Graduate Macro Theory II:
The Real Business Cycle Model

Eric Sims
University of Notre Dame

Spring 2017

1 Introduction

This note describes the canonical real business cycle model. A couple of classic references here are Kydland and Prescott (1982), King, Plosser, and Rebelo (1988), and King and Rebelo (2000). The model is essentially just the neoclassical growth model augmented to have variable labor supply.

2 The Decentralized Model

I will set the problem up as a decentralized model, studying first the behavior of households and then the behavior of firms.

There are two primary ways of setting the model up, which both yield identical solutions. In both households own the firms, but management and ownership are distinct, and so households behave as though firm profits are given. In one formulation firms own the capital stock, and issue both debt and equity to households. In another formulation, households own the capital stock and rent it to firms; firms still issue debt and equity to households. We will go through both formulations. Intuitively these set-ups have to be the same because, either way, households own the capital stock (either directly or indirectly).

In both setups I abstract from trend growth, which, as we have seen, does not really make much of a difference anyway.

2.1 Firms Own the Capital Stock

Here we assume that firms own the capital stock. We begin with the household problem.

2.1.1 Household Problem

There is a representative household. It discounts the future by $\beta < 1$. It supplies labor (measured in hours), N_t, and consumes, C_t. It gets utility from consumption and leisure; with the time endowment normalized to unity, leisure is $1 - N_t$. It earns a wage rate, w_t, which it takes as given. It holds bonds, B_t, which pay interest rate r_{t-1}. r_{t-1} is the interest rate known at $t-1$ which pays out in t; r_t is the interest rate known in t which pays out in $t+1$. $B_t > 0$ means that the household has a positive stock of savings; $B_t < 0$ means the household has a stock of debt. Note that "savings" is a stock; "saving" is a flow. The household takes the interest rate as given. Its budget constraint says that each period, total expenditure cannot exceed total income. It earns wage income, $w_t N_t$, profit distributions in the form of dividends, Π_t, and interest income on existing bond holds, $r_{t-1}B_t$ (note this can be negative, so that there is an interest cost of servicing debt). Household expenditure is composed of consumption, C_t and saving, $B_{t+1} - B_t$ (i.e. the accumulation of new savings). Hence we can write the constraint:

$$C_t + (B_{t+1} - B_t) \leq w_t N_t + \Pi_t + r_{t-1} B_t \tag{1}$$

Note a timing convention – r_{t-1} is the interest you have to pay today on existing debt. r_t is what you will have to pay tomorrow, but you choose how much debt to take into tomorrow today. Hence, we assume that the household observes r_t in time t. Hence we can treat r_t as known from the perspective of time t. The household chooses consumption, work effort, and the new stock of savings each period to maximize the present discounted value of flow utility:

$$\max_{C_t, N_t, B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t)\right)$$

s.t.

$$C_t + B_{t+1} \leq w_t N_t + \Pi_t + (1 + r_{t-1}) B_t$$

We can form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) + \lambda_t (w_t N_t + \Pi_t + (1 + r_{t-1}) B_t - C_t - B_{t+1})\right)$$

The first order conditions characterizing an interior solution are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow u'(C_t) = \lambda_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow v'(1 - N_t) = \lambda_t w_t \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1} (1 + r_t) \tag{4}$$

These can be combined together to yield:

$$u'(C_t) = \beta E_t \left(u'(C_{t+1})(1 + r_t) \right) \tag{5}$$

$$v'(1 - N_t) = u'(C_t) w_t \tag{6}$$

(5) and (6) have very intuitive, intermediate micro type interpretations. (5) says to equate the marginal rate of substitution between consumption today and tomorrow (i.e. $\frac{u'(c_t)}{\beta E_t u'(c_{t+1})}$) to the relative price of consumption today (i.e. $1+r_t$). (6) says to equate the marginal rate of substitution between leisure and consumption (i.e. $\frac{v'(1-n_t)}{u'(c_t)}$) to the relative price of leisure (i.e. w_t).

In addition, there is the transversality condition:

$$\lim_{t \to \infty} \quad \beta^t B_{t+1} u'(C_t) = 0 \tag{7}$$

2.1.2 The Firm Problem

There is a representative firm. The firm wants to maximize the present discounted value of (real) net revenues (i.e. cash flows). It discounts future cash flows by the stochastic discount factor. The way I'll write the stochastic discount factor puts cash flows (measured in goods) in terms of *current* consumption (we take the current period to be $t = 0$). Define the stochastic discount factor as:

$$M_t = \beta^t \frac{E_0 u'(C_t)}{u'(C_0)}$$

The firm discounts by this because this is how consumers value future dividend flows. One unit of dividends returned to the household at time t generates $u'(C_t)$ additional units of utility, which must be discounted back to the present period (which we take to be 0), by β^t. Dividing by $u'(C_0)$ gives the period 0 consumption equivalent value of the future utils. The firm produces output, Y_t, according to a constant returns to scale production function, $Y_t = A_t F(K_t, N_t)$, with the usual properties. It hires labor, purchases new capital goods, and issues debt. I denote its debt as D_t, and it pays interest on its debt, r_{t-1}. Its revenue each period is equal to output. Its costs each period are the wage bill, investment in new physical capital, and servicing costs on its debt. It can raise its cash flow by issuing new debt (i.e. $D_{t+1} - D_t$ raises cash flow). Its problem can be written as:

$$\max_{N_t, I_t, D_{t+1}, K_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(A_t F(K_t, N_t) - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t \right)$$

s.t.

$$K_{t+1} = I_t + (1 - \delta) K_t$$

We can re-write the problem by imposing that the constraint hold each period:

$$\max_{N_t,K_{t+1},D_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(A_t F(K_t, N_t) - w_t N_t - K_{t+1} + (1-\delta)K_t + D_{t+1} - (1+r_{t-1})D_t\right)$$

The first order conditions are as follows:

$$\frac{\partial V_0}{\partial N_t} = 0 \Leftrightarrow A_t F_N(K_t, N_t) = w_t \tag{8}$$

$$\frac{\partial V_0}{\partial K_{t+1}} = 0 \Leftrightarrow u'(C_t) = \beta E_t(u'(C_{t+1})((A_{t+1}F_K(K_{t+1}, N_{t+1}) + (1-\delta)) \tag{9}$$

$$\frac{\partial V_0}{\partial D_{t+1}} = 0 \Leftrightarrow u'(C_t) = \beta E_t(u'(C_{t+1})(1+r_t)) \tag{10}$$

