combine (8) with (9) to solve for C^* in terms of the steady state capital to labor ratio:

$$C^* = \frac{1}{B} (1-\alpha) \left(\frac{K^*}{N^*} \right)^{\alpha} \tag{16}$$

Now go to the aggregate accounting identity, which can be written as:

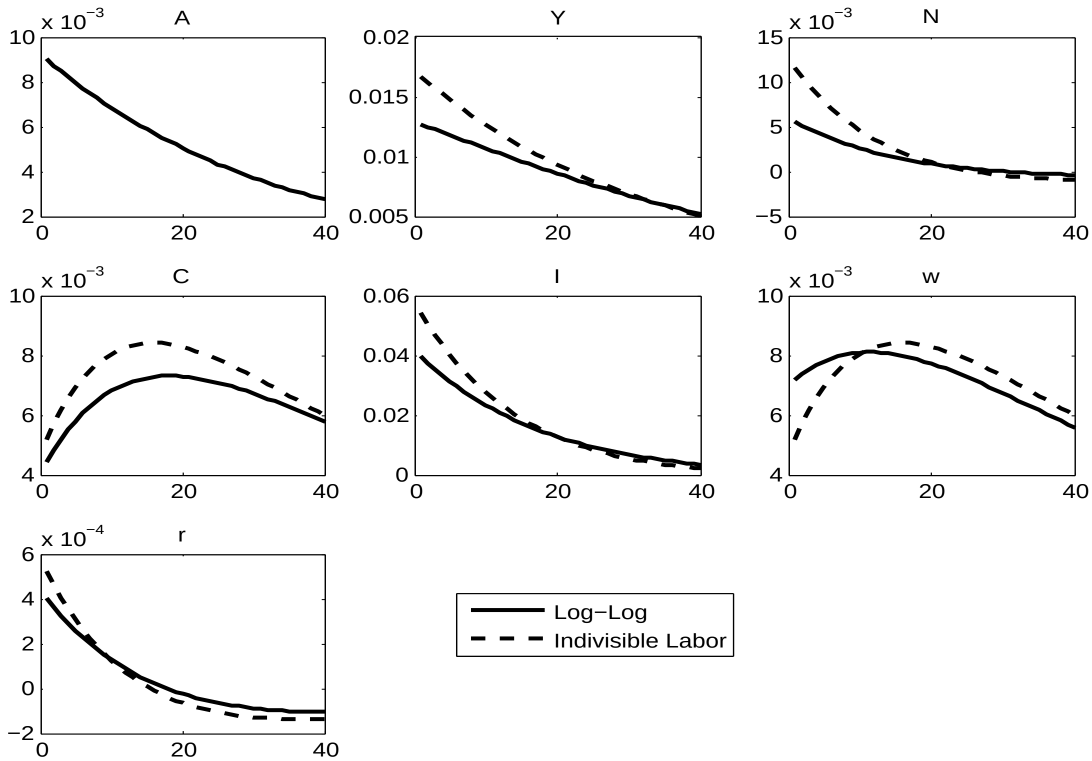
$$C^* = N^* \left(\left(\frac{K^*}{N^*} \right)^{\alpha} - \delta \left(\frac{K^*}{N^*} \right) \right) \tag{17}$$

Since (16) and (17) must both hold, we can set them equal to one another and solve for B, taking $N^* = \frac{1}{3}$ as a given target in the calibration. We get:

$$B = \frac{(1-\alpha) \left(\frac{K^*}{N^*} \right)^{\alpha}}{N^* \left(\left(\frac{K^*}{N^*} \right)^{\alpha} - \delta \left(\frac{K^*}{N^*} \right) \right)} \tag{18}$$

If we use our now "standard" calibrations of $\alpha = 0.33$, $\beta = 0.99$, and $\delta = 0.025$, then we see that $\frac{K^*}{N^*} = 28.35$. This implies that $B = 2.63$.

I solve both this model and the standard RBC model (with log preferences over leisure and θ calibrated to guarantee $N^* = 1/3$). All other parameters are the same across both models. Below are impulse response to a technology shock in each model:



We see significantly more amplification in the indivisible labor case than in the log-log case, as shown by the differences between the dashed and solid lines. Output and hours both increase by significantly more on impact. As a result of this, consumption and investment both go up by more initially. Because the indivisible labor case is isomorphic to the Frisch labor supply being infinite, the labor supply curve is perfectly horizontal here. This is why we get a bigger increase in labor hours (and a smaller increase in wages) after the productivity shock.

Quantitatively, indivisible labor improves the fit of the model along several dimensions. First, it provides greater amplification – I get output volatility of 2.2 percent with indivisible labor, as opposed to 1.7 percent in the standard case. This means that I can match the output volatility in US data with smaller TFP shocks. In addition, indivisible labor increases the relative volatility of hours substantially. In the benchmark RBC case, the relative volatility of hours is 0.43. In the indivisible labor case it is 0.69. This is a large improvement, though it is still quite far from the data. Furthermore, indivisible labor makes wages somewhat less volatile (volatility of 0.008 instead of 0.010) and somewhat less procyclical (correlation with output of 0.92 instead of 0.99).

2.2 Non-Separability in Preferences

In our basic specification we have assumed two kinds of separability in preferences – separability between leisure and consumption (intratemporal separability) and separability of both leisure and consumption across time (intertemporal separability). We consider both of these in turn.

2.2.1 Intratemporal Non-Separability: King, Plosser, and Rebelo (1988) and Greenwood, Hercowitz, and Hoffman (1988)

The generic definition of balanced growth path is a situation in which all variables growth at a constant rate over time (though this rate need not be the same across variables). A special case of a balanced growth path is a steady state, in which the growth rate of all variables is equal to zero. In our benchmark specification above there is no explicit trend growth, though we could fairly easily modify the model in such a way that we get (essentially) the same first order conditions in the redefined variables which are detrended.

In any balanced growth path, feasibility requires that hours not grow. The intuition for this is straightforward – if hours were declining, we would eventually hit zero and have no output. If hours were growing, we would eventually hit 1, which is the upper bound on hours. It is straightforward to show, under the assumptions about technology and production we have made, consumption and the real wage must grow at the same rate along the balanced growth path, irrespective of the kinds of preferences. Consider a generic, possibly non-separable within-period utility function: $u(C_t, 1 - N_t)$. The only assumptions are that it is increasing and concave in its arguments. The generic static labor supply condition is as follows;

$$-u_N(C_t, 1 - N_t) = u_C(C_t, 1 - N_t)w_t$$

This is really just an MRS = price ratio condition between consumption and leisure. To satisfy the conditions laid out above (namely that consumption and the wage grow at the same rate and hours not grow), it must be the case that this first order condition reduce to something like:

$$f(N_t) = \frac{w_t}{C_t}$$

In other words, the left hand side must be a function only of N_t, and the right hand side must feature the wage over consumption. With wages and consumption growing at the same right, the right hand side will be constant along a balanced growth path. Then with the left hand side only a function of N_t (and, of course, parameters), there will be a unique solution for N^* that is not growing.

Consider a standard iso-elastic preference specification:

$$u(C_t, 1 - N_t) = \frac{C_t^{1-\sigma} - 1}{1 - \sigma} + \theta \frac{(1 - N_t)^{1-\xi} - 1}{1 - \xi}$$

The intratemporal first order condition for labor supply would then be:

$$\theta(1 - N_t)^{-\xi} = C_t^{-\sigma} w_t$$

The only way that we end up with $\frac{w_t}{C_t}$ on the right hand side is if $\sigma = 1$: in other words, for these preferences to be consistent with balanced growth, it must be the case that utility over consumption is log. This is potentially problematic, because it imposes that the coefficient of relative risk aversion is 1, which is much lower than what is needed to explain things like the equity premium.

King, Plosser, and Rebelo (1988) show that preferences must take the following form to be consistent with balanced growth:

$$u(C_t, N_t) = \frac{(C_t v(1-N_t))^{1-\sigma} - 1}{1-\sigma} \quad \text{if} \quad \sigma \neq 1$$
$$u(C_t, N_t) = \ln C_t + \ln v(1-N_t) \quad \text{if} \quad \sigma = 1$$

The second step follows from application of L'Hopital's rule. We require that $v(1-N_t)$ be an increasing and concave function of its argument, leisure (one minus labor). So as to make this all consistent with our original specification, suppose that $v(\cdot)$ takes the following form:[1]

$$v(1-N_t) = \exp\left(\theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}\right)$$

With this specification, if $\sigma = 1$, then we get:

$$u(C_t, 1-N_t) = \ln C_t + \theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}$$

Then, if $\xi = 1$, by L'Hopital's rule we would get: $u(C_t, 1-N_t) = \ln C_t + \theta \ln(1-N_t)$. If $\xi = 0$, we would get $u(C_t, 1-N_t) = \ln C_t + \theta(1-N_t)$, which is essentially the indivisible labor model. Thus, we can nest all of these specifications in terms of this general functional form.

For the general case in which $\sigma \neq 1$ and $\xi \neq 1$, we can verify that these preferences will be consistent with constant labor hours in steady state. Let's find the marginal utilities:

$$u_C(C_t, 1-N_t) = \left(C_t \exp\left(\theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}\right)\right)^{-\sigma} \exp\left(\theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}\right)$$
$$u_N(C_t, 1-N_t) = -\left(C_t \exp\left(\theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}\right)\right)^{-\sigma} C_t \exp\left(\theta \frac{(1-N_t)^{1-\xi} - 1}{1-\xi}\right)(1-N_t)^{-\xi}$$

Then for the generic first order condition, we get:

$$-\frac{u_N(C_t, 1-N_t)}{u_C(C_t, 1-N_t)} = \frac{\theta(1-N_t)^{-\xi}}{C_t} = w_t \Rightarrow \theta(1-N_t)^{-\xi} = \frac{w_t}{C_t}$$

In other words, the static first order condition for labor supply ends up looking *exactly* like it does in the case of log consumption with these preferences. Hours will be stationary. θ and ξ will have exactly the same interpretations as in the basic model (θ will determine N^* and ξ will determine the Frisch elasticity).

[1] Note that you'll often see KPR preferences written with $v(1-N_t) = (1-N_t)^\theta$ or some variation thereof, so that $u(C_t, N_t) = \frac{(C_t(1-N_t)^\theta)^{1-\sigma} - 1}{1-\sigma}$. When $\sigma \to 1$, this collapses to $\ln C_t + \theta \ln(1-N_t)$; this effectively imposes the "log-log" preference specification. By writing the $v(\cdot)$ function with the exp operator, I permit a more general specification of the utility from leisure.

What does σ govern? It is still going to have the interpretation as the elasticity of intertemporal substitution. The first order conditions for the household side of the model for consumption and bonds can be written:

$$\lambda_t = C_t^{-\sigma}\left(\exp\left(\theta\frac{(1-N_t)^{1-\xi}-1}{1-\xi}\right)\right)^{1-\sigma} \tag{19}$$
$$\lambda_t = \beta E_t(\lambda_{t+1}(1+r_t)) \tag{20}$$
$$\lambda_t = \beta E_t\lambda_{t+1}(R_{t+1}+(1-\delta)) \tag{21}$$

Log-linearize:

$$\begin{aligned}\ln\lambda_t &= \ln\beta + \ln\lambda_{t+1} + \ln(1+r_t)\\ \frac{\lambda_t-\lambda^*}{\lambda^*} &= \frac{\lambda_{t+1}-\lambda^*}{\lambda^*} + \frac{r_t-r^*}{1+r^*}\\ \widetilde{\lambda}_t &= E_t\widetilde{\lambda}_{t+1} + \beta\widetilde{r}_t\end{aligned}$$

The last line follows from the fact that we define $\widetilde{r}_t$ as the actual deviation from steady state, not percentage deviation. Now log-linearize the expression for λ:

$$\begin{aligned}\ln\lambda_t &= -\sigma\ln C_t + (1-\sigma)\theta\left(\frac{(1-N_t)^{1-\xi}-1}{1-\xi}\right)\\ \frac{\lambda_t-\lambda^*}{\lambda^*} &= -\sigma\frac{C_t-c^*}{c^*} + (1-\sigma)\theta(1-N^*)^{-\xi}(N_t-N^*)\\ \widetilde{\lambda}_t &= -\sigma\widetilde{C}_t + (1-\sigma)\theta(1-N^*)^{-\xi}N^*\widetilde{N}_t\end{aligned}$$

Now combine these two expressions:

$$-\sigma\widetilde{C}_t + (1-\sigma)\theta(1-N^*)^{-\xi}N^*\widetilde{N}_t = E_t\left(-\sigma\widetilde{C}_{t+1} + (1-\sigma)\theta(1-N^*)^{-\xi}N^*\widetilde{N}_{t+1}\right) + \beta\widetilde{r}_t$$

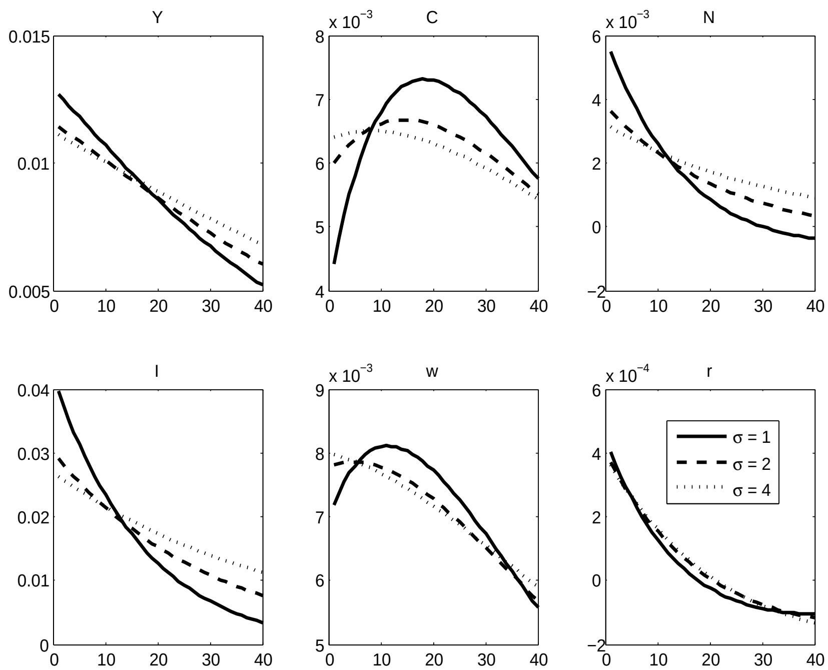
Simplify:

$$E_t\left(\widetilde{C}_{t+1}-\widetilde{C}_t\right) = \frac{\beta}{\sigma}\widetilde{r}_t + \left(\frac{(1-\sigma)}{\sigma}\theta(1-N^*)^{-\xi}N^*\right)\left(E_t\left(\widetilde{N}_{t+1}-\widetilde{N}_t\right)\right) \tag{22}$$

If we approximate $\beta\approx 1$, then this says that the elasticity of intertemporal substitutions is $\frac{1}{\sigma}$, just like in the case with separable utility. There is just an additional term now that depends on expected employment growth, though if $\sigma = 1$ this term drops out and we are in the normal case.

What this specification of preferences thus does is allow us to consider different parameterizations of σ different from one while still having preferences that are consistent with balanced growth. Loosely speaking, σ governs the household's desire to smooth consumption. If σ is very large, the household will want consumption (in expectation) to be very smooth, whereas if σ is quite small then the household will be quite willing to allow consumption to not be smooth (again in expectation).