(9) and (10) follow from the fact that $M_t = \beta^t u'(C_t)/u'(C_0)$ and $E_t M_{t+1} = \beta^{t+1} E_t u'(C_{t+1})/u'(C_0)$. Note that (10) is the same as (5), and therefore must hold in equilibrium as long as the household is optimizing. This means that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of D_{t+1}. This is essentially the Modigliani-Miller theorem – it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

2.1.3 Closing the Model

To close the model we need to specify a stochastic process for the exogenous variable(s). The only exogenous variable in the model is a_t. We assume that it is well-characterized as following a mean zero AR(1) in the log (we have abstracted from trend growth):

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{11}$$

2.1.4 Equilibrium

A competitive equilibrium is a set of prices (r_t, w_t) and allocations $(C_t, N_t, K_{t+1}, D_{t+1}, B_{t+1})$ taking K_t, D_t, B_t, A_t and the stochastic process for A_t as given; the optimality conditions (5) - (7), (8)-(10), and the transversality condition holding; the labor and bonds market clearing ($N_t^d = N_t^s$ and $B_t = D_t$ in all periods, so $B_{t+1} = D_{t+1}$ as well); and both budget constraints holding with equality.

Let's consolidate the household and firm budget constraints:

$$C_t + (B_{t+1} - B_t) = w_t N_t + r_{t-1}B_t + A_t F(K_t, N_t) - w_t N_t - I_t + D_{t+1} - (1+r_{t-1})D_t \tag{12}$$

$$\Rightarrow \tag{13}$$

$$A_t F(K_t, N_t) = C_t + I_t \tag{14}$$

In other words, bond market-clearing plus both budget constraints holding just gives the standard accounting identity that output must be consumed or invested.

If you combine the household's first order condition for labor supply with the firm's condition, you get:

$$v'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t)$$

The first order condition for bonds/debt along with the first order condition for the firm's choice of its next period's capital stock imply that:

$$E_t u'(C_{t+1}) \left(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \right) = E_t u'(C_{t+1})(1 + r_t)$$

This can be written:

$$1 + r_t = \frac{E_t u'(C_{t+1}) \left(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \right)}{E_t u'(C_{t+1})}$$

I can perform this operation because r_t is known at t, and hence can be taken out of the expectations operator. One would be tempted to cancel the $E_t u'(C_{t+1})$ terms on the right hand side, but note that you cannot do that because you cannot distribute the E_t in the numerator. Suffice it to say r_t is closely tied to the marginal product of capital – this makes sense in that bonds and capital are substitutable savings vehicles.

2.2 Households Own the Capital Stock

Now we consider a version of the decentralized problem in which the households own the capital stock and rent it to firms. Otherwise the structure of the problem is the same.

2.2.1 Household Problem

As before, the household consumes and supplies labor. Now it also owns the capital stock. It earns a rental rate for renting out the capital stock to firms each period, R_t. The household budget constraint is:

$$C_t + K_{t+1} - (1 - \delta)K_t + B_{t+1} - B_t = w_t N_t + R_t K_t + r_{t-1} B_t + \Pi_t \tag{15}$$

The household has income comprised of labor income, capital income, interest income, and profits (again it takes profits as given). It can consume this, accumulate more capital (this is the $K_{t+1} - (1 - \delta)K_t$ term), or accumulate more saving. Its problem is:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) \right)$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t = w_t N_t + R_t K_t + r_{t-1}B_t + \Pi_t$$

Form a current value Lagrangian:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1-N_t) + \lambda_t(w_t N_t + R_t K_t + (1+r_{t-1})B_t + \Pi_t - C_t - K_{t+1} + (1-\delta)K_t - B_{t+1})\right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow u'(C_t) = \lambda_t \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \Leftrightarrow v'(1-N_t) = \lambda_t w_t \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(R_{t+1} + (1-\delta)) \tag{18}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow \lambda_t = \beta E_t \lambda_{t+1}(1+r_t) \tag{19}$$

These first order conditions can be combined to yield:

$$v'(1-N_t) = u'(C_t)w_t \tag{20}$$

$$u'(C_t) = \beta E_t u'(C_{t+1})(R_{t+1} + (1-\delta)) \tag{21}$$

$$u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t) \tag{22}$$

Note that (20) and (21) are the same as (5) and (9). (22) is the same as (6). One is tempted to claim that $r_t + \delta = R_{t+1}$ given that (19) and (20) must both hold. This is not quite right. r_t is known at time t; R_{t+1} is not. Hence one can take $1 + r_t$ outside of the expectations operator in (22) to get:

$$1 + r_t = \frac{u'(C_t)}{\beta E_t u'(C_{t+1})}$$

But once cannot do the same for (21). Intuitively, $E_t(u'(C_{t+1})(R_{t+1}+(1-\delta)) = E_t(u'(C_{t+1})E_t(R_{t+1}+(1-\delta))+cov(u'(C_{t+1}), R_{t+1})$. In general, that covariance term is not going to be zero. It is likely to be negative – as we will see, $E_t R_{t+1}$ is the expected marginal product of capital. When the marginal product of capital is high (so R_{t+1} is high), then consumption is likely to be high (because MPK being high probably means that productivity is high), which means that marginal utility of consumption is low. To be compensated for holding an asset whose return covaries negatively with consumption, the household would demand a premium over the safe, riskless return r_t.

In a linearization of the model, that covariance term would drop out, and we could say that

$r_t = E_t R_{t+1} - \delta$, but in general there is another term that is essentially the equity premium.

2.2.2 The Firm Problem

The firm problem is similar to before, but now it doesn't choose investment. Rather, it chooses capital *today* given the rental rate, R_t. Note that the firm can vary capital today even though the household cannot given that capital is predetermined. The labor choice and debt choice are similar. In fact, because the amount of the debt is going to end up being indeterminate, it is common to just assume that firms don't issue/hold debt and just solve a static problem. Again, the firm wants to maximize the present discounted value of cash flows.

$$\max_{n_t, k_t, d_{t+1}} \quad V_0 = E_0 \sum_{t=0}^{\infty} M_t \left(A_t F(K_t, N_t) - w_t N_t - R_t K_t + D_{t+1} - (1 + r_{t-1}) D_t \right)$$

The first order conditions are:

$$\frac{\partial V_0}{\partial N_t} = 0 \Leftrightarrow A_t F_K(K_t, N_t) = w_t \tag{23}$$

$$\frac{\partial V_0}{\partial K_t} = 0 \Leftrightarrow A_t F_K(K_t, N_t) = R_t \tag{24}$$

$$\frac{\partial V_0}{\partial D_{t+1}} = 0 \Leftrightarrow u'(C_t) = \beta E_t u'(C_{t+1})(1 + r_t) \tag{25}$$

(25) follows from the definition of the stochastic discount factor, and again is automatically satisfied; so again the amount of debt is indeterminate.

2.2.3 Equivalence to the Other Setup

Plug (24) into (21) and you get:

$$u'(C_t) = \beta E_t (u'(C_{t+1})(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta)) \tag{26}$$

This is identical to (9). Also, (20) is equivalent to (6); (5) is equivalent to (22); and (23) is the same as (8). Hence, all the first order conditions are the same. The definition of equilibrium is the same. Both the firm and household budget constraints holding again give rise to the accounting identity (14). Hence, these setups give rise to identical solutions. It simply does not matter whether households own the capital stock and lease it to firms or whether firms own the capital stock. Since households own firms, these are equivalent ownership structures.

3 Equilibrium Analysis of the Decentralized Model

We can combined first order conditions from the firm and household problems (in either setup) to yield the equilibrium conditions:

$$u'(C_t) = \beta E_t \left(u'(C_{t+1})(A_{t+1}F_K(k_{t+1}, N_{t+1}) + (1-\delta)\right) \tag{27}$$

$$v'(1-N_t) = u'(C_t)A_tF_N(K_t, N_t) \tag{28}$$

$$K_{t+1} = A_t f(K_t, N_t) - C_t + (1-\delta)K_t \tag{29}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t \tag{30}$$

$$Y_t = A_t f(K_t, N_t) \tag{31}$$

$$Y_t = C_t + I_t \tag{32}$$

$$u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t) \tag{33}$$

$$w_t = A_t F_N(K_t, N_t) \tag{34}$$

$$R_t = A_t F_K(K_t, N_t) \tag{35}$$

(27) can essentially be interpreted as an investment-saving equilibrium. (28) characterizes equilibrium in the labor market, since the wage is equal to the marginal product of labor. (29) is just the capital accumulation equation, and (31) is the exogenous process for technology. (31) defines output and (32) defines investment. (33)-(35) just give us back the equilibrium factor prices. We have 1 truly forward-looking variable (consumption); two state/exogenous variables (capital and productivity); and six static/redundant variables (hours, output, investment, the real interest rate, the real wage, and the real rental rate). That's a total of nine variables and we have nine equations.

We need to specify functional forms. For simplicity, assume that $u(C_t) = \ln C_t$ and $v(1-N_t) = \theta \ln(1-N_t)$. Assume that the production function is Cobb-Douglas: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$.

Given these parameter values we can analyze the steady state. The steady state is a situation in which $A^* = 1$ (its non-stochastic unconditional mean), $K_{t+1} = K_t = K^*$, and $C_{t+1} = C_t = C^*$. Given the steady state values of these variables, the steady state values of the static variables can be backed out. We can most easily solve for the steady state by beginning with the dynamic Euler equation, (27).

$$1 = \beta(\alpha K^{*\alpha-1} N^{*1-\alpha} + (1-\delta))$$

Let's use this to solve for the steady state capital to labor ratio (life is much easier if you do it this way):

$$\frac{1}{\beta} - (1-\delta) = \alpha \left(\frac{K^*}{N^*}\right)^{\alpha-1}$$
$$\Rightarrow$$
$$\frac{K^*}{N^*} = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{36}$$

This is the same expression we had for the steady state capital stock in the model with labor inelastically supplied at one. Given the steady state capital-labor ratio, we now have the steady state factor prices:

$$W^* = (1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha} \tag{37}$$
$$R^* = \alpha \left(\frac{K^*}{N^*}\right)^{\alpha-1} \tag{38}$$
$$r^* = R^* - \delta \tag{39}$$

From the capital accumulation equation evaluated in steady state it is clear that $I^* = \delta k^*$. Use this and look at the production function combined with the accounting identity:

$$\left(\frac{K^*}{N^*}\right)^{\alpha} N^* = C^* + \delta K^*$$

Divide everything by N^*, and use this to express the consumption-hours ratio as a function of now-known things (the capital to hours ratio):

$$\frac{C^*}{N^*} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta \left(\frac{K^*}{N^*}\right) \tag{40}$$

Hold on to this. Now go to (28), the intratemporal consumption-leisure tradeoff condition, evaluated at steady state:

$$\frac{\theta}{1-N^*} = \frac{1}{C^*}(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha}$$

Multiply and divide both sides by N^* and re-arrange to expression the consumption-hours ratio on the right hand side:

$$\frac{C^*}{N^*} = \left(\frac{1-N^*}{N^*}\right)\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} \tag{41}$$

(40) and (41) essentially constitute two equations in two unknowns – the consumption-hours ratio and hours. Set them equal:

$$\left(\frac{1-N^*}{N^*}\right)\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} = \left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right)$$

Now solve for N^*:

$$N^* = \frac{\frac{1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha}}{\frac{\theta+1-\alpha}{\theta}\left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right)} \tag{42}$$

Now that we have N^*, the rest of this is pretty easy to compute:

$$I^* = \delta\left(\frac{K^*}{N^*}\right)N^* \tag{43}$$

Steady state output is:

$$Y^* = \left(\frac{K^*}{N^*}\right)^{\alpha}N^* \tag{44}$$

Steady state consumption then comes from the accounting identity:

$$C^* = N^*\left(\left(\frac{K^*}{N^*}\right)^{\alpha} - \delta\left(\frac{K^*}{N^*}\right)\right) \tag{45}$$

To do a quantitative analysis one would need to specify parameter values. Before doing that, let's think about analyzing the model qualitatively first. I'm going to log-linearize the first order conditions about the non-stochastic steady state. Start by taking logs of the static labor supply condition (28):

$$\ln\theta - \ln(1-N_t) = -\ln C_t + \ln(1-\alpha) + \ln A_t + \alpha\ln K_t - \alpha\ln N_t$$