Below are impulse responses to a standard technology shock for different values of σ. I fix all other parameter values at their "baseline" values. I consider the following values of σ: 1, 2, and 4.



As we might expect, the initial jump in consumption is increasing in σ (note again that large σ means you want consumption to be smooth *in expectation*, not necessarily in response to a shock). This means that the jump in labor is decreasing in σ. Why is that? Think back the labor supply and demand curves. When TFP increases, labor demand shifts right, the amount by which is independent of σ. When consumption increases, labor supply shifts left. The bigger is the consumption increase, the bigger is this inward shift in labor supply, and therefore the smaller is the hours response in equilibrium and the larger is the wage response. That's exactly what we see in terms of the impulse responses: when σ is bigger, the hours jump is smaller, the output jump is smaller, and the wage jump is larger.

This all suggests that one way to make the model better fit the data is to make σ smaller – we see here that higher values of σ make the amplification problem in the model worse – we get less of a jump in labor because consumption jumps more, which means output goes up by less when productivity goes up. In particular, we get more employment volatility and hence more amplification for $\sigma < 1$. The problem with this is that most micro evidence does not support such a claim – there estimates of σ are typically *far* greater than one. In particular, Hall (1988) says "... supporting the strong conclusion that the elasticity (of intertemporal substitution, the inverse of σ)

is unlikely to be much above 0.1, and may well be zero." This would mean that $\sigma \geq 10$! A number of papers in the asset pricing literature rely upon very large values of σ in order to generate the excess returns on equity over debt that we see in the data. If we take values of σ much greater than 1, the RBC model begins to fit the data even worse than in the log case (in terms of amplification and relative volatility of hours). It is worth mentioning, however, that most of these estimates that find very large values of σ are based on time series data. Gruber (2006) finds a much smaller value of σ (more like 0.5) using micro data from looking at tax variation.

Greenwood, Hercowitz, and Hoffman (1988) propose another popular utility specification that features non-separability between consumption and leisure/labor. Unlike KPR preferences, GHH preferences are not consistent with balanced growth. What GHH preferences do is eliminate the wealth effect on labor supply – as we will see in a moment, this means that the FOC for labor gives labor as a function only of the wage (no consumption showing up). This means that there is no wealth effect – so, for example, when labor demand shifts out because of a technology shock, there is no inward shift of labor supply because of c_t increasing. This will result in more amplification.

I will write utility in terms of disutility from labor instead of utility from leisure. The GHH preference specification can be written:

$$U(N_t, N_t) = \frac{1}{1-\sigma}\left(C_t - \theta\frac{N_t^{1+\chi}}{1+\chi}\right)^{1-\sigma}$$

The marginal utilities are:

$$U_C(C_t, N_t) = \left(C_t - \theta\frac{N_t^{1+\chi}}{1+\chi}\right)^{-\sigma}$$

$$u_N(C_t, N_t) = -\theta N_t^{\chi}\left(C_t - \theta\frac{N_t^{1+\chi}}{1+\chi}\right)^{-\sigma}$$

The generic first order condition for labor supply is:

$$-u_N(C_t, N_t) = u_C(C_t, N_t)w_t$$

With these marginal utilities, this works out to:

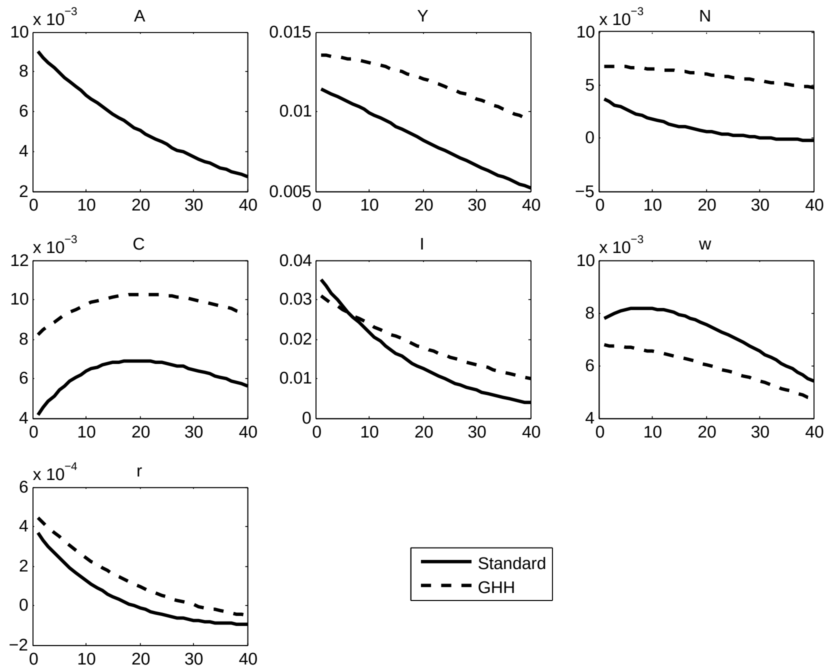
$$\theta N_t^{\chi} = w_t$$

Here we see, as promised, that the consumption term drops out altogether on the right hand side – labor is only a function of the wage. We can write the Euler equations for bonds and capital by defining auxiliary variables $\lambda_t = u_c$:

$$\lambda_t = \beta E_t \lambda_{t+1}(R_{t+1} + (1-\delta))$$

$$\lambda_t = \beta E_t \lambda_{t+1}(1 + r_t)$$

I solve the model using values $\sigma = 1$ (which corresponds to utility taking the form $\ln\left(C_t - \theta\frac{N_t^{1+\chi}}{1+\chi}\right)$) and $\chi = 1$, using standard values I've been using. Below, I compare the impulse responses under standard preferences (utility takes the form here $\ln C_t - \theta\frac{N_t^{1+\chi}}{1+\chi}$); in each case I solve for θ such that steady state labor hours are 0.33.



We see here that we get much larger amplification with GHH preferences – output and hours rise by substantially more (and appear to be more persistent). Naturally, the wage rises by less (this occurs because there is no inward shift of labor supply following the productivity shock). Interestingly, we see that consumption jumps up by more and actually investment increases by less. In terms of volatilities, under the standard setup I get output and labor volatilities (HP Filtered) of 0.015 and 0.005, for a relative volatility of about 1/3. For the GHH specification, I get output volatility of 0.0176 and labor volatility of 0.009, for about a relative volatility of 0.5. I could make these numbers look even better by lowering χ.

2.2.2 Intertemporal Non-Separability: Habit Formation

Another important kind of non-separability is non-separability across time. This usually goes by the name "habit formation", with the idea that people get utility not from the level of consumption, but from the level of consumption relative to past consumption. The idea is that one becomes accustomed to a certain level of consumption (i.e. a "habit") and utility becomes relative to that. Habit formation has been included in macro models for a variety of reasons. In particular, habit formation can help resolve some empirical failings of the PIH. For example, habit formation can help resolve the "excess smoothness" puzzle because, the bigger is habit formation, the smaller consumption will jump in response to news about permanent income. Another area where habit formation has gained ground is in asset pricing, in particular with regard to the equity premium puzzle. A large degree of habit formation, in essence, makes consumers behave "as if" they are extremely risk averse, and can thereby help explain a large equity premium without necessarily resorting to extremely large coefficients of relative risk aversion (see the previous subsection).

Assume intratemporal separability so that utility from consumption is logarithmic. Let the within period utility function be given by:

$$u(C_t, 1 - N_t) = \ln(C_t - \phi C_{t-1}) + \theta \ln(1 - N_t)$$

ϕ is the habit persistence parameter; if $\phi = 0$ we are in the "normal" case, and as $\phi \to 1$ agents get utility not from the level of consumption, but from the change in consumption. For computational purposes we need to restrict $\phi < 1$ – if it is exactly 1 then marginal utility in the steady state would be ∞.

Let's setup the household's problem using a Lagrangian. Assume that households own the capital stock:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln(C_t - \phi C_{t-1}) + \theta \ln(1 - N_t) + \dots \right.$$
$$\left. \dots \lambda_t \left(w_t N_t + R_t K_t + \Pi_t + (1 + r_{t-1}) B_t - C_t - K_{t+1} + (1 - \delta) K_t - B_{t+1} \right) \right)$$

The first order conditions are:

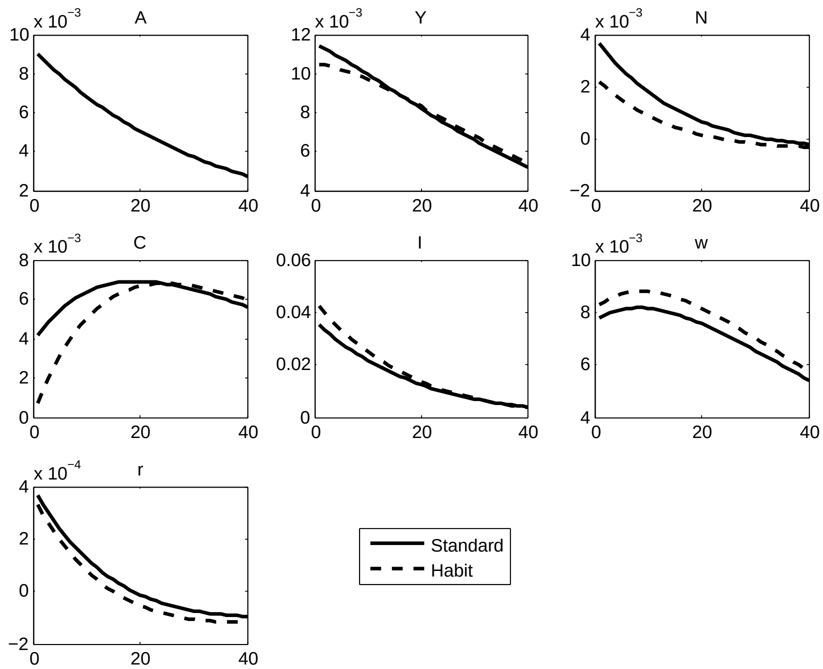
$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \lambda_t = \frac{1}{C_t - \phi C_{t-1}} - \beta \phi E_t \frac{1}{C_{t+1} - \phi C_t} \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow \frac{\theta}{1 - N_t} = \lambda_t w_t \tag{24}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t(\lambda_{t+1}(R_{t+1} + (1 - \delta)) \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t(\lambda_{t+1}(1 + r_t)) \tag{26}$$

The only first order condition that is different is the one that defines λ_t; if $\phi = 0$ we are back in the usual case. It is easiest to solve this model by *not* substituting out for the Lagrange multiplier – just treat it as another endogenous variable. Below are impulse responses – using our otherwise standard calibration of a RBC model – for comparing a value of $\phi = 0$ (the standard case) with $\phi = 0.9$ (the habit formation case).



We observe that the main difference is that consumption jumps up by less on impact the bigger is ϕ. The intuition for this is high consumption today lowers utility tomorrow, other things being equal, the bigger is ϕ. Hence people will behave "cautiously" in essence by not adjusting consumption by much. The other impulse responses are reasonably similar across parameterizations, though hours don't jump up by much, the real wage jumps up by a lot, and output doesn't jump up by much (i.e. this is not going to improve the fit of the model along those dimensions). The main dimension along which the inclusion of habit formation does help the model match the data is not in terms of unconditional moments, but rather in terms of conditional impulse response functions. Most estimated impulse responses to identified shocks (say, monetary policy shocks) show "hump-shaped" responses of consumption. This is difficult to generate without habit formation.

Note that you can combine this kind of habit formation with different preferences specifications – e.g. I could embed this into the non-separable KPR or GHH preferences; I would just replace C_t with $(C_t - \phi C_{t-1})$ everywhere in both cases.

Another form of habit formation is sometimes what is called "external habit formation" or "Catching Up with the Joneses" (Abel, 1990). Here the idea is that utility from consumption depends not on consumption relative to own lagged consumption, but rather on consumption relative to lagged *aggregate* consumption – the idea being that you care about your consumption relative to that of your neighbor. Now, of course, in a representative agent framework own and aggregate end up being the same. The difference is that external habit formation simplifies the problem, because the consumer does not take into account the effect of current consumption decisions on the habit stock (essentially the second term in the expression for λ above drops out).

2.3 Capital and Investment Adjustment Costs

The standard RBC model has the implication that the price of capital goods relative to consumption is 1 (e.g. Hayashi's q is always 1), which in turn says that the value of a firm is just the physical capital stock. The standard RBC model also typically doesn't generate "hump-shaped" impulse responses to shocks, which seems to be a feature of the data. In this subsection we introduce convex costs to adjusting the capital stock. The idea is that there is some cost to adjusting the capital stock (or the level of investment) relative to what is normal. This will have the effect of breaking the $q = 1$ result in the basic model and will also impart some interesting dynamics in the model.

I'll begin with what I'll call "capital adjustment costs" because they follow the form set by Hayashi (1982). In particular, assume that the capital accumulation equation can be written:

$$K_{t+1} = I_t - \frac{\phi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t$$

When $\phi = 0$, this is the standard accumulation equation. If $\phi > 0$, then doing investment different than steady state (in steady state $\frac{I_t}{K_t} = \delta$) results in a cost which essentially makes your capital depreciate faster. Note that this cost is (i) denominated in units of current physical capital and (ii) is symmetric (so doing too little investment relative to steady state also costs you some capital, which may seem a bit funny).