The linearization (here I'm going to ignore the evaluation at steady state, which cancels out) is:

$$\frac{N_t-N^*}{1-N^*} = -\frac{C_t-C^*}{C^*} + \frac{A_t-A^*}{A^*} + \alpha\frac{K_t-K^*}{K^*} - \alpha\frac{N_t-N^*}{N^*}$$

Simplify into our "tilde" notation:

$$\left(\frac{N^*}{1-N^*}\right)\widetilde{N}_t = -\widetilde{C}_t + \widetilde{A}_t + \alpha\widetilde{K}_t - \alpha\widetilde{N}_t$$

So as to economize on notation, let's denote $\gamma = \frac{N^*}{1-N^*} > 0$. Then we get:

$$\widetilde{N}_t = -\left(\frac{1}{\gamma+\alpha}\right)\widetilde{C}_t + \left(\frac{1}{\gamma+\alpha}\right)\widetilde{A}_t + \left(\frac{\alpha}{\gamma+\alpha}\right)\widetilde{K}_t \tag{46}$$

Now let's linearize the accumulation equation. Begin by taking logs:

$$\ln K_{t+1} = \ln(A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta)K_t)$$

Now linearize, again ignoring the evaluation at steady state part, which cancels out anyway:

$$\begin{aligned}\frac{K_{t+1}-K^*}{K^*} = \frac{1}{K^*}\big(&K^{*\alpha}N^{*1-\alpha}(A_t - A^*) + \alpha K^{*\alpha-1}N^{*1-\alpha}(K_t - K^*) + \dots \\ &+(1-\alpha)K^{*\alpha}N^{*-\alpha}(N_t - N^*) - (C_t - C^*) + (1-\delta)(K_t - K^*)\big)\end{aligned}$$

Now simplify and use our "tilde" notation to denote the percentage deviation of a variable from its steady state:

$$\begin{aligned}\widetilde{K}_{t+1} &= \left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{N}_t\right) - \frac{C^*}{K^*}\widetilde{C}_t + (1-\delta)\widetilde{K}_t \\ \widetilde{K}_{t+1} &= \frac{1}{\beta}\widetilde{K}_t + \left(\frac{K^*}{N^*}\right)^{\alpha-1}\widetilde{A}_t + (1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\widetilde{N}_t - \frac{C^*}{K^*}\widetilde{C}_t\end{aligned}$$

The last simplification follows from the fact that $\alpha K^{*\alpha-1}N^{*1-\alpha} + (1-\delta) = \frac{1}{\beta}$. Now substitute the log-linearized expression for employment into this expression:

$$\widetilde{K}_{t+1} = \left(\frac{1}{\beta} + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\widetilde{K}_t + \left(\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{A}_t - \left(\frac{C^*}{K^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\widetilde{C}_t \tag{47}$$

Now we need to log-linearize the consumption Euler equation. Begin by taking logs:

$$-\ln C_t = \ln\beta - \ln C_{t+1} + \ln\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + (1-\delta)\right)$$

Now linearize, ignoring the evaluation at steady state and making use of the fact that $\alpha K^{*\alpha-1}N^{*1-\alpha} + (1-\delta) = \frac{1}{\beta}$:

$$\begin{aligned}-\frac{C_t - C^*}{C^*} = -\frac{C_{t+1}-C^*}{C^*} + \beta\big(&\alpha K^{*\alpha-1}N^{*1-\alpha}(A_{t+1} - A^*) + (\alpha-1)\alpha K^{*\alpha-2}N^{*1-\alpha}(K_{t+1} - K^*) + \dots \\ &+(1-\alpha)\alpha K^{*\alpha-1}N^{*-\alpha}(N_{t+1} - N^*)\big)\end{aligned}$$

This can be simplified using our tilde notation:

$$-\widetilde{C}_t = -\widetilde{C}_{t+1} + \beta\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\widetilde{A}_{t+1} + (\alpha-1)\widetilde{K}_{t+1} + (1-\alpha)\widetilde{N}_{t+1}\right) \tag{48}$$

Now eliminate $\widetilde{N}_{t+1}$ using (41):

$$-\widetilde{C}_t = -\widetilde{C}_{t+1} + \beta\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\widetilde{A}_{t+1} + \beta\alpha(\alpha-1)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\widetilde{K}_{t+1} + \dots$$
$$\dots + \beta\alpha(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(-\left(\frac{1}{\gamma+\alpha}\right)\widetilde{C}_{t+1} + \left(\frac{1}{\gamma+\alpha}\right)\widetilde{A}_{t+1} + \left(\frac{\alpha}{\gamma+\alpha}\right)\widetilde{K}_{t+1}\right)$$

Simplifying:

$$-\widetilde{C}_t = -\left(1 + \beta\alpha(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\frac{1}{\gamma+\alpha}\right)\right)\widetilde{C}_{t+1} + \left(\beta\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\left(\frac{1+\gamma}{\gamma+\alpha}\right)\widetilde{A}_{t+1} + \dots$$
$$\dots - \left(\beta\alpha(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\left(\frac{\gamma}{\gamma+\alpha}\right)\widetilde{K}_{t+1} \tag{49}$$

Equations (47) and (49) (plus the exogenous process for TFP) define a system of linearized difference equations. Let's try to think about this in the context of a phase diagram. The $\widetilde{c}_{t+1} = \widetilde{c}_t$ isocline can be solved for from (44):

$$\beta\alpha(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\frac{1}{\gamma+\alpha}\right)\widetilde{C}_{t+1} = \left(\beta\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\left(\frac{1+\gamma}{\gamma+\alpha}\right)\widetilde{A}_{t+1} - \left(\beta\alpha(1-\alpha)\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\left(\frac{\gamma}{\gamma+\alpha}\right)\widetilde{K}_{t+1}$$

This simplifies greatly to yield:

$$\widetilde{C}_{t+1} = \left(\frac{1+\gamma}{1-\alpha}\right)\widetilde{A}_{t+1} - \gamma\widetilde{K}_{t+1}$$

To make things easier, evaluate this at t (we again engage in the abuse of terminology in treating the two time periods as approximately the same):

$$\widetilde{C}_t = \left(\frac{1+\gamma}{1-\alpha}\right)\widetilde{A}_t - \gamma\widetilde{K}_t \tag{50}$$

This is the $\widetilde{C}_{t+1} = \widetilde{C}_t = 0$ isocline – i.e. the set of $(\widetilde{C}_t, \widetilde{K}_t)$ pairs where consumption is constant. In $(\widetilde{C}_t, \widetilde{K}_t)$ space it is downward sloping, and it will shift up if $\widetilde{A}_t$ were to change. How does this compare to what we had in the neoclassical model? Re-arrange terms to write this:

$$\widetilde{K}_t = \frac{1+\gamma}{\gamma}\frac{1}{1-\alpha}\widetilde{A}_t - \frac{1}{\gamma}\widetilde{C}_t$$

In the neoclassical model we normalized $N^* = 1$. If this is the case, then $\gamma \to \infty$. This means that the C_t term drops out and $\frac{1+\gamma}{\gamma} = 1$. So the isocline would be:

$$\widetilde{K}_t = \frac{1}{1-\alpha}\widetilde{A}_t$$

This would be a vertical line, just like in the neoclassical growth model case. What makes the $\widetilde{C}_{t+1} = \widetilde{C}_t$ isocline downward sloping is $\gamma < \infty$. As we will see below, γ has the interpretation of the inverse Frisch labor supply elasticity. So as long as labor supply isn't perfectly inelastic ($\gamma = \infty$), this isocline will be downward-sloping (at least locally in the region of the steady state, since we are taking a linear approximation).

Now go to (47) to find the $\widetilde{k}_{t+1} = \widetilde{k}_t = 0$ isocline:

$$\left(\frac{C^*}{K^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\widetilde{C}_t = \left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\widetilde{K}_t + \left(\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{A}_t$$

Simplify a bit:

$$\widetilde{C}_t = \left(\frac{C^*}{K^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)^{-1}\left(\left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\widetilde{K}_t + \left(\left(\frac{K^*}{N^*}\right)^{\alpha-1}\left(\frac{1+\gamma}{\gamma+\alpha}\right)\right)\widetilde{A}_t\right) \quad (51)$$

This is upward-sloping in $\widetilde{K}_t$. Furthermore, we actually know that the coefficient on the capital stock must be less than one. The coefficient can be written as follows:

$$\left(\frac{C^*}{K^*} + \frac{1-\alpha}{\gamma+\alpha}\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)^{-1}\left(\left(\frac{1}{\beta} - 1 + \frac{1-\alpha}{\gamma+\alpha}\alpha\left(\frac{K^*}{N^*}\right)^{\alpha-1}\right)\right)$$

We know that $\frac{1-\alpha}{\gamma+\alpha} > \frac{\alpha(1-\alpha)}{\gamma+\alpha}$ since $0 < \alpha < 1$. Hence, we can prove that this coefficient is less than unity if we can show that $\frac{1}{\beta} - 1$ is less than $\frac{C^*}{K^*}$ (because this would be sufficient to show that the numerator is less than the denominator, and hence the total coefficient is less than unity). We can find an expression for the consumption-capital ratio in the steady state by looking at the accounting identity and simplifying:

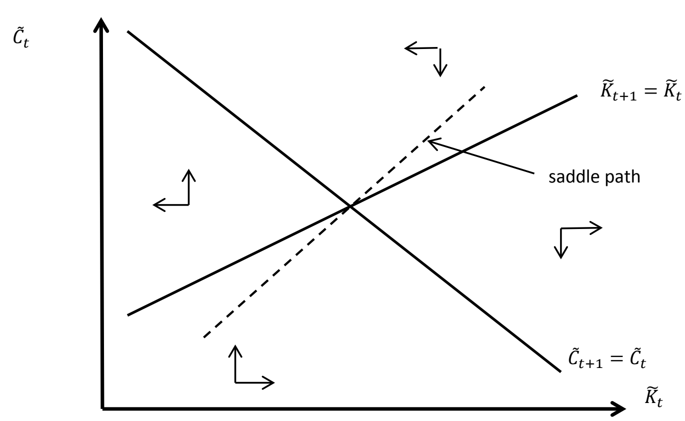
$$\begin{aligned}
C^* &= K^{*\alpha} n^{*1-\alpha} - \delta K^* \\
\frac{C^*}{K^*} &= \left(\frac{K^*}{N^*}\right)^{\alpha-1} - \delta \\
\frac{C^*}{K^*} &= \frac{\frac{1}{\beta} - (1-\delta)}{\alpha} - \delta \\
\frac{C^*}{K^*} &= \frac{\frac{1}{\beta} - 1}{\alpha} + \delta\left(\frac{1-\alpha}{\alpha}\right)
\end{aligned}$$

This has to be greater than $\frac{1}{\beta} - 1$, since $0 < \alpha < 1$ and $\delta > 0$. Note that we are taking a local, linear approximation around the steady state. As we saw in the neoclassical growth model with fixed labor, the isocline for capital was hump-shaped. In the region of the steady state, however, it was upward-sloping, just as it is here.

We thus have two isoclines. We want to plot these in a plane with $\widetilde{C}_t$ on the vertical axis and $\widetilde{K}_t$ on the horizontal axis. When $\widetilde{A}_t = 0$ (i.e. we were at the unconditional mean of technology), it is clear that the two isoclines cross at the point $\widetilde{c}_t = \widetilde{k}_t = 0$ – i.e. they cross at the non-stochastic steady state. We have that the $\widetilde{C}_{t+1} - \widetilde{C}_t = 0$ isocline is downward sloping and that the $\widetilde{K}_{t+1} - \widetilde{K}_t = 0$ isocline is downward sloping.

We want to examine the dynamics off of the isoclines so as to locate the saddle path. Quite intuitively, if $\widetilde{K}_t$ is "too big" relative to what it would be when $\widetilde{C}_{t+1} - \widetilde{C}_t = 0$ – i.e. we are to the right of the $\widetilde{C}_{t+1} - \widetilde{C}_t = 0$ iscoline – then consumption will be expected to decline over time. The intuition for this is straight from the Euler equation – if the capital stock is "too big", then the marginal product of capital (and hence the real interest rate) is "too small" given concavity of the production function. The low interest rate means that consumption is high but expected to fall. Hence, anywhere to the right of $\widetilde{C}_{t+1} - \widetilde{C}_t = 0$ isocline the arrows point down. If $\widetilde{C}_t$ is "too big" relative to where it would be along the $\widetilde{K}_{t+1} - \widetilde{K}_t = 0$ isocline, then capital will be expected to decline – intuitively, people aren't saving enough. Hence, anywhere above the $\widetilde{K}_{t+1} - \widetilde{K}_t = 0$ iscoline, arrows point left; anywhere below, arrows point right.