Household preferences are standard. Instead of combining the flow budget constraint with the accumulation, let's treat them as separate (this turns out to be easier). The household problem is:

$$\max_{C_t, I_t, N_t, K_{t+1}, B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi}\right)$$

s.t.

$$C_t + I_t + B_{t+1} \leq w_t N_t + R_t K_t + \Pi_t + (1 + r_{t-1})B_t$$

$$K_{t+1} = I_t - \frac{\phi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t$$

Set up a Lagrangian with two constraints:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} + \lambda_t \left(w_t N_t + R_t K_t + \Pi_t + (1+r_{t-1})B_t - C_t - I_t - B_{t+1}\right) + \dots \right.$$

$$\left. \dots + \mu_t \left(I_t - \frac{\phi}{2}\left(\frac{I_t}{K_t} - \delta\right)^2 K_t + (1-\delta)K_t - K_{t+1} \right) \right\}$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow \theta N_t^{\chi} = \lambda_t w_t$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow \lambda_t = \mu_t \left(1 - \phi\left(\frac{I_t}{K_t} - \delta\right)\right)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \mu_t = \beta E_t \left[R_{t+1}\lambda_{t+1} - \mu_{t+1}\frac{\phi}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)^2 + \mu_{t+1}\phi\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)\frac{I_{t+1}}{K_{t+1}} + \mu_{t+1}(1-\delta)\right]$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1 + r_t)$$

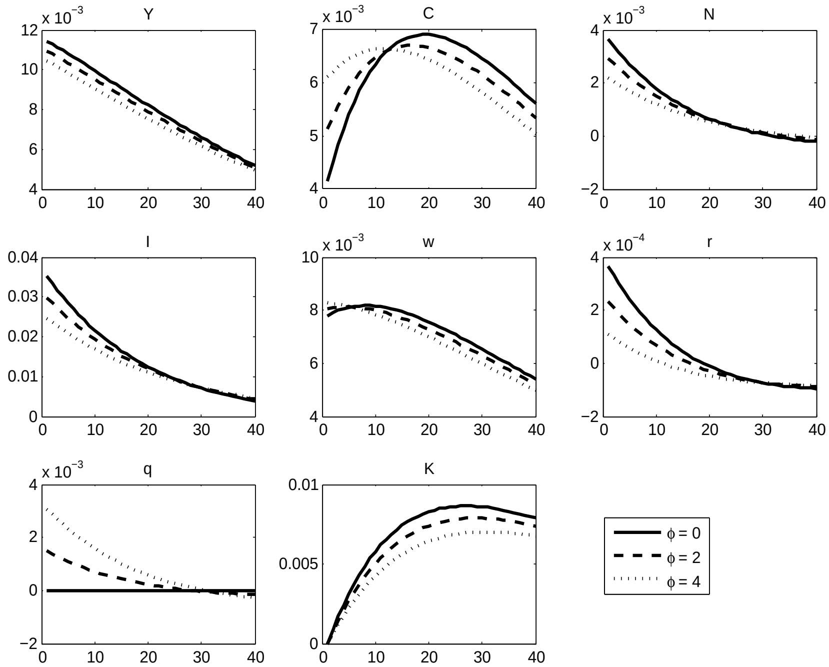
Let's define $q_t \equiv \frac{\mu_t}{\lambda_t}$. μ_t is the marginal utility of having come extra installed capital (K_{t+1}), and λ_t is the marginal utility of having some extra consumption. The ratio is then how much consumption you would give up to have some extra future capital – i.e. it is the relative price of capital in terms of consumption. If $\phi = 0$, we see that $\lambda_t = \mu_t$, so $q_t = 1$ always. If $\phi > 0$, q_t can be different from 1. Using this formulation, we can write these FOC as:

$$q_t = \left(1 - \phi\left(\frac{I_t}{K_t} - \delta\right)\right)^{-1}$$

$$q_t = \beta E_t \frac{C_t}{C_{t+1}}\left[R_{t+1} + q_{t+1}\left((1-\delta) + \phi\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)\frac{I_{t+1}}{K_{t+1}} - \frac{\phi}{2}\left(\frac{I_{t+1}}{K_{t+1}} - \delta\right)\right)\right]$$

The first expression establishes that investment to capital, $\frac{I_t}{K_t}$, is an increasing function of q_t: for this to be bigger than δ, q_t must exceed one. The second is functionally a difference equation in q_t: current q_t is a discounted value of the future marginal product of capital, future adjustment costs, and future q_{t+1}, where the discounting is by the household's stochastic discount factor, $\beta E_t \frac{C_t}{C_{t+1}}$.

I solve the model using my standard parameter values (here $\chi = 1$) for three different values of ϕ: 0, 2, and 4. The impulse responses are below



As predicted, if $\phi = 0$ then $q_t = 1$ always and the responses are the same as in the basic RBC model. As ϕ gets bigger, q_t rises more in response to the productivity shock. Quite naturally, we observe that investment goes up by less, and capital accumulates more slowly, the bigger is ϕ. Because it is costly to adjust the capital stock quickly, investment doesn't jump as much, which means consumption jumps more; this mechanically results in a smaller increase in employment than we would get because labor supply shifts in more. We also see that the real interest rate rises by less (significantly so). $\phi \neq 0$ breaks the tight connection between the real interest rate and the marginal product of capital; this is a good thing from the perspective of the model, since we'd really rather r_t not rise when hit with a productivity shock, since in the data the real interest rate is essentially acyclical.

An alternative adjustment cost specification is based on Christiano, Eichenbaum, and Evans (2005). I refer to this is an "investment adjustment cost" (as opposed to a capital adjustment cost). I call it an investment adjustment cost because (i) the adjustment cost is measured in units of investment, not units of capital as above, and (ii) the adjustment cost doesn't depend on the size of investment relative to the capital stock, but rather on the growth rate of investment. Let the capital accumulation equation be:

$$K_{t+1} = \left[1 - \frac{\phi}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right] I_t + (1-\delta)K_t$$

In steady state $\frac{I_t}{I_{t-1}} = 1$, so this reverts to the standard accumulation equation. As we can see, the cost depends on the growth rate of investment and is measured in units of investment rather than units of capital.

The household problem is otherwise the same as before. As we did earlier, form a Lagrangian with two constraints:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} + \lambda_t \left(w_t N_t + R_t K_t + \Pi_t + (1+r_{t-1})B_t - C_t - I_t - B_{t+1} \right) + \dots \right.$$
$$\left. \dots + \mu_t \left(\left[1 - \frac{\phi}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right] I_t + (1-\delta)K_t - K_{t+1} \right) \right\}$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow \theta N_t^{\chi} = \lambda_t w_t$$

$$\frac{\partial \mathcal{L}}{\partial I_t} = 0 \Leftrightarrow \lambda_t = \mu_t \left(1 - \frac{\phi}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \phi \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) + \beta E_t \mu_{t+1} \phi \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \mu_t = \beta E_t \left(\lambda_{t+1} R_{t+1} + (1-\delta)\mu_{t+1} \right)$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1+r_t)$$

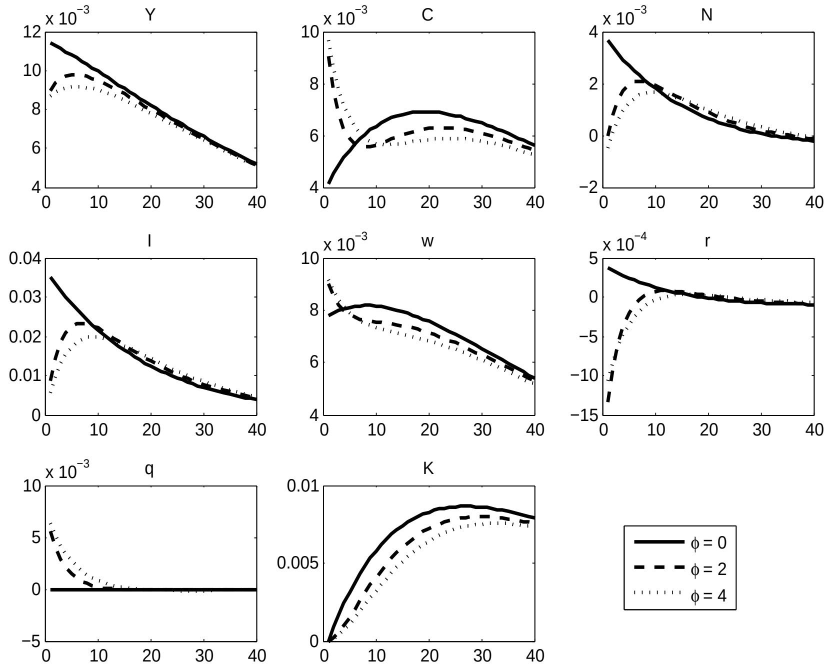
We can again define $q_t = \frac{\mu_t}{\lambda_t}$. Doing so, we can write:

$$q_t = \beta E_t \frac{C_t}{C_{t+1}} \left(R_{t+1} + (1-\delta) q_{t+1} \right)$$

$$1 = q_t \left(1 - \frac{\phi}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 - \phi \left(\frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) + \beta E_t \frac{C_t}{C_{t+1}} q_{t+1} \phi \left(\frac{I_{t+1}}{I_t} - 1 \right) \left(\frac{I_{t+1}}{I_t} \right)^2$$

Above I have made use of the fact that $\lambda_t = \frac{1}{C_t}$. Relative to the capital adjustment cost case earlier, the FOC for K_{t+1} is much simpler and defines q_t is the present discounted value of the rental rate on capital; the FOC for I_t is substantially more complicated. But you Can see that if I_t is large relative to steady state, then q_t will be greater than 1.

I solve the model quantitatively for this specification of adjustment costs for different values of the adjustment cost parameter ϕ. The impulse responses are below:



In terms of the dynamics of q_t and K_t, this specification of adjustment costs plays a fairly similar role to the capital adjustment cost specification – q_t rises more and K_t accumulates more slowly the bigger is ϕ. But the other dynamics are quite different. First, note that for positive values of ϕ both the investment and output impulse responses are "hump-shaped." This means that the *growth rates* of investment and output are autocorrelated, which is actually a feature of the data that the basic RBC model is incapable of matching. You can see why you get this in terms of investment growth rates just from the specification of the adjustment cost. Since there is a convex cost in the investment growth rate, you want to slowly adjust investment growth – – this gives you the hump-shaped investment response, which partially carries over into output. Because investment jumps up so little, consumption responds more to the productivity shock with bigger adjustment costs, which mechanically feeds into a smaller response of employment.

What is perhaps most marked in these responses is the response of the real interest rate. With these adjustment costs, the real interest rate actually declines after the productivity shock rather than increasing. As with the "capital adjustment cost" this investment adjustment cost breaks the connection between the real interest rate and the marginal product of capital – here R_t increases but r_t decreases. With these adjustment costs the breaking of this connection is much stronger than in the capital adjustment cost specification. This is useful because a major failure of the basic RBC model is the strong procylicality of the real interest rate predicted by the model, whereas in the data real interest rates are either acyclical or mildly countercyclical.

In DSGE models featuring many of these additions, it is now most common to include the investment adjustment cost specification in place of the Hayashi style capital adjustment cost specification, precisely because it can generate hump-shaped impulse responses (positive autocorrelation of growth rates) and works to brake the procyclicality of the real interest rate.

2.4 Non-Stationary Productivity

In the specifications we have thus far looked at, we have (implicitly, most of the time) assumed that the non-stationary series of the model are *trend stationary*, because we assumed that labor augmenting technology followed a deterministic, linear time trend, while the other productivity term followed a stationary AR(1) process:

$$
\begin{aligned}
Y_t &= A_t K_t^{\alpha} (Z_t N_t)^{1-\alpha} \\
Z_t &= (1+g_z)^t Z_0 \\
\ln A_t &= \rho \ln A_{t-1} + \varepsilon_t
\end{aligned}
$$

When writing down the model with no growth, as I have been doing, I've been implicity setting $g_z = 0$ and $z_0 = 1$. Allowing $g_z > 0$ requires re-writing the variables, but would only very mildly affect the equilibrium conditions (in essence, would have a small effect on the discount factor).

A priori, some people object to the notion of temporary productivity shocks – if one thinks of A_t as representing knowledge, does it make sense to forget things we've learned. Let's instead suppose that technology follows a stochastic trend. We can get rid of Z_t altogether and write the model as:

$$
\begin{aligned}
Y_t &= a A_t K_t^{\alpha} N_t^{1-\alpha} \\
\Delta \ln A_t &= (1-\rho_A) g_z + \rho_A \Delta \ln A_{t-1} + \varepsilon_t \\
\Delta \ln A_t &= \ln A_t - \ln A_{t-1}
\end{aligned}
$$

Here what I've assumed is that A_t follows an AR(1) process in the *growth rate*, with $0 \leq \rho_a < 1$. This means that shocks, ε_t, will have permanent effects on the level of a_t. g_z is the mean growth rate; if I set $\rho_a = 0$, then $\ln A_t$ would follow a random walk with drift.

We could write this process in the levels as:

$$
\left(\frac{A_t}{A_{t-1}} \right) = \exp(g_z)^{1-\rho_A} \left(\frac{A_{t-1}}{A_{t-2}} \right)^{\rho_A} \exp(\varepsilon_t)
$$

If you take logs of this you get back the process written above. Because we only want one period of leads/lags in writing down the equilibrium conditions, it is useful to introduce a new variable, call it $g_t \equiv \frac{A_t}{A_{t-1}}$. We can then write the process above as:

$$
g_t = \exp(g_z)^{1-\rho_a} g_{t-1}^{\rho_A} \exp(\varepsilon_t)
$$

Or in logs:

$$\ln g_t = (1 - \rho_A) g_z + \rho_A \ln g_{t-1} + \varepsilon_t$$

Let's figure out how to transform the variables of this model. Start with the production function, then take logs, and then first difference so as to get in growth rate form:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$
$$\ln Y_t = \ln A_t + \alpha \ln K_t + (1-\alpha) \ln N_t$$
$$(\ln Y_t - \ln Y_{t-1}) = (\ln A_t - \ln A_{t-1}) + \alpha(\ln K_t - \ln K_{t-1}) + (1-\alpha)(\ln N_t - \ln N_{t-1})$$

Along a balanced growth path hours will not grow and capital will grow at the same rate as output (capital and output must grow at the same rate because the real interest rate is constant along a balanced growth path, and the real interest rate in the long run is tied to the capital/output ratio). Using these facts, we have:

$$(\ln Y_t - \ln Y_{t-1}) = (\ln A_t - \ln A_{t-1}) + \alpha(\ln Y_t - \ln Y_{t-1})$$
$$\ln Y_t - \ln Y_{t-1} = \frac{1}{1-\alpha}(\ln A_t - \ln A_{t-1})$$

This says that, along a balanced growth path, output will grow at $\frac{1}{1-\alpha}$ times the rate of technological progress (if we had written this as labor augmenting technological progress, as opposed to neutral, they would grow at the same rate . . . these two setups are equivalent provided we re-define the trend growth rate appropriately).

Play around with the above:

$$\ln\left(\frac{Y_t}{Y_{t-1}}\right) = \ln\left(\frac{A_t}{A_{t-1}}\right)^{\frac{1}{1-\alpha}}$$
$$\frac{Y_t}{Y_{t-1}} = \left(\frac{A_t}{A_{t-1}}\right)^{\frac{1}{1-\alpha}}$$
$$\frac{Y_t}{A_t^{\frac{1}{1-\alpha}}} = \frac{Y_{t-1}}{A_{t-1}^{\frac{1}{1-\alpha}}}$$

In other words, along the balanced growth path output divided by $A_t^{\frac{1}{1-\alpha}}$ does not grow – i.e. it is stationary. Hence, we can induce stationarity into the model by dividing through by this.