We can see that the saddle path must be upward-sloping, moving from southwest to northeast in the picture. Note also that the slope of the saddle path must be steeper than the slope of the $\widetilde{K}_{t+1} - \widetilde{K}_t$ isocline. One can show (albeit tediously) that the $\widetilde{K}_{t+1} - \widetilde{K}_t$ isocline has slope less than 1, which we did above.



This phase diagram tells us what $\widetilde{C}_t$ needs to be given $\widetilde{K}_t$ and $\widetilde{A}_t$ so as to be consistent with the first order conditions and the transversality condition holding. Given the two states, once we know $\widetilde{C}_t$, we can back out the values of the static variables (employment, output, and consumption) and the factor prices (the real wage and the real interest rate).

Begin by going to the labor market. Labor supply is implicitly determined by (6). Using our functional forms, this is:

$$\frac{\theta}{1-N_t} = \frac{1}{C_t} w_t$$

In other words, the amount of labor households want to supply depends negatively on consumption and positively on the real wage. Loosely speaking, we can think about consumption being in there as picking up wealth effects – when you're wealthier you want more leisure, which means less work. Let's log-linearize this equation:

$$\ln \theta - \ln(1-N_t) = -\ln C_t + \ln w_t$$
$$-\frac{N_t - N^*}{1-N^*} = -\frac{C_t - C^*}{C^*} + \frac{w_t - w^*}{w^*}$$

Simplify using the "tilde" notation to get:

$$\left(\frac{N^*}{1-N^*}\right) \widetilde{N}_t = -\widetilde{C}_t + \widetilde{w}_t \tag{52}$$

The term $\frac{1-N^*}{N^*}$ has a special name in economics – it is called the Frisch labor supply elasticity. It gives the percentage change in employment for a percentage change in the real wage, *holding the marginal utility of wealth (i.e. the Lagrange multiplier from the household's problem) fixed, which with these preferences is like holding consumption fixed.* If $N^* = \frac{1}{3}$, for example, then the Frisch elasticity would be 2. In terms of a graphical representation of the linearized labor supply function, it would be upward sloping in $\widetilde{w}_t$, with slope equal to the Frisch elasticity, and would shift in whenever consumption goes up (or out whenever consumption goes down). Using our notation from above, $\gamma = \frac{N^*}{1-N^*}$ will be the inverse Frisch labor supply elasticity.

Now go to the firm's first order condition which implicitly define a labor demand curve (i.e. equation (8)). Using our functional form assumptions, this is:

$$A_t(1-\alpha)K_t^{\alpha} N_t^{-\alpha} = w_t$$

Let's log-linearize this:

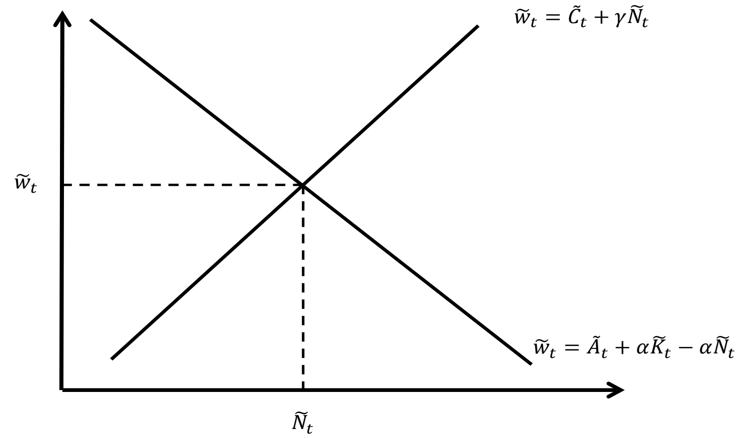
$$\ln A_t + \ln(1-\alpha) + \alpha \ln K_t - \alpha \ln N_t = \ln w_t$$
$$\frac{w_t - w^*}{w^*} = \frac{A_t - A^*}{A^*} + \alpha \frac{K_t - K^*}{K^*} - \alpha \frac{N_t - N^*}{N^*}$$

Using the "tilde" notation, this simplifies to:

$$\widetilde{w}_t = \widetilde{A}_t + \alpha \widetilde{K}_t - \alpha \widetilde{N}_t \tag{53}$$

In terms of a graph, this is downward sloping in $\widetilde{w}_t$, and will shift out whenever $\widetilde{A}_t$ or $\widetilde{K}_t$ increase. We graph this below. The intersection of the labor supply and demand curves determines

$\widetilde{w}_t$ and $\widetilde{N}_t$.



$\widetilde{A}_t$ and $\widetilde{K}_t$ are given. Once we know $\widetilde{C}_t$ from the phase diagram, we can determine the position of the labor supply curve. The position of the labor demand curve is given once we know capital and productivity. The intersection of the curves determines the real wage and level of employment.

Next we can determine output, given employment. Using our functional form assumptions, $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$. Let's log-linearize this:

$$\begin{aligned} \ln Y_t &= \ln A_t + \alpha \ln K_t + (1-\alpha) \ln N_t \\ \frac{Y_t - Y^*}{Y^*} &= \frac{A_t - A^*}{A^*} + \alpha \frac{K_t - K^*}{K^*} + (1-\alpha) \frac{N_t - N^*}{N^*} \end{aligned}$$

Using the "tilde" notation, we get:

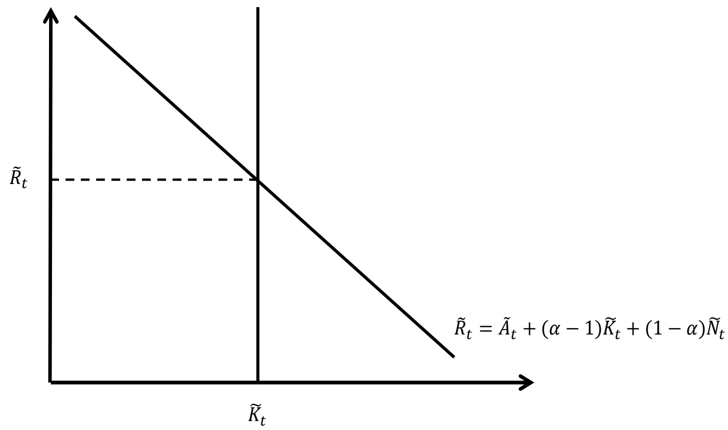
$$\widetilde{Y}_t = \widetilde{A}_t + \alpha \widetilde{K}_t + (1-\alpha) \widetilde{N}_t \tag{54}$$

The basic idea is the following. Once we know $\widetilde{C}_t$ from the phase diagram, we can determine $\widetilde{N}_t$ from the intersections of labor supply and demand. Once we know $\widetilde{N}_t$, given $\widetilde{A}_t$ and $\widetilde{K}_t$, we can determine $\widetilde{Y}_t$.

Now let's log-linearize the capital demand equation (from the setup in which households own the capital stock and lease it to firms on a period-by-period basis):

$$\begin{aligned} \ln R_t &= \ln \alpha + \ln A_t + (\alpha - 1) \ln K_t + (1-\alpha) \ln N_t \\ \widetilde{R}_t &= \widetilde{A}_t + (\alpha - 1) \widetilde{K}_t + (1-\alpha) \widetilde{N}_t \end{aligned}$$

Capital supply is fixed, at least within period. Hence, the capital supply curve is vertical. Hence the intersection of capital demand and supply determines the real rental rate on capital as follows:



Once we have determined $\widetilde{N}_t$ above, we can use this graph to determine $\widetilde{R}_t$.

How can we determine the real interest rate? Recall the Euler equation for bonds:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}}(1 + r_t)$$

Let's log-linearize this:

$$-\ln C_t = \ln \beta - \ln C_{t+1} + \ln(1 + r_t)$$

Using the approximation that $\ln(1 + r_t) \approx r_t$, we can write this in tilde notation as:

$$\widetilde{r}_t = E_t \widetilde{C}_{t+1} - \widetilde{C}_t$$

In doing this linearization, note that we interpret $\widetilde{r}_t = r_t - r^*$ (i.e. the absolute deviation from steady state, not the percentage deviation, since r_t is already measured in percentage units. Hence, the real interest rate is equal to the expected growth rate of consumption. This can be determined from the saddle path in the phase diagram.

Another way to think about the behavior of the real interest rate is as follows. Look at the Euler equation for capital (in the setup where the households own the capital stock and lease it to firms). This can be written:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left(R_{t+1} + (1 - \delta)\right)$$

Let's log-linearize this:

$$-\ln C_t = \ln \beta - \ln C_{t+1} + \ln \left(R_{t+1} + (1 - \delta)\right)$$

$$-\widetilde{C}_t = -E_t \widetilde{C}_{t+1} + \beta d R_{t+1}$$

Or:

$$-\widetilde{C}_t = -E_t\widetilde{C}_{t+1} + \beta R^* E_t \widetilde{R}_{t+1}$$

Combining this with what we have above, we must have:

$$\widetilde{r}_t = \beta R^* E_t \widetilde{R}_{t+1}$$

In other words, there is a very close connection between the real interest rate on bonds and the expected return on capital.

Finally, we can determine investment from the accounting identity. In log-linear form:

$$\widetilde{Y}_t = \frac{C^*}{Y^*}\widetilde{C}_t + \frac{I^*}{Y^*}\widetilde{I}_t$$

Since the phase diagram determined the value of $\widetilde{C}_t$, and once we know that we know $\widetilde{N}_t$ and hence $\widetilde{Y}_t$, we can back out $\widetilde{I}_t$. We could also determine $\widetilde{I}_t$ from the phase diagram, which tells us the expected trajectory of the capital stock.

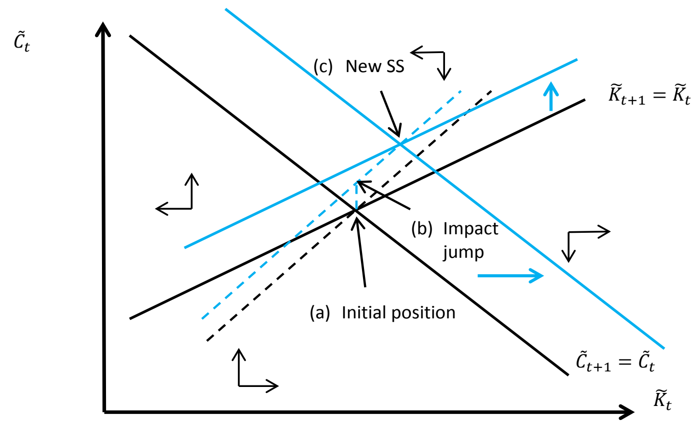
Hence, one can think about there being a "causal ordering". Given productivity and capital, determine consumption from the phase diagram. Given consumption, determine employment and the real wage. Once you know employment, you know output. Given employment and productivity, determine the rental rate on capital. Given the rental rate, determine the interest rate. . Then given output and consumption, determine investment:

$$\widetilde{A}_t \ \& \ \widetilde{K}_t \ \text{exogenous} \rightarrow \widetilde{C}_t \rightarrow \widetilde{N}_t \ \& \ \widetilde{w}_t \rightarrow \widetilde{Y}_t \& \widetilde{R}_t \rightarrow \widetilde{r}_t \rightarrow \widetilde{I}_t$$

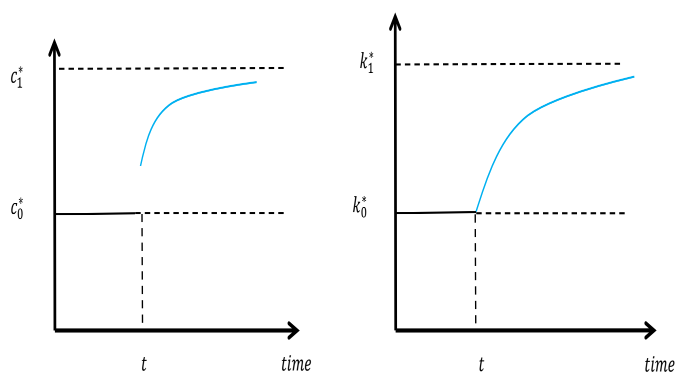
4 Dynamic Analysis in Response to TFP Shocks

We want to qualitatively characterize the dynamic responses of the endogenous variables of the model to shocks to productivity/TFP (the only source of stochastic variation in the model as it currently stands).