Define the following stationarity inducing transformations:

$$\widehat{Y}_t \equiv \frac{Y_t}{A_t^{\frac{1}{1-\alpha}}}$$
$$\widehat{I}_t \equiv \frac{K_t}{A_{t-1}^{\frac{1}{1-\alpha}}}$$
$$\widehat{I}_t \equiv \frac{I_t}{A_t^{\frac{1}{1-\alpha}}}$$
$$\widehat{w}_t \equiv \frac{w_t}{A_t^{\frac{1}{1-\alpha}}}$$
$$\widehat{C}_t \equiv \frac{C_t}{A_t^{\frac{1}{1-\alpha}}}$$

There is one very slight modification due to a timing assumption – we need to divide by K_t by A_{t-1}. Intuitively, this is because K_t is chosen at $t-1$, not t. We need to write it this way so that $\widehat{K}_t$ is a predetermined state variable in the model; scaling K_t by A_t instead would also render the model stationary, but would induce issues with the solution. We can use these transformations to alter the first order conditions of the basic model as needed. Begin with the production function, dividing both sides by the scaling factor $A_t^{\frac{1}{1-\alpha}}$:

$$\widehat{Y}_t = A_t^{\frac{-\alpha}{1-\alpha}} K_t^{\alpha} N_t^{1-\alpha}$$

Now, multiply and divide by $A_{t-1}^{\frac{\alpha}{1-\alpha}}$ to get the capital stock in the correct terms:

$$\widehat{Y}_t = A_t^{\frac{-\alpha}{1-\alpha}} \left(\frac{K_t}{A_{t-1}^{\frac{1}{1-\alpha}}} \right)^{\alpha} A_{t-1}^{\frac{\alpha}{1-\alpha}} N_t^{1-\alpha}$$
$$\widehat{Y}_t = \left(\frac{A_t}{A_{t-1}} \right)^{\frac{-\alpha}{1-\alpha}} \widehat{K}_t^{\alpha} N_t^{1-\alpha}$$

Using our change of variables, we can write this as:

$$\widehat{Y}_t = g_t^{-\frac{\alpha}{1-\alpha}} \widehat{K}_t^{\alpha} N_t^{1-\alpha}$$

Next go to the capital accumulation equation, and divide both sides by the scaling factor at date t.

$$\frac{K_{t+1}}{A_t^{\frac{1}{1-\alpha}}} = \frac{I_t}{A_t^{\frac{1}{1-\alpha}}} + (1-\delta)\frac{K_t}{A_t^{\frac{1}{1-\alpha}}}$$

$$\widehat{K}_{t+1} = \widehat{I}_t + (1-\delta)\left(\frac{K_t}{A_{t-1}^{\frac{1}{1-\alpha}}}\right)\left(\frac{A_{t-1}}{A_t}\right)^{\frac{1}{1-\alpha}}$$

We can simplify this further by noting our change of variable:

$$\widehat{K}_{t+1} = \widehat{I}_t + (1-\delta)g_t^{-\frac{1}{1-\alpha}}\widehat{K}_t$$

The accounting identity is the same in terms of the transformed variables as always: $\widehat{y}_t = \widehat{c}_t + \widehat{I}_t$. Next, consider expressions for factor prices:

$$\begin{aligned} w_t &= (1-\alpha)A_t K_t^{\alpha} N_t^{-\alpha} \\ R_t &= \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \end{aligned}$$

Transform these:

$$\frac{w_t}{A_t^{\frac{1}{1-\alpha}}} = (1-\alpha)A_t^{-\frac{\alpha}{1-\alpha}} K_t^{\alpha} N_t^{-\alpha}$$

$$\widehat{w}_t = (1-\alpha)A_t^{-\frac{\alpha}{1-\alpha}} A_{t-1}^{\frac{\alpha}{1-\alpha}} \left(\frac{K_t}{A_{t-1}^{\frac{1}{1-\alpha}}}\right)^{\alpha} N_t^{-\alpha}$$

$$\widehat{w}_t = (1-\alpha)g_t^{-\frac{\alpha}{1-\alpha}} \widehat{K}_t^{\alpha} N_t^{-\alpha}$$

For the rental rate, we have:

$$R_t = \alpha A_t A_{t-1}^{\frac{\alpha-1}{1-\alpha}} \left(\frac{K_t}{A_{t-1}^{\frac{1}{1-\alpha}}}\right)^{\alpha-1} N_t^{1-\alpha}$$

$$R_t = \alpha\left(\frac{A_t}{A_{t-1}}\right)\widehat{K}_t^{\alpha-1} N_t^{1-\alpha}$$

$$R_t = \alpha g_t \widehat{K}_t^{\alpha-1} N_t^{1-\alpha}$$

Now what about the Euler equations? Start with the one for bonds:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} (1 + r_t)$$

$$\frac{A_t^{\frac{1}{1-\alpha}}}{C_t} = \beta E_t \frac{A_t^{\frac{1}{1-\alpha}}}{C_{t+1}} (1 + r_t)$$

$$\frac{A_t^{\frac{1}{1-\alpha}}}{C_t} = \beta E_t \frac{A_{t+1}^{\frac{1}{1-\alpha}}}{A_{t+1}^{\frac{1}{1-\alpha}}} \frac{A_t^{\frac{1}{1-\alpha}}}{C_{t+1}} (1 + r_t)$$

$$\frac{1}{\widehat{C}_t} = \beta E_t g_{t+1}^{-\frac{1}{1-\alpha}} \frac{1}{\widehat{C}_{t+1}} (1 + r_t)$$

Since the rental rate on capital is stationary, the Euler equation for capital is going to look the same in terms of transformed variables:

$$\frac{1}{\widehat{C}_t} = \beta E_t g_{t+1}^{-\frac{1}{1-\alpha}} \frac{1}{\widehat{C}_{t+1}} (R_{t+1} + (1 - \delta))$$

The labor supply condition is straightforward to re-write in terms of transformed variables – since $\frac{w_t}{C_t}$ shows up on the right hand side, and we divide both variables by the same scaling factor, we can simply write:

$$\theta \frac{1}{1 - N_t} = \frac{1}{\widehat{C}_t} \widehat{w}_t$$

Then the full set of equilibrium conditions are:

$$\frac{1}{\widehat{C}_t} = \beta E_t g_{t+1}^{-\frac{1}{1-\alpha}} \frac{1}{\widehat{C}_{t+1}} (1 + r_t)$$

$$\frac{1}{\widehat{C}_t} = \beta E_t g_{t+1}^{-\frac{1}{1-\alpha}} \frac{1}{\widehat{C}_{t+1}} (R_{t+1} + (1 - \delta))$$

$$\theta \frac{1}{1 - N_t} = \frac{1}{\widehat{C}_t} \widehat{w}_t$$

$$\widehat{w}_t = (1 - \alpha) g_t^{-\frac{\alpha}{1-\alpha}} \widehat{K}_t^{\alpha} N_t^{-\alpha}$$

$$R_t = \alpha g_t \widehat{K}_t^{\alpha - 1} N_t^{1-\alpha}$$

$$\widehat{Y}_t = \widehat{C}_t + \widehat{I}_t$$

$$\widehat{K}_{t+1} = \widehat{I}_t + (1 - \delta) g_t^{-\frac{1}{1-\alpha}} \widehat{K}_t$$

$$\widehat{Y}_t = g_t^{-\frac{\alpha}{1-\alpha}} \widehat{K}_t^{\alpha} N_t^{1-\alpha}$$

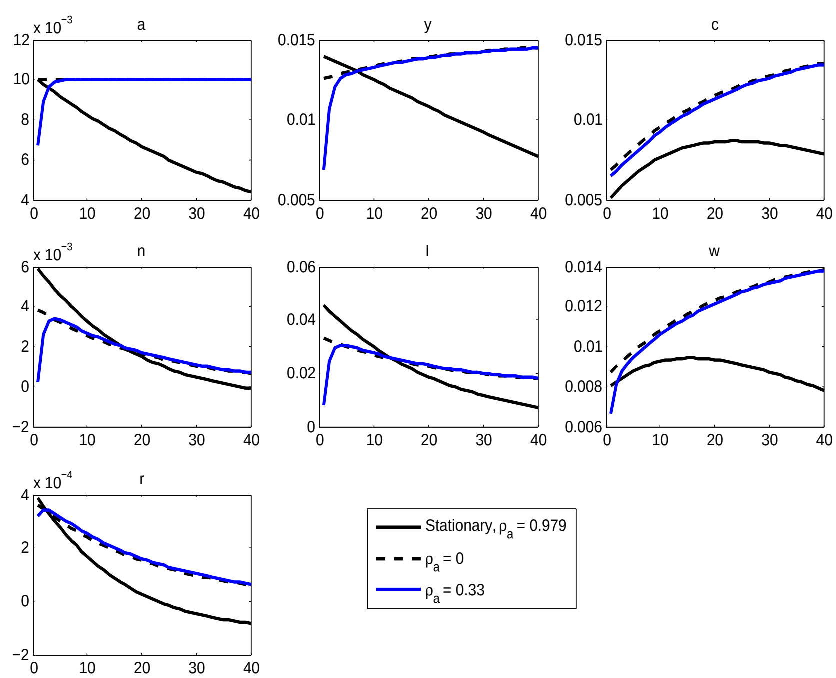
$$\ln g_t = (1 - \rho_A) g_z + \rho_A \ln g_{t-1} + \varepsilon_t$$

This is nine equations in nine variables.

I solve the model using our standard parameter values. Dynare will produce impulse responses of the detrended variables. To construct impulse responses of the regular variables, I need to transform the responses produced by Dynare. First, I can find the response of the log level of A_t by cumulating the response of the log growth rate, $\ln g_t$. Then once I have the impulse response of A_t, I can "add back" $\frac{1}{1-\alpha}\ln A_t$ to the impulse responses of the other variables to get them back in log level form (recall that Dynare will produce the impulse response of the logs, so $\ln \widehat{x}_t$. Since $\widehat{x}_t \equiv \frac{x_t}{A_t^{\frac{1}{1-\alpha}}}$, I can add $\frac{1}{1-\alpha}\ln A_t$ to $\ln \widehat{x}_t$ to recover the log level of the variable of interest; I only need to do this for variables that are non-stationary – so not for the real interest rate, the rental rate on capital, or the level of employment).

I compute impulse responses under some different parameter configurations in order to make a few points. First, I show IRFs under a stationary shock with $\rho_A = 0.979$. Second, I show two different permanent shocks, one with $\rho_A = 0$ (random walk) and the other with $\rho_A = 0.33$ (so some positive autocorrelation in the growth rate). I set the size of the transitory shock to 0.01 (so productivity increases by one percent); in the case of the two permanent shocks, I set the size of the shocks such that the long run effect on productivity is also 0.01 (so one percent); this means that the size of the shock is 0.01 in the random walk case, but $(1-\rho_A)0.01$ more generally. So for the shock that is persistent in growth rates, I'm actually hitting the economy with a smaller initial shock.

Below, the solid black lines show the responses to the transitory productivity shock; the dashed black line shows the responses to the random walk productivity shock; the blue solid line shows the responses to the persistent growth rate shock.



The main take-away from this picture is that the impact effects on output, hours, and investment are smaller the more persistent is the productivity shock. For the shock that is autocorrelated in growth rates, we see that labor hours essentially do not react. Also worth noting is that after about 10 periods the responses with the random walk shock and the persistent growth rate shock are very similar.

What is going on here? It's the same intuition from labor demand-supply. If the productivity shock is permanent, you get a much bigger initial jump in consumption than you do in the transitory shock case; this translates to a bigger inward shift of the labor supply curve, which for a given labor demand shift results in a smaller increase in N_t. A smaller increase in N_t means that output goes up by less on impact; this combined with a bigger increase in C_t means a smaller increase in investment. One thing that is interesting to note is that the impact increase in C_t is actually smaller in the case of the autocorrelated shock than in the case of the random walk shock. What is going on here? Consumption is very forward-looking; by the way I constrained the size of the shocks, the long run effects are identical in each case. Because you get more of the extra productivity sooner in the random walk case than later in the persistent growth rate case, the wealth effect is actually bigger in the random walk case, so C_t jumps by more (although not by a lot). Why then do we observe a smaller increase in N_t in the autocorrelated growth rates case than the random walk case if the wealth effect / jump in c_t is smaller? It's because the outward shift in labor demand is also smaller given that I constrained the long run effect on productivity to be the same in both cases; when the shock is autocorrelated, the immediate effect on productivity (and hence labor demand) is smaller. This combines to result in a much smaller increase in N_t, a smaller initial increase in Y_t, and a much smaller initial increase in I_t.

The bottom line here is that it is (relatively) straightforward to augment the model to allow for permanent productivity shocks, but this will only make the model fit the data worse along the dimensions of the relative volatility of hours as well as the overall lack of amplification.

2.5 Variable Capital Utilization

In this section we consider adding another amplification mechanism to the model – variable capital utilization. The idea here is that while the capital stock may be predetermined within period, but we can "work" the capital more intensively (or not) depending on conditions. One could also think about variable labor utilization, but for this to work we'd need to make labor at least partially predetermined – you'll only get a unique value of utilization if you can't completely change the factor (capital in our main case, in this case labor).

There are different ways to model capital utilization and the costs associated with it. What I'm going to do here assumes (i) the households own the capital stock, (ii) the households choose the level of utilization, and (iii) the household leases "capital services" (the product of utilization and physical capital) to firms at rental rate R_t. The cost of utilization is faster depreciation.

Define u_t to be utilization and $\widehat{K}_t \equiv u_t K_t$ as capital services (K_t is the physical capital stock). The depreciation rate is now a function of capital utilization, which we want to normalize to be one in steady state. In particular:

$$\delta(u_t) = \delta_0 + \phi_1(u_t - 1) + \frac{\phi_2}{2}(u_t - 1)^2$$

The household problem is otherwise standard. We can write it as:

$$\max_{C_t, N_t, K_{t+1}, u_t} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\ln C_t + \theta \ln(1 - N_t)\right)$$

s.t.

$$C_t + K_{t+1} - (1 - \delta(u_t))K_t + B_{t+1} \leq w_t N_t + R_t u_t K_t + \Pi_t + (1 + r_{t-1})B_t$$

The first order conditions for bonds/consumption and labor work out to be the same as we had earlier:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}}(1 + r_t)$$

$$\theta \frac{1}{1 - N_t} = \frac{1}{C_t} w_t$$

The first order condition for utilization works out to:

$$\delta'(u_t) = R_t$$

Or:

$$\phi_1 + \phi_2(u_t - 1) = R_t$$

The Euler equation for capital is:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left(R_{t+1} u_{t+1} + (1 - \delta(u_{t+1}))\right)$$

The firm produces output using:

$$Y_t = A_t \widehat{K}_t^{\alpha} N_t^{1-\alpha}$$

Note that the firm acts as though it gets to choose capital services at rental rate R_t, even though household can separately choose utilization and capital. The first order conditions of the firm problem are otherwise standard:

$$w_t = (1 - \alpha) A_t \widehat{K}_t^{\alpha} N_t^{-\alpha}$$

$$R_t = \alpha A_t \widehat{K}_t^{\alpha - 1} N_t^{-\alpha}$$

The aggregate resource constraint is standard and we assume that productivity follows an AR(1) in the log. The full set of equilibrium conditions can be written (getting ride of $\widehat{K}_t$ by replacing it with $u_t K_t$):

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}}(1 + r_t)$$

$$\theta \frac{1}{1 - N_t} = \frac{1}{C_t} w_t$$

$$\phi_1 + \phi_2(u_t - 1) = R_t$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left(R_{t+1} u_{t+1} + (1 - \delta(u_{t+1}))\right)$$

$$w_t = (1 - \alpha) A_t u_t^{\alpha} K_t^{\alpha} N_t^{-\alpha}$$

$$R_t = \alpha A_t u_t^{\alpha-1} K_t^{\alpha-1} N_t^{-\alpha}$$

$$Y_t = A_t u_t^{\alpha} K_t^{\alpha} N_t^{1-\alpha}$$

$$Y_t = C_t + I_t$$

$$K_{t+1} = I_t + (1 - \delta(u_t))K_t$$

$$\delta(u_t) = \delta_0 + \phi_1(u_t - 1) + \frac{\phi_2}{2}(u_t - 1)^2$$

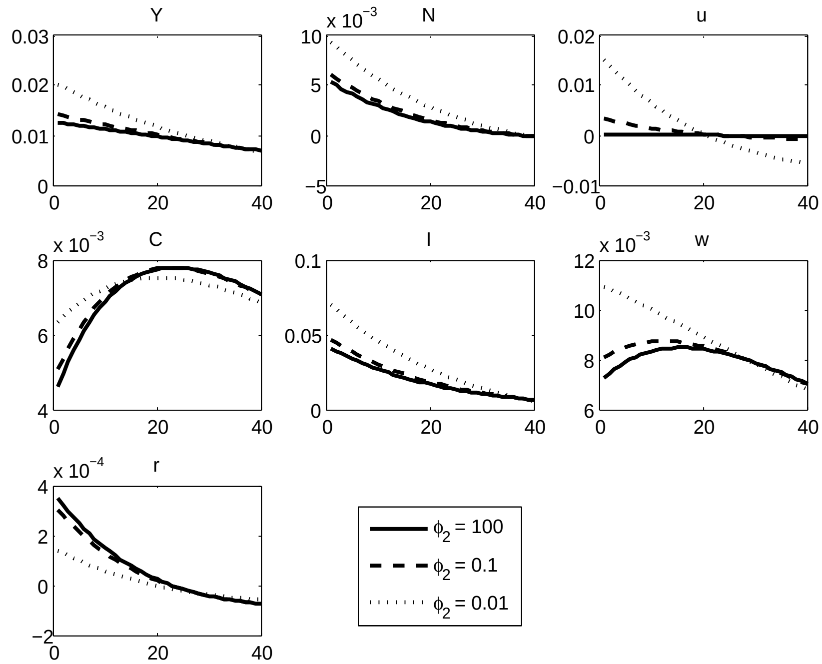
$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t$$

This is 11 equations in 11 variables. The parameters of the utilization cost function are not entirely free. To normalize $u = 1$ in steady state, we must have: $R = \frac{1}{\beta} - (1 - \delta_0)$. But from the FOC for utilization this requires:

$$\frac{1}{\beta} - (1 - \delta_0) = \phi_1$$

In other words, ϕ_1 must be set to pin down steady state utilization at 1; ϕ_2 is a free parameter, and δ_0 is the steady state depreciation rate, which we can calibrate to match the steady state investment-output ratio as before.

We can think about $\phi_2 \to \infty$ as fixing $u_t = 1$, which puts us back in the RBC model. This may not be obvious, because one would be tempted to look at the above expression and think this fixed R_t at its steady state value, $\frac{1}{\beta} - (1 - \delta)$. This is not so, because ∞ times 0 is something finite.



We can see that there is significantly more amplification of the productivity shock when the costs of utilization are lower (i.e. when ϕ_2 is smaller): output, employment, consumption, and investment rise by significantly more with variable utilization than without. Without variable utilization ($\phi_2 = 100$), HP filtered output volatility is about 0.017; as I decrease ϕ_2, I get output volatility of 0.019 and 0.027 ($\phi_2 = 0.1$ and $\phi_2 = 0.01$, respectively). This means that, in principle, we'd need smaller productivity shocks to generate the observed output volatility we observe in the data, which many people find attractive. We also see that the inclusion of variable utilization (slightly) increases the relative volatility of employment: its volatility relative to output goes from 0.43 in the no utilization case to 0.49 with $\phi_2 = 0.01$.

Finally, and importantly, it is worth pointing out that variable capital utilization invalidates standard "growth accounting" techniques to get measured TFP. Measured TFP is defined as $\ln \widehat{A}_t = \ln Y_t - \alpha \ln K_t - (1 - \alpha) \ln N_t$. In this model, this is equal to $\ln A_t + \alpha \ln u_t$. Hence, to the extent to which utilization moves around, the volatility of measured TFP will be an overstatement of the volatility of the model counterpart, $\ln A_t$. We see this in the quantitative simulations here. The volatility of actual $\ln A_t$ is 0.0117; with no utilization the volatility of measured TFP is the same, with $\phi_2 = 0.1$ it is 0.0131, and with $\phi_2 = 0.01$ it is 0.0184. This is a big difference.

Because the model is only driven by one shock (the productivity shock), the inclusion of variable capital utilization doesn't do much to change the cyclicality of measured TFP versus true productivity – they are both very positively correlated with output. Suppose there is some shock which raises N_t (like a government spending shock, or perhaps a preference shock). Because labor and capital services are complementary in the model, anytime N_t goes up you'd like to also increase u_t. But from above, this means that measured TFP will rise (along with output) even though actual $\ln A_t$ won't change. Thus, variable capital utilization will change both the volatility and the cyclicality of measured TFP – measured TFP will be both more volatile and more positively correlated with output with variable capital utilization than the true exogenous productivity variable, $\ln A_t$, is.

2.6 Preference Shocks

The basic RBC model focuses on the productivity shock, but it is possible and fruitful to consider other sources of shocks. It is popular to write down models where there are preference shocks – shocks to how agents get utility from consumption and/or leisure/labor. I will write down the standard model with two such shocks – an intertemporal preference shock that governs how you weight current utility relative to future utility, and an intratemporal preference shock that governs how you value utility from labor/leisure.

Consider the following household problem:

$$\max_{C_t, K_{t+1}, N_t, B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \psi_t \left(\ln C_t - \nu_t \theta \frac{N_t^{1+\chi}}{1+\chi} \right)$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t \leq w_t N_t + R_t K_t + \Pi_t + r_{t-1} B_t$$

I assume that both ψ_t and ν_t follow mean zero AR(1)s in the log (so that the non-stochastic levels are unity):

$$\ln \psi_t = \rho_\psi \ln \psi_{t-1} + \varepsilon_{\psi,t}$$

$$\ln \nu_t = \rho_\nu \ln \nu_{t-1} + \varepsilon_{\nu,t}$$

The exogenous variable ψ_t is an intertemporal preference shock – it doesn't impact you value utility from consumption versus utility from leisure, but rather how you value utility today versus utility in the future. If ψ_t increases, for example you place relatively more weight on present utility than the future. ν_t is an intratemporal preference shock – it affects how you value utility from consumption relative to disutility from labor (or utility from leisure).

The first order conditions of the model can be written:

$$\frac{1}{C_t} = \beta E_t \left[\frac{\psi_{t+1}}{\psi_t} \frac{1}{C_{t+1}} \left(R_{t+1} + (1-\delta) \right) \right]$$

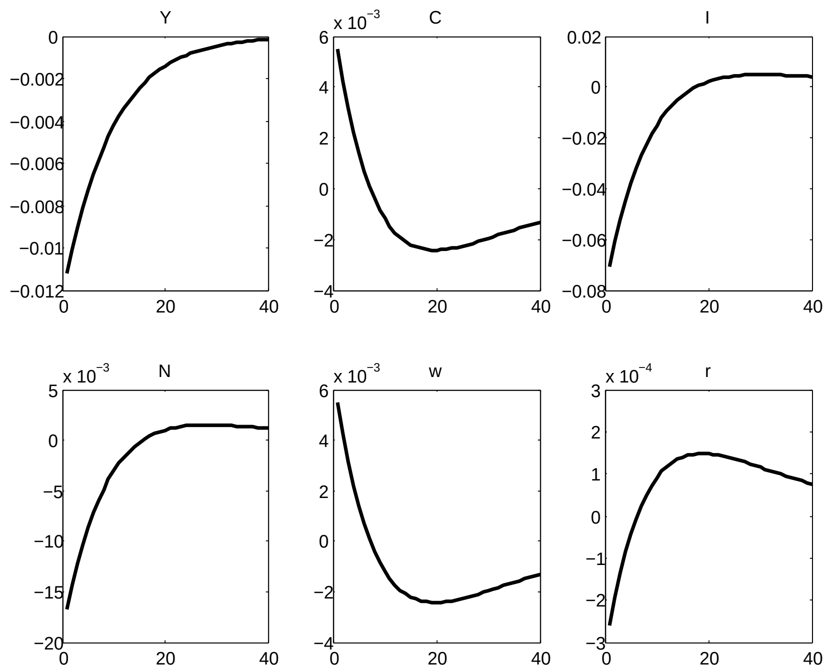
$$\frac{1}{C_t} = \beta E_t \left[\frac{\psi_{t+1}}{\psi_t} \frac{1}{C_{t+1}} \left(1 + r_t \right) \right]$$

$$\nu_t \theta N_t^{\chi} = \frac{1}{C_t} w_t$$

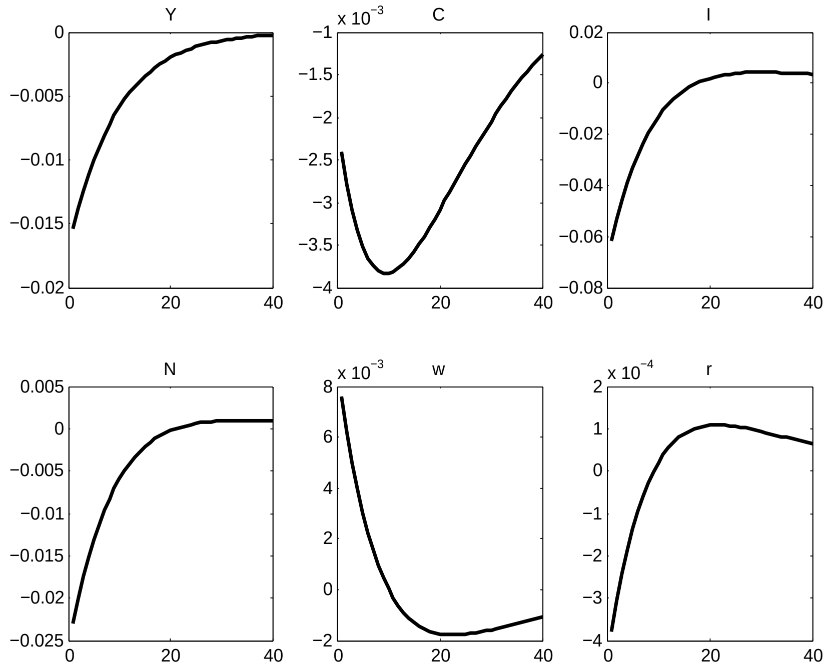
Two things to point out. First, ψ_t does not show up in the labor supply condition: higher ψ_t increases the marginal utility of both consumption and the marginal disutility of labor, but these cancel out. Second, since ψ_t is mean reverting, when ψ_t is high we will have $E_t \frac{\psi_{t+1}}{\pi_t}$ be relatively low. This means that an increase in ψ_t is isomorphic to a temporary reduction in β – it means you are relatively more impatient. Third, ν_t shows up in the labor supply condition in a way analagous to a distortionary tax rate on labor income – if you divide both sides by ν_t you see that an increase in ν_t is functionally equivalent to an increase in τ_t^n (since $(1 - \tau_t^n)$ would show up on the right hand side of the condition).

I solve the model quantitatively using a first order log-linear approximation using our "standard" parameter values (a Frisch elasticity of 1, $\chi = 1$). I assume that the AR coefficients on the two preference shocks are 0.9 (i.e. $\rho_\psi = \rho_\nu = 0.9$) and that the standard deviations of each shock are one percent (i.e. 0.01). I'm not trying to parameterize these in any serious way; and for impulse response analysis in a first order approximation, the size of the shocks is irrelevant for the shape of the IRFs, thought it may have large effects on unconditional moments.

Below are the responses to the intertemporal preference shock, ψ_t. We see that output, hours, investment, and the real interest rate all decline immediately, while consumption and the real wage increase. What is going on here is the following. The increase in ψ_t is effectively like a decrease in the discount factor – households value current utility relatively more than future utility. This means they want to consume more in the present and work less – hence the increase in C_t and decline in N_t (in a mechanical sense from the FOC for labor the increase in C_t shifts the labor supply curve in). The inward shift of labor supply along a stable labor demand curve leads to an increase in the wage. Falling hours with no immediate change in A_t or K_t means that Y_t must fall. Output falling with consumption increasing means that investment must fall. The real interest rate must fall immediately. There are two ways to see this. First, since consumption is high and falling, the real interest rate must fall for the Euler equation to hold. Second, the fall in N_t lowers the marginal product of capital, and with capital fixed this means that the rental rate on capital must decline, and without adjustment costs r_t moves in the same direction as R_t. Overall, while this shock produces interesting dynamics, it does not produce positive co-movement between consumption and output, and hence cannot be the primary driving force of the business cycle.



Next, consider the intratemporal preference shock. This leads to a reduction in C_t, N_t, Y_t, and I_t, with an increase in w_t. The increase in ψ_t means that people dislike labor relatively more – this means naturally that they want to work less. This shifts the labor supply curve in; along a stable labor demand curve, this means that the wage must rise. Lower employment means lower output. Consumption also falls – this occurs naturally because household income declines. Investment also falls.



We see here that, unlike the intertemporal preference shock, the intratemporal preference shock can produce co-movement between consumption and output and employment. I've said before that the FOC for labor makes it difficult for consumption and employment to move together unless A_t changes. In a mechanical sense here, an increase in ν_t functionally plays a similar role (although in a different way). What you need to get co-movement between C_t and N_t is for *either* the labor demand or supply curves to shift for a reason other than pure wealth effect of C_t shifting the labor supply curve. ν_t will do the trick, as would a change in the tax rate on labor income (see discussion above).

2.7 Investment Shocks

The standard RBC model assumes that shocks to neutral productivity are the primary (or sole) driver of business cycle fluctuations. Another kind of disturbance that has recently received a good deal of attention is a shock to the marginal efficiency of investment (MEI), or just "investment shock" for short. The investment shock makes the economy more productive at transforming investment into new physical capital (in a way somewhat analogous to how the productivity shock, A_t, makes you more productive at transforming capital and labor into output).

Let Z_t denote the investment shock. It enters the capital accumulation equation as follows:

$$K_{t+1} = Z_t I_t + (1 - \delta) K_t$$

Here, an increase in Z_t means you get more K_{t+1} for a given amount of I_t – i.e. this shock increases the efficiency of investment. Some authors have argued that this shock is a reduced form proxy for modeling the health of the financial system – the financial system essentially turns investment into capital, so the higher (or lower) Z_t is, the better (or worse) the financial system is.

Let's again assume that the household owns the capital stock and leases it to firms. The household problem can be written with two constraints:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln C_t - \theta \frac{N_t^{1+\chi}}{1+\chi} \right\}$$

$$\text{s.t.}$$

$$C_t + I_t + B_{t+1} - B_t \leq w_t N_t + R_t K_t + \Pi_t + r_{t-1} B_t$$

$$K_{t+1} = Z_t I_t + (1-\delta) K_t$$

These constraints can be combined into one:

$$C_t + \frac{K_{t+1}}{Z_t} - (1-\delta)\frac{K_t}{Z_t} + B_{t+1} - B_t \leq w_t N_t + R_t K_t + \Pi_t + r_{t-1} B_t$$

The first order conditions for bonds, labor, and capital can be written:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (1 + r_t) \right)$$

$$\theta N_t^{\chi} = \frac{1}{C_t} w_t$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}} \frac{Z_t}{Z_{t+1}} \left(Z_{t+1} R_{t+1} + (1-\delta) \right) \right]$$

The rest of the equilibrium conditions are standard:

$$Y_t = C_t + I_t$$

$$K_{t+1} = Z_t I_t + (1-\delta) K_t$$

$$w_t = \alpha A_t K_t^{\alpha} N_t^{-\alpha}$$

$$R_t = (1-\alpha) A_t K_t^{\alpha-1} N_t^{1-\alpha}$$

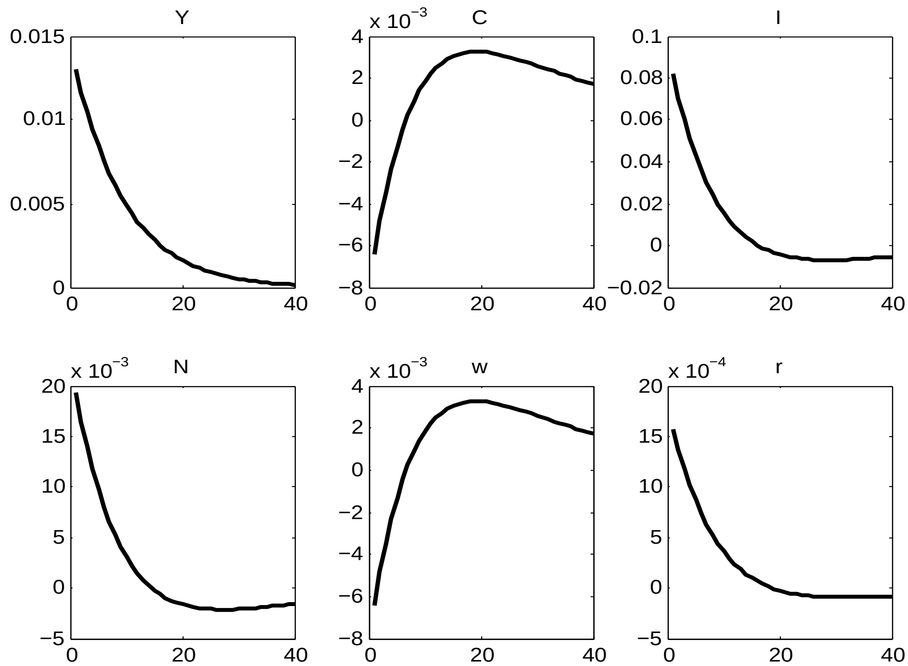
$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

I assume that both A_t and Z_t follow mean zero AR(1) processes in the log:

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t}$$

$$\ln Z_t = \rho_z \ln Z_{t-1} + \varepsilon_{z,t}$$

I use a standard parameterization of the model. Here I assume that $\rho_z = 0.9$ and that the standard deviation of the investment shock is 0.01. The impulse responses to the investment shock are shown below.



We observe that the investment shock leads to a sizeable increase in output, hours, and investment, with reductions in consumption and the real wage. The intuition for what is going on is as follows. When Z_t increases, since turning investment into capital is more efficient, it makes sense to save more through capital. Hence, consumption jumps down. This results in an inward shift of the labor supply curve (along an initially stable labor demand curve), which leads an increase in N_t and a reduction in w_t. The increase in N_t results in an increase in Y_t, which combined with the reduction in C_t means I_t is higher. The real interest rate rises because the marginal product of capital is initially higher. As we go further out, we start to accumulate more capital, and the Z_t shock fades away, consumption beings to increase, which shifts labor supply back in, driving down N_t and the wage up. We do not generate co-movement between C_t and N_t (and Y_t and I_t) here for reasons that have been mentioned above: with our standard labor supply relationship, absent a change in A_t consumption and hours cannot move together (this holds the capital stock fixed, which is a safe approximation at short forecast horizons). Hence, while the investment shock produces interesting dynamics here, in the model as currently presented it cannot be a major source of business cycle fluctuations as it does not get the co-movement right. In difference setups where the labor supply condition is broken this is not necessarily the case.

A final caveat is in order here. The investment shock as presented is closely related to a different kind of shock that is often called "investment-specific technology" (or IST). This is a shock which affects the transformation of consumption goods into investment goods, whereas the investment shock laid out here impacts the transformation of investment goods into capital goods. Since investment is non-consumed output, in terms of the capital accumulation equation both the IST and MEI shocks show up the same way. But the two kinds of shocks have different implications for the relative price of investment and consumption. In our setup, the relative price of investment to consumption is 1; in the IST setup, the relative price of investment is the inverse of the IST shock. The relative price of investment to consumption has trend significantly down in the post-war era, suggesting that IST shocks are an important long run feature of the data. But the relative price of investment does not move a ton at cyclical frequencies, so mean-reverting shocks to IST cannot be super important business cycle shocks. Since the MEI shock here doesn't affect the relative price of investment, you can't rule out that it's important over the business cycle.

2.8 Imperfect Competition

We now deviate from the assumption of perfect competition. Although it is not necessary, it is helpful to break production up into two sectors. The first is the "final goods" sector and is competitive, so we can think about there being a representative final goods firm. This firm doesn't use any factors of production, but rather "bundles" intermediate goods into a final good. The intermediate goods use capital and labor to produce. There are a continuum of intermediate goods firms who populate the unit interval. This is just a convenient normalization – the point is that there are a "lot" of intermediate good firms, but they produce differentiated goods.

The final good is a constant elasticity of substitution aggregate of intermediate goods. The "production" technology is:

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}} \tag{27}$$

Remember that an integral is just the sum – this is the sum of each intermediate input raised to a power, with the whole sum raised to a power that is the inverse of the power on each intermediate input. ν is a parameter assumed to be positive and it governs the degree of substitutability among intermediate inputs. As it goes to infinity, this just becomes the sum of intermediate goods (i.e. goods are perfect substitutes). As it goes to zero, the production technology becomes Leontief (perfect complements). For $\nu = 1$, there is a "unit elasticity of substitution" and the production technology is Cobb-Douglas (the product of the intermediate inputs). Assume for what follows that $\nu > 1$.

The final goods firm wants to maximize (nominal) profits, given a final good price, P_t, and taking intermediate good prices, $P_{j,t}$, as given:

$$\max_{Y_{j,t}} \quad \Pi_t^F = P_t \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}} - \int_0^1 P_{j,t} Y_{j,t} dj$$

The first order conditions are found by differentiating with respect to $y_{i,t}$ and setting equal to zero:

$$\frac{\partial \Pi_t^F}{\partial y_{i,t}} = 0 \Leftrightarrow P_t \frac{\nu}{\nu-1} \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}-1} \frac{\nu-1}{\nu} Y_{j,t}^{\frac{\nu-1}{\nu}-1} = Pp_{j,t}$$

$$P_t \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{1}{\nu-1}} Y_{j,t}^{\frac{-1}{\nu}} = P_{j,t}$$

$$Y_{j,t}^{\frac{-1}{\nu}} = \left(\frac{P_{j,t}}{P_t} \right) \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{-1}{\nu-1}}$$

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\nu} \left(\int_0^1 Y_{j,t}^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}$$

Using the definition of the aggregate final goods production technology, this reduces nicely to:

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\nu} Y_t \tag{28}$$

In words, the relative demand for differentiated intermediate good j depends on its relative price, with ν the price elasticity of demand.

We can now solve for the aggregate price index. The nominal value of the final good is just the sum of prices times quantities of intermediate goods, using the above demand specification:

$$P_t Y_t = \int_0^1 P_{j,t} Y_{j,t} dj = \int_0^1 P_{j,t} \left(\frac{P_{j,t}}{P_t} \right)^{-\nu} Y_t dj$$

$$P_t Y_t = \int_0^1 P_{j,t}^{1-\nu} P_t \nu Y_t dj = P_t^{\nu} Y_t \int_0^1 P_{j,t}^{1-\nu} dj$$

Simplifying, we get:

$$P_t = \left(\int_0^1 P_{j,t}^{1-\nu} dj \right)^{\frac{1}{1-\nu}} \tag{29}$$

The intermediate goods firms produce output using capital and labor, according to a standard production technology:

$$Y_{j,t} = A_t K_{j,t}^{\alpha} N_{j,t}^{1-\alpha} \tag{30}$$

A_t is aggregate technology and is common across intermediate goods firms. It follows that aggregate capital and aggregate employment are just the sum of these factors across intermediate goods firms:

$$K_t = \int_0^1 K_{j,t} dj$$
$$N_t = \int_0^1 N_{j,t} dj$$

Assume that there are no debt instruments and that the intermediate goods firm rents capital from households. These firms all face the same factor prices (rental rate and wage rate). The firms do, however, have the ability to set their own price, given that they face downward sloping demand curves (as long as ν is not ∞). Hence, they want to solve the following constrained problem:

$$\max_{Y_{j,t}, P_{j,t}, K_{j,t}, N_{j,t}} \quad P_{j,t} Y_{j,t} - w_t N_{j,t} - R_t K_{j,t}$$

s.t.

$$Y_{j,t} = A_t K_{j,t}^{\alpha} N_{j,t}^{1-\alpha}$$
$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t} \right)^{-\nu} Y_t$$

Set the problem up using a Lagrangian, with two multipliers, $\lambda_{1,t}^j$ and $\lambda_{2,t}^j$.

$$\mathcal{L} = P_{j,t} Y_{j,t} - w_t N_{j,t} - R_t K_{j,t} + \lambda_{1,t}^j \left(A_t K_{j,t}^{\alpha} N_{j,t}^{1-\alpha} - Y_{j,t} \right) + \lambda_{2,t}^j \left(\left(\frac{P_{j,t}}{P_t} \right)^{-\nu} Y_t - Y_{j,t} \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial Y_{j,t}} = 0 \Leftrightarrow P_{j,t} = \lambda_{1,t}^j + \lambda_{2,t}^j \tag{31}$$
$$\frac{\partial \mathcal{L}}{\partial P_{j,t}} = 0 \Leftrightarrow Y_{j,t} = \nu \lambda_{2,t}^j P_{j,t}^{-\nu-1} P_t^{\nu} Y_t \tag{32}$$
$$\frac{\partial \mathcal{L}}{\partial K_{j,t}} = 0 \Leftrightarrow R_t = \lambda_{1,t}^j \alpha A_t K_{j,t}^{\alpha-1} N_{j,t}^{1-\alpha} \tag{33}$$
$$\frac{\partial \mathcal{L}}{\partial N_{j,t}} = 0 \Leftrightarrow w_t = \lambda_{1,t}^j (1-\alpha) A_t K_{j,t}^{\alpha} N_{j,t}^{\alpha} \tag{34}$$

The first order condition for the price can be simplified:

$$Y_{j,t} = \nu \lambda_{2,t}^{j} \left(\frac{P_{j,t}}{P_t}\right)^{-\nu} y_t P_{j,t}^{-1}$$
$$P_{j,t} = \nu \lambda_{2,t}^{j}$$
$$\lambda_{2,t}^{j} = \frac{P_{j,t}}{\nu}$$

Now plug this into the first order condition for output:

$$P_{j,t} = \lambda_{1,t}^{j} + \frac{P_{j,t}}{\nu}$$

Simplify:

$$P_{j,t} = \frac{\nu}{\nu - 1} \lambda_{1,t}^{j} \tag{35}$$

Now that $\frac{\nu}{\nu-1} \geq 1$. What is the interpretation of this statement? $\lambda_{1,t}^{j}$ can be interpreted as marginal cost. It is the shadow value on the first constraint: if I make you produce a little less, by how much do your profits go up (equivalently how much do your costs go down). Hence, this expression says that the optimal pricing rule is to set price equal to a "markup" of price over marginal cost, with the markup defined as $\varphi = \frac{\nu}{\nu-1}$. The less substitutable the intermediate goods are (i.e. the smaller is ν) the bigger the markup will be.

Plug this into the first order conditions for capital and labor; this will allow these conditions to be written in terms of the real product wage and the real product rental rate (the "product" qualifier means that we divide the nominal factor price by the price of the product, not the price level of all goods . . . this is the real factor price relevant for firm decision making):

$$\frac{w_t}{P_{j,t}} = \frac{\nu - 1}{\nu} \alpha A_t K_{j,t}^{\alpha} N_{j,t}^{-\alpha} \tag{36}$$
$$\frac{R_t}{P_{j,t}} = \frac{\nu - 1}{\nu} (1 - \alpha) A_t K_{j,t}^{\alpha - 1} N_{j,t}^{1 - \alpha} \tag{37}$$

Because $\frac{\nu-1}{\nu} \leq 1$, factors will be paid less than their marginal products; this gives rise to economic profits for the intermediate goods firms.

Now use the first order conditions for labor and capital to eliminate $\lambda_{1,t}^{j}$:

$$\lambda_{1,t}^{j} = \frac{R_t}{\alpha A_t K_{j,t}^{\alpha-1} N_{j,t}^{1-\alpha}}$$

$$w_t = \frac{R_t}{\alpha A_t K_{j,t}^{\alpha-1} N_{j,t}^{1-\alpha}} (1-\alpha) A_t K_{j,t}^{\alpha} N_{j,t}^{-\alpha}$$

$$w_t = R_t \frac{1-\alpha}{\alpha} \frac{K_{j,t}}{N_{j,t}}$$

$$\frac{K_{j,t}}{N_{j,t}} = \frac{\alpha}{1-\alpha} \frac{w_t}{R_t}$$

This last condition is important. It says that all forms will hire capital and labor in the same ratio, since the wage, the rental rate, and α are common to all firms. Use this fact to go back to the expression for $\lambda_{1,t}^{j}$, which again has the interpretation as marginal cost:

$$\lambda_{1,t}^{j} = \frac{R_t}{\alpha A_t \left(\frac{K_{j,t}}{N_{j,t}}\right)^{\alpha-1}}$$

Since all firms will hire capital and labor in the same ratio, this means that they all have the same marginal cost. But going back to the pricing rule, if they all have the same marginal cost, then they all will charge the same price. Then using the formula for the aggregate price level, we see:

$$P_{j,t} = P_t \quad \forall \ j \tag{38}$$

In other words, all firms charge the same price, which is equal to the final goods price. From the demand specification, if all firms charge the same price, they must produce the same amount of output:

$$Y_{j,t} = Y_t \quad \forall \ j \tag{39}$$

This may seem a little odd, but this is the advantage of defining firms as existing over the unit interval – the output of any one firm is equal to the aggregate output which is equal to average output. The individual production function is:

$$Y_{j,t} = A_t \left(\frac{K_{j,t}}{N_{j,t}}\right)^{\alpha} N_{j,t}$$

Since all firms hire capital and labor in the same ratio, and also produce the same amount of output, we can see that they must all hire the same amount of labor, and hence the same amount of capital:

$$K_{j,t} = K_t \quad \forall \ j$$
$$N_{j,t} = N_t \quad \forall \ j$$

This means that we can think of there being an aggregate production function (for the final good) that is identical to the production function of any intermediate good firm:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{40}$$

Because all firms charge the same price, the relative price of all goods comes out to be 1 in equilibrium. The level of prices is indeterminate without specifying some process for money (i.e. we could easily do that). Hence, we can normalize all prices to be one; this means that there is no difference between real and nominal factor prices. The factor demand equations become:

$$w_t = \frac{\nu - 1}{\nu}(1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{41}$$
$$R_t = \frac{\nu - 1}{\nu}\alpha a_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{42}$$

The household side of the model is the same as in our benchmark case, and has the same first order conditions. The entire set of first order conditions characterizing the equilibrium of this model are:

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (R_{t+1} + (1-\delta) \right) \tag{43}$$
$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (1 + r_{t+1}) \right) \tag{44}$$
$$\frac{\theta}{1 - N_t} = \frac{1}{C_t} w_t \tag{45}$$
$$w_t = \frac{\nu - 1}{\nu}(1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{46}$$
$$R_t = \frac{\nu - 1}{\nu}\alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{47}$$
$$Y_t = C_t + I_t \tag{48}$$
$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{49}$$
$$K_{t+1} = I_t + (1-\delta) K_t \tag{50}$$
$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{51}$$

We can see that these are exactly the same first order conditions which obtain in the basic RBC model, with the exception of the inverse of the price markup in the factor demand equations. If we assume that ν is constant, then the only thing that will be different about this model is the steady state – in particular, $\nu < \infty$ will distort the steady state values. In a linearization of the model, the impulse responses will be identical. Essentially the imperfect competition is a steady state distortion; to a first order approximation it does not impact the dynamics of the model. The competitive equilibrium steady state allocations will be different than what a planner would choose – a welfare optimizing government would want to equate marginal products of capital and labor to their factor prices, and could eliminate the distortion with Pigouvian taxes.

We can, however, entertain fluctuations in ν. We can effectively think of these as being markup shocks. As above, define $\varphi = \frac{\nu}{\nu-1}$. Suppose that the log of this follows a stationary AR(1):

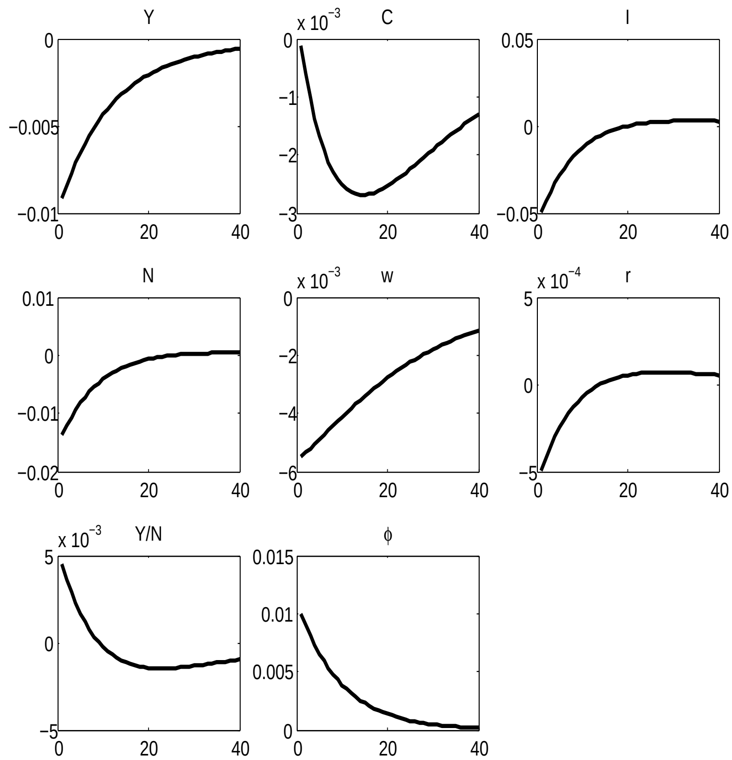
$$\ln \varphi_t = (1 - \rho_\varphi)\varphi^* + \rho_\varphi \ln \varphi_{t-1} + \varepsilon_{\varphi,t}$$

I can re-write the factor demand equations as:

$$w_t = \frac{1}{\varphi_t}(1 - \alpha)A_t K_t^\alpha N_t^{-\alpha} \tag{52}$$

$$R_t = \frac{1}{\varphi_t}\alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{53}$$

I'm just going to first create parameterization of the process for φ out of thin air: let's assume that $\varphi^* = 1.2$ (equivalently $\nu^* = 5$, that $\rho_\varphi = 0.9$, and that the standard deviation of the innovation is 0.01 (i.e. 1 percent). Below are impulse responses to a markup shock in the model with our benchmark parameterization (note that, to match the steady state hours of $\frac{1}{3}$, I would need to adjust θ to reflect φ^* . . . I don't do that here, as it doesn't affect the dynamics).



There are a couple of things evident from these responses. First, the markup shock causes consumption, hours, output, and investment to all decline together (i.e. positive co-movement). But second, average labor productivity goes up, which means that hours are falling by more than output. Adding markup shocks is thus going to do two things for us that the basic RBC model struggles with – it will reduce the cyclicality of labor productivity (which is way too high in the basic model relative to the data), and it will increase the volatility of hours relative to output (which is way too low in the model relative to the data). In many respects the markup shock is very similar (but not identical) to the intratemporal shock to labor supply or a shock to a tax rate on labor.

2.9 Money

We have abstracted from money thus far. Isn't economics all about money? We will give money a functional definition – it is anything which is used as a medium of exchange, serves as a unit of account, and serves as a store of value. The existence of money eliminates the problems presented by the double coincidence of wants presented by a system of exchange based on barter. In the model we have presented thus far the presence of money is somewhat trivial, since there is only one good. But in a multi-good world (i.e. reality) money is obviously important.

That being said, it turns out to be fairly difficult to get agents to hold money. Agents will not willingly hold money in equilibrium for its store of value function – agents can also "save" through capital or bonds, which pay interest. What differentiates money is that it does not pay interest. Hence, the "cards are stacked" against money. It *must* be the exchange motive that gets people to hold money.

There are three ways in which to get agents to hold money, and we will consider two of them. The one we will not consider is the "money search" literature. This is a super micro-founded literature that considers money as rising endogenously in a search-theoretic framework. It is beyond the scope of this course. We will consider two "shortcuts" – cash in advance and money in the utility function. Both get at the exchange role of money. Cash in advance assumes that cash is *required* to purchase goods. This can be thought of as a "technological" constraint and is a reduced-form way of getting at the exchange role of money. The other approach is "money in the utility function". In this case we assume that agents get utility from holding money. This is also a reduced form way of getting at the idea that holding money makes conducting exchange "easier". These approaches yield similar results but they are not exactly the same. We consider each in turn.

2.9.1 The Budget Constraint

Before proceeding we need to write out a budget constraint that includes money. This is because money is a store of value. Let M_{t-1} denote the nominal holdings brought into period t – this is predetermined. Let M_t denote new money holdings (determined at time t) that will be brought into $t+1$. Let p_t denote the nominal price of goods – this is the price of goods measured in units of money. Let i_{t-1} denote the nominal interest rate on nominal bonds, B_{t+1}, observed at time t that pays off in time $t+1$. It pays off in dollars.

The household earns real income on work ($w_t N_t$), real income from leasing capital ($R_t K_t$), and nominal interest earned on bonds brought into this period ($i_{t-1} B_t$). We can convert this income from holding bonds into real terms by dividing by the price level, P_t. With this real income the household can (i) consume, C_t; (ii) purchase more capital, K_{t+1}, (iii) buy more (real) bonds, B_{t+1}/P_t, or)iv) accumulate more money, $(M_t - M_{t-1})/P_t$. The household also pays a lump sum tax, T_t, which I return to below:

$$C_t + K_{t+1} - (1-\delta)K_t + \left(\frac{B_{t+1} - B_t}{P_t}\right) + \left(\frac{M_t - M_{t-1}}{P_t}\right) = w_t N_t + R_t K_t - T_t + \Pi_t + i_{t-1}\frac{B_t}{P_t} \quad (54)$$

In either of the following setups, the household can freely choose the real variables C_t and N_t. It can freely choose the bond and money holdings it carries over into the future, M_t and B_{t+1}. It takes all prices (w_t, R_t, i_t, and P_t as given). Firms are unaffected by money, since we can model their problem as completely static (they technically have the ability to operate in debt markets, but this ends up being indeterminate anyway so we can abstract from that part of the problem).

The firm problem is always standard:

$$\max_{K_t, N_t} \quad A_t F(K_t, N_t) - w_t N_t - R_t K_t$$

The first order conditions are:

$$w_t = A_t F_N(K_t, N_t)$$
$$R_t = A_t F_K(K_t, N_t)$$

There exists a central bank that sets the money supply in both set ups. Let's suppose that the exogenous process for the money supply follows an AR(1) in the growth rate (first difference of the log). This specification will generate positive trend inflation:

$$\ln M_t - \ln M_{t-1} = (1-\rho_m)\pi^* + \rho_m\left(\ln M_{t-1} - \ln M_{t-2}\right) + \varepsilon_{m,t} \tag{55}$$

Here π^* is the steady state growth rate of the money supply. This will end up being equal to steady state inflation in both models.

Because the government prints money, it effectively earns some revenue (which we call seignorage). We are going to assume that there is no government spending and that the government does not operate in the bond market (which will mean that in equilibrium $B_t = 0$); this is without loss of generality because Ricardian equivalence holds in this setup as long as taxes are lump sum. The budget constraint is:

$$0 \leq T_t + \frac{M_t - M_{t-1}}{P_t}$$

This says that government expenditure (which I assume is zero) cannot exceed revenue, which is lump sum taxes plug the change in the money supply divided by the price level. $\frac{M_t - M_{t-1}}{P_t}$ is "seignorage revenue" in the sense that the change in money that the government produces divided by the price level is essentially like a tax. If the government budget constraint holds with equality, this means that:

$$T_t = -\frac{M_t - M_{t-1}}{P_t}$$

When imposing equilibrium, plugging this into the household budget constraint causes the terms involving M_t and M_{t-1} to cancel out, so we get the standard resource constraint that $Y_t = C_t + I_t$.

2.9.2 Money in the Utility Function

In this specification households get utility from consumption, leisure, and holding real money balances – M_t/P_t. Note the timing convention here – M_t is how much money the household chooses to hold today to carry into tomorrow. The idea here is that the more money one has (relative to the price level), the "easier" conducting transactions is. As before, we will go ahead and make functional form assumptions that permit a quantitative solution of the model. The household problem is:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}, M_t} \quad E_0 \sum_{t=0}^{\infty} \left\{ \ln C_t + \theta \ln(1 - N_t) + \psi \frac{\left(\frac{M_t}{P_t}\right)^{1-\zeta} - 1}{1 - \zeta} \right\}$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + \left(\frac{B_{t+1} - B_t}{P_t}\right) + \left(\frac{M_t - M_{t-1}}{P_t}\right) = w_t N_t + R_t K_t - T_t + \Pi_t + i_{t-1}\frac{B_t}{P_t}$$

Form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \left\{ \ln C_t + \theta \ln(1 - N_t) + \psi \frac{\left(\frac{M_t}{P_t}\right)^{1-\zeta} - 1}{1 - \zeta} + \dots \right.$$

$$\left. \dots + \lambda_t \left(w_t N_t + R_t K_t - T_t + \Pi_t + (1 + i_t)\frac{B_t}{P_t} - C_t - K_{t+1} + (1-\delta)K_t - \frac{M_t}{P_t} + \frac{M_{t-1}}{P_t} - \frac{B_{t+1}}{P_t} \right) \right\}$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t \tag{56}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow \frac{\theta}{1 - N_t} = \lambda_t w_t \tag{57}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \left(\lambda_{t+1} (R_{t+1} + (1 - \delta)) \right) \tag{58}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \left(\lambda_{t+1} (1 + i_t) \frac{P_t}{P_{t+1}} \right) \tag{59}$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = 0 \Leftrightarrow \psi \left(\frac{M_t}{P_t}\right)^{-\zeta} \frac{1}{P_t} = \frac{\lambda_t}{P_t} - \beta E_t \frac{\lambda_{t+1}}{P_{t+1}} \tag{60}$$

The first four equations can be re-arranged to yield the *exactly* the same first order conditions which obtain in the standard RBC model:

$$\frac{\theta}{1 - N_t} = \frac{1}{C_t} w_t \tag{61}$$

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (R_{t+1} + (1 - \delta)) \right) \tag{62}$$

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \left((1 + i_t) \frac{P_t}{P_{t+1}} \right) \right) \tag{63}$$

This is identical to the previous setup because of the Fisher relationship, which says that $1 + r_t = (1 + i_t)\frac{P_t}{P_{t+1}}$. We can greatly simplify the first order condition for holdings of money by using the first order condition for bonds:

$$\left(\frac{M_t}{P_t}\right)^{-\zeta} = \lambda_t - \beta E_t \lambda_{t+1} \frac{P_t}{P_{t+1}}$$
$$\beta E_t \lambda_{t+1} \frac{P_t}{P_{t+1}} = \frac{\lambda_t}{1+i_t}$$
$$\Rightarrow$$
$$\left(\frac{M_t}{P_t}\right)^{-\zeta} = \lambda_t \left(1 - \frac{1}{1+i_t}\right)$$

Define $m_t = \frac{M_t}{P_t}$ as "real money balances". This essentially says how much current consumption you are giving up by carrying money from today into tomorrow. Simplifying the above we have:

$$\psi m_t^{-\zeta} = \frac{1}{C_t}\left(\frac{i_t}{1+i_t}\right)$$

This can be simplified further to yield:

$$m_t = \psi^{\zeta} C_t^{\zeta} \left(\frac{1+i_t}{i_t}\right)^{\zeta} \tag{64}$$

This is quite intuitive. It says that the demand for money (i) increases one for one in the price level; (ii) is increasing in consumption; and (iii) is decreasing in the nominal interest rate. The nominal interest rate is the opportunity cost of holding money – if you didn't save in money, you could have saved in bonds, earning nominal interest. The fact that it is increasing in consumption essentially just says that money is a "normal" good in this setup – the wealthier you are, the more consumption you want and the more money you want to hold. The fact that the demand for money increases one for one with the price level gets at the idea that you get utility from real money balances, so an increase in the price level (which affects nothing else in the model) leads one to desire to hold more money.

To close the model out we need to deal with the non-stationarity inherent in the assumed process for the money supply. In particular, we want to write it in terms of real balances (which will be stationary). Hence, we need to play around with the exogenous process for money by adding and subtracting logs of the price level at various leads and lags.

$$\ln M_t - \ln M_{t-1} = (1-\rho_m)\pi^* + \rho_m \left(\ln M_{t-1} - \ln M_{t-2}\right) + \varepsilon_{m,t}$$
$$\ln M_t - \ln P_t + \ln P_t - \ln P_{t-1} - \ln M_{t-1} + \ln P_{t-1} = (1-\rho_m)\pi^* \dots$$
$$\dots + \rho_m \left(\ln M_{t-1} - \ln P_{t-1} + \ln P_{t-1} - \ln P_{t-2} - \ln M_{t-2} + \ln P_{t-2}\right) + \varepsilon_{m,t}$$

We have $\ln m_t = \ln M_t - \ln P_t$, and define $\pi_t = \ln P_t - \ln P_{t-1}$. We can write this as:

$$\Delta \ln m_t + \pi_t = (1-\rho_m)\pi^* + \rho_m \pi_{t-1} + \rho_m \Delta \ln m_{t-1} + \varepsilon_{m,t} \tag{65}$$

The full set of conditions characterizing the model's equilibrium are then:

$$\frac{\theta}{1-N_t} = \frac{1}{C_t} w_t \tag{66}$$

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (R_{t+1} + (1-\delta)) \right) \tag{67}$$

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (1 + r_t) \right) \tag{68}$$

$$R_t = \alpha A_t K_t^{\alpha-1} N_t^{1-\alpha} \tag{69}$$

$$w_t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{70}$$

$$Y_t = A_t K_t^{\alpha} N_t^{\alpha} \tag{71}$$

$$K_{t+1} = I_t + (1-\delta) K_t \tag{72}$$

$$Y_t = C_t + I_t \tag{73}$$

$$\Delta \ln m_t = (1-\rho_m)\pi^* - \pi_t + \rho_m \pi_{t-1} + \rho_m \Delta \ln m_{t-1} + \varepsilon_{m,t} \tag{74}$$

$$m_t = \psi^{\zeta} C_t^{\zeta} \left(\frac{1+i_t}{i_t} \right)^{\zeta} \tag{75}$$

$$1 + r_t = (1+i_t) E_t (1+\pi_{t+1})^{-1} \tag{76}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{77}$$

$$\Delta \ln m_t = \ln m_t - \ln m_{t-1} \tag{78}$$

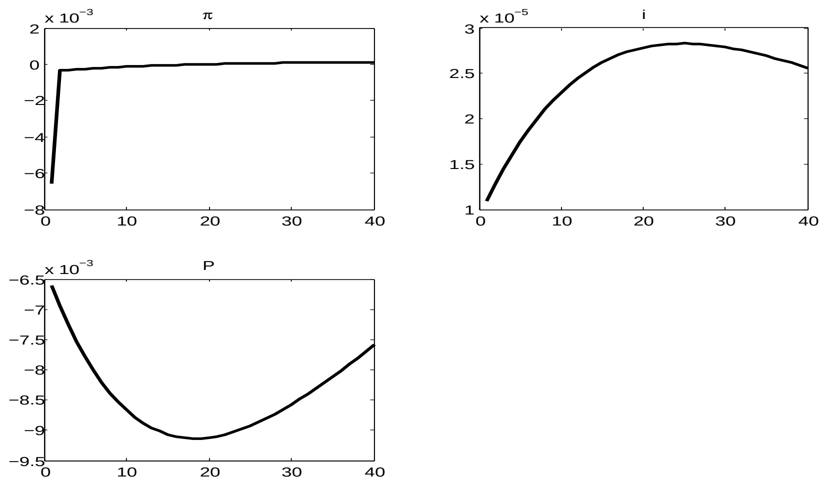
This is 13 equations in 13 variables – N_t, C_t, R_t, r_t, w_t, K_t, A_t, Y_t, I_t, $\Delta \ln m_t$, m_t, i_t, and π_t. The equations determining the real variables of the model are *exactly* the same as in the basic RBC model – you can determine N_t, C_t, w_t, R_t, r_t, w_t, K_t, Y_t, and A_t independently of m_t, π_t, or i_t. Intuitively, this means that the response of the real variables to a technology shock will be identical in this set up to earlier, and real variables will not respond to monetary shocks. Put differently, money is completely neutral with respect to real variables, and the classical dichotomy holds – real variables are determined first and then nominal variables are determined.

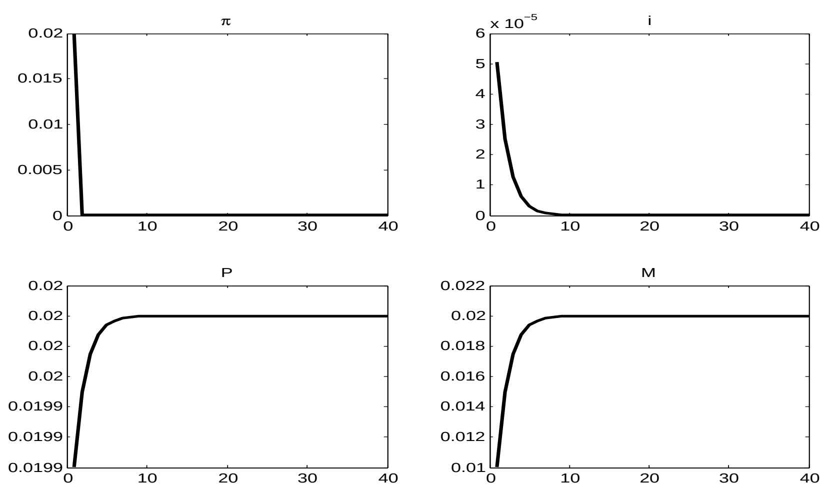
I parameterize the model exactly the same as before in the basic RBC notes. There are a few new parameters to be set, however. I set $\rho_m = 0.5$, and the standard deviation of the monetary policy shock to 0.01 (i.e. 1 percent). I set $\pi^* = 0.00$, so that there is no inflation in the steady state. I set $\psi = 1$ and $\zeta = 2$. The steady state growth rate of money, equal to the steady state inflation rate, does not have interesting short run dynamic effects here

Below I show impulse response functions (just of the nominal variables, since the responses of the real variables to a technology shock are the same as in the baseline RBC model and their response to the monetary shock is zero) to both shocks. I construct the responses of the price level and the level of the nominal money supply using the facts that:

$$\ln p_t = \pi_t + \ln p_{t-1}$$
$$\ln M_t = \ln m_t + \ln p_t$$

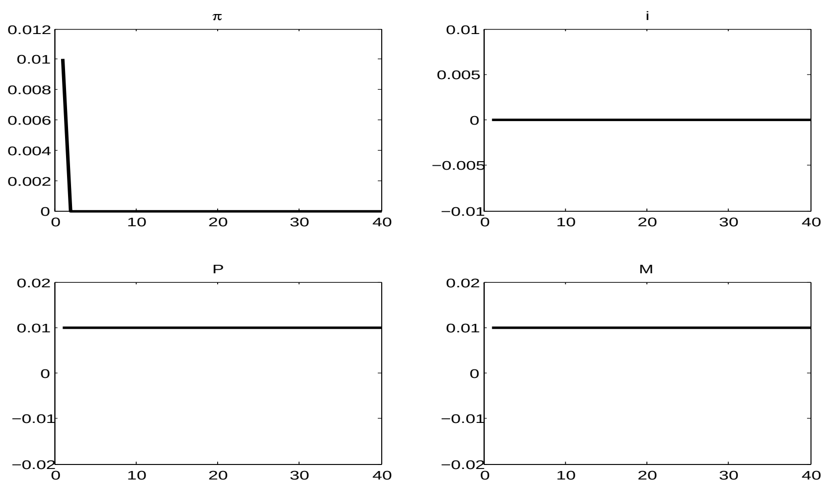
Since the model is linearized, impulse responses don't depend on initial conditions, so I can normalize $\ln p_{t-1} = 0$ in constructing those responses.





These have features we would more or less expect – inflation (and hence the price level) fall in response to a technology shock and rise in response to a monetary shock. The nominal interest rate rises when the money supply increases at an unexpectedly fast rate; the nominal interest rate also rises after the technology shock. Further, we can immediately tell that the price level will be countercyclical – this is because technology shocks (which raise output) lower inflation and the price level, inducing a negative correlation. That correlation is consistent with the data.

It's an interesting exercise to see what happens if I make $\rho_m = 0$, which means that the money supply follows a random walk (with drift, if $\pi^* \neq 0$). In response to a monetary shock, the only effect is for the price level to immediately jump up by the amount of the change in M – there is no change in the nominal interest rate. The logic for this is as follows. Since nothing real changes, the real interest rate, r_t, will not respond to the monetary shock. But since there is no persistence to the shock, the price level will just jump up by the increase in M_t, and since there is no further change in M_t in period $t+1$ or beyond, nothing more will happen to the price level after period t. This means that expected inflation between t and $t+1$ will not react – e.g. π_{t+1} will not change. But if π_{t+1} doesn't change, and r_t doesn't change, then from the Fisher relationship i_t doesn't change either. In response to a monetary shock, i_t will simply move with $E_t\pi_{t+1}$ in such a way as to keep r_t unchanged. We can see this in the impulse responses shown below:



2.9.3 Cash in Advance

Now we undertake another assumption that allows us to get money into our basic RBC model. It ends up having fairly similar implications to the money in the utility model specification but it is not identical. In particular, in this framework money is not completely neutral and the classical dichotomy does not hold.

The cash in advance constraint says that one must have enough money on hand to finance all nominal purchases of consumption goods. In particular:

$$M_{t-1} \geq P_t C_t \tag{79}$$

Otherwise the problem is the standard real business cycle model, modified to have money entering the budget constraint as a store of value. We can write out the household problem as:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}, M_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \{\ln C_t + \theta \ln(1 - N_t)\}$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + \left(\frac{B_{t+1} - B_t}{P_t}\right) + \left(\frac{M_t - M_{t-1}}{P_t}\right) = w_t N_t + R_t K_t - T_t + \Pi_t + i_{t-1}\frac{B_t}{P_t}$$
$$M_{t-1} \geq p_t C_t$$

We can form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln C_t + \theta \ln(1 - N_t) + \mu_t \left(\frac{M_{t-1}}{P_t} - C_t \right) \dots \right.$$
$$\dots + \lambda_t \left(w_t N_t + R_t K_t - T_t + \Pi_t + (1 + i_{t-1})\frac{B_t}{P_t} - C_t - K_{t+1} + (1-\delta)K_t - \frac{B_{t+1}}{P_t} - \frac{M_t}{P_t} + \frac{M_{t-1}}{P_t} \right)$$

This is a similar setup to before, except now there is no money in the utility function and there is an extra constraint, with Lagrange multiplier given by μ_t. The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow \frac{1}{C_t} = \lambda_t + \mu_t \tag{80}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow \frac{\theta}{1 - N_t} = \lambda_t w_t \tag{81}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \left(\lambda_{t+1}(R_{t+1} + (1-\delta)) \right) \tag{82}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} \left((1 + i_t) \left(\frac{P_t}{P_{t+1}} \right) \right) \tag{83}$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = 0 \Leftrightarrow -\frac{\lambda_t}{P_t} + \beta E_t \frac{\mu_{t+1}}{P_{t+1}} + \beta E_t \frac{\lambda_{t+1}}{p_{t+1}} = 0 \tag{84}$$

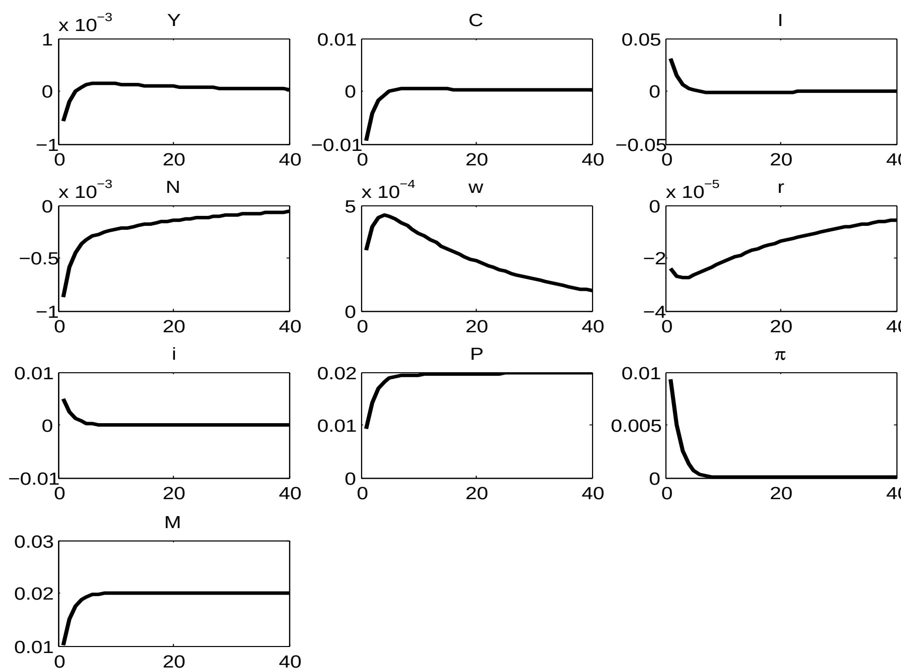
The final first order condition can be simplified to yield:

$$\lambda_t = \beta E_t \left(\mu_{t+1} \frac{P_t}{P_{t+1}} + \lambda_{t+1} \frac{P_t}{P_{t+1}} \right) \tag{85}$$

The first four of these first order conditions are identical to the money in the utility function setup. Take a look at the last one. Suppose that one had enough money (i.e. M_t was sufficiently big relative to consumption) that the cash in advance constraint was never binding. This would mean that $E_t \mu_{t+1} = 0$ for all time. Plugging this in to the last first order condition and comparing with the previous first order condition, we see that the only way the two could simultaneously hold is if $i_t = 0$. This makes sense – as long as bonds pay non-zero interest, one would *never* want to hold money. Hence, we would be at a corner solution. In the presence of the cash in advance constraint, however, this will not be true.

I solve the model in Dynare assuming that the cash in advance constraint always binds. The firm problem, money growth rule, and stochastic process for technology are identical to above. It is helpful to *not* eliminate the Lagrange multipliers when solving this problem – that's fine, as it just introduces more variables. I also have to solve the model using inflation and real money balances, as the nominal money supply and price level are non-stationary.

The impulse responses of real variables to a technology shock are *exactly* the same in the cash in advance model as in the money in the utility function model, which are, in turn, exactly the same as in the basic RBC model. It is in this sense that abstracting from money altogether in that model is fine. It turns out here, however, that money does have real effects, although these are small. The impulse responses of the real variables to a monetary policy shock are shown below:



Here we see something perhaps not very intuitive. Not only does the monetary shock have real effects, it actually causes an output contraction (albeit it is very small). What is the intuition for this? Inflation is essentially a tax on holders on money. In the absence of the technological constraint requiring them to hold money (the cash in advance constraint), people thus wouldn't hold it at all. But given that they do have to hold it, an increase in the rate of growth of the money supply – which causes more inflation – makes people want to "get out of" money because it's essentially a tax on money. Since consumption requires money, they can't substitute from money to consumption, so they substitute from money to leisure. Hence, there is a reduction in labor supply and a reduction in consumption, which leads to an output decline, real wage increase, and investment increase.

That being said, the real effects of money in this model are pretty small (in comparison with the responses to a productivity shock) and are not very persistent. In particular, for the parameterization I used, money explains less than 1 percent of the variance of output, about 6 percent of the variance of investment, about 1 percent of the variance of horus, and about 4 percent of the variance of consumption. Hence, money is *approximately* neutral in this model. To get large monetary non-neutrality, one needs to introduce other kinds of frictions (like price stickiness).

2.9.4 Optimal Long Run Inflation

I've thus far ignored the optimal trend growth rate of the money supply (which is in turn equal to the long run steady state growth rate of prices), simply taking it is given. What would be the optimal long run inflation rate in either of these models?

The so-called Friedman rule is to set the nominal interest rate on bonds equal to zero $i_t = 0$. It turns out that this is optimal (from the perspective of steady state welfare) in both the cash in advance and money in the utility function models. Friedman's original intuition was straightforward. Money is a "good" thing in the sense of reducing transactions frictions and therefore increases welfare. It is (essentially) costless to produce. The nominal interest rate being positive imposes a tax on the holders of money, which distorts welfare. Put differently, the private marginal cost of holding money is the nominal interest rate, while the public marginal cost of producing money is (essentially) zero. To bring about efficiency we need to bring these into equality by reducing the distortion.

At a more formal level, we can see why this is optimal in both of these specifications. It is perhaps easiest to see in the CIA model. What you would like is for the cash in advance constraint to not bind – if the constraint doesn't bing, agents have to be weakly better off than if it does bind. The constraint not binding would mean that $\mu_t = 0$. The only way for the first order conditions to all hold with $\mu_t = 0$ is if $i_t = 0$. Hence, setting $i = 0$ in the long run is optimal, as Friedman conjectured. From the Fisher relationship, since $r = \frac{1}{\beta} - 1$, $i = 0$ requires that $1 + \pi = (1/\beta - 1)^{-1}$, which means $\pi < 0$. Hence, you want deflation in the steady state.