Consider first an unexpected, *permanent* increase in A_t. Assume that the economy initially sits in a steady state. This means we will end up in a new steady state, which means that the new steady state of the linearized variables will *not* be zero, as we linearized about the old steady state. From (50) and (51), we know that the two isoclines must both shift "up" when $\widetilde{A}_t$ suddenly increases. The new isoclines must cross at a point with a higher (relative to the initial) steady state capital and consumption (one can show this analytically). There will also be a new saddle path associated with the "new" system. At time t, consumption must jump to the new saddle path, from which point it must be expected to "ride" it all the way to the new steady state. The initial jump in consumption turns out to be ambiguous – it could increase, decrease, or not change at all (similarly to the basic neoclassical case with fixed labor input). That being said, for plausible parameterizations, consumption will jump up on impact (permanent income intuition), and hence that's how I'm going to draw it. See the phase diagram below:

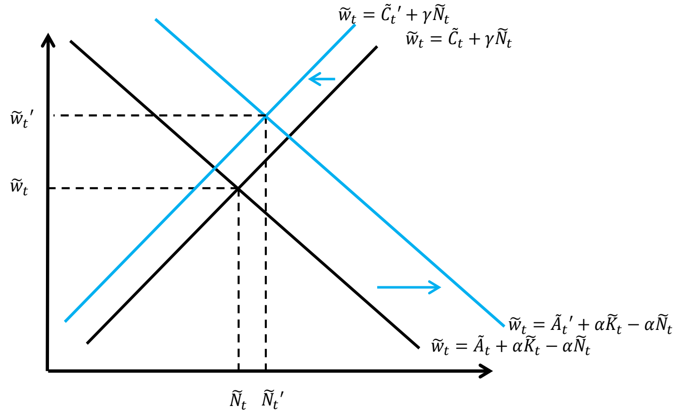


We can see from the picture that $\widetilde{C}_t$ jumps up on impact, from its initial steady state. From thereafter it must ride the new saddle path to the new steady state, which in terms of these linearized variables will feature $\widetilde{K}^* > 0$ and $\widetilde{C}^* > 0$, since we linearized about the steady state associated with the old level of TFP. Both consumption and capital must be expected to increase along the new transitional dynamics. We can therefore glean that the impulse responses of consumption and capital ought to look something like what they did in the neoclassical growth model:

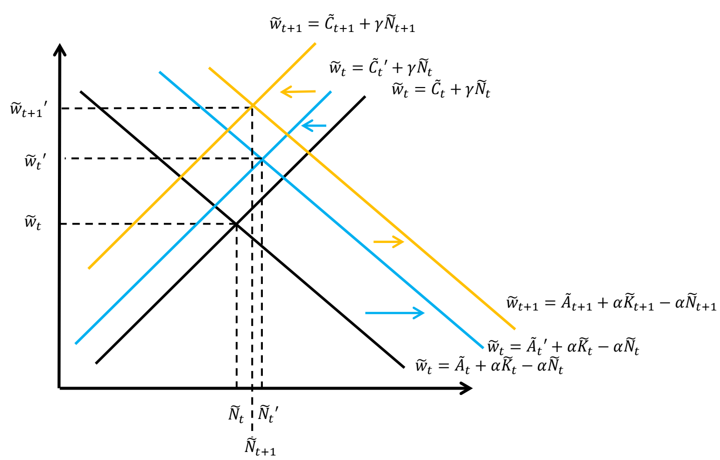


These are qualitatively very similar to what we would get in the neoclassical growth model with fixed labor.

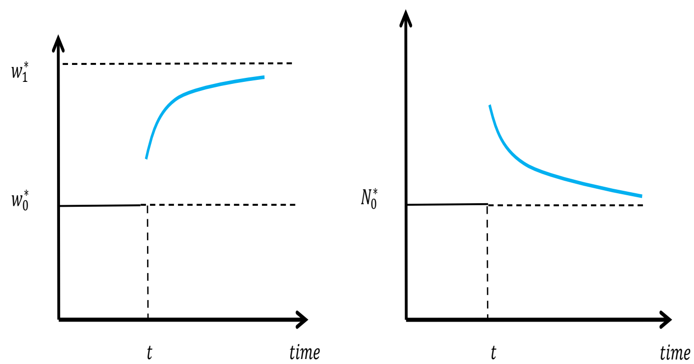
Now that we know consumption, go to the labor market. Higher productivity shifts labor demand out. Higher consumption shifts labor supply in. These effects are shown in blue in the diagram below. The net effect is for the real wage to definitely be higher, but there is an ambiguous effect on employment – it could go up, down, or not change at all. I'm going to draw it as going up, as this seems to be the plausible case.



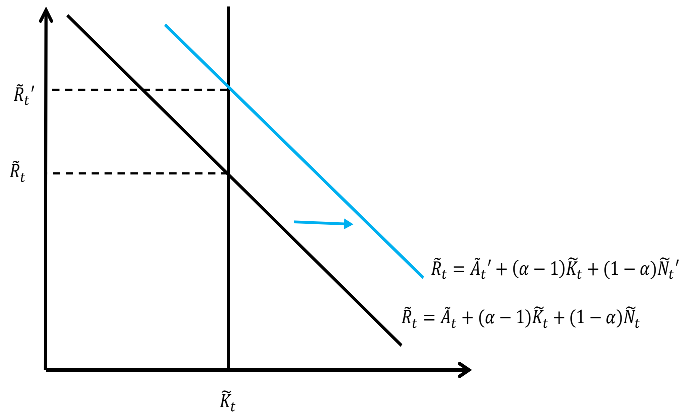
Now what happens dynamically in expectation? As we ride the new saddle path to the new higher steady state, consumption and capital are both increasing (A_t jumps up and in expectation stays up). Capital increasing is going to continue to shift the labor demand curve to the right, whereas consumption increasing is going to shift the labor supply curve inward. These effects are shown below in orange for the next period, $t+1$. The net effect is that the real wage continues to rise after period t and into $t+1$ (and similarly going forward). Depending on the magnitudes of the shifts, labor input could be increasing relative to its t value or decreasing; I have drawn the diagram where labor input decreases relative to its higher value in $t+1$. These effects are shown in the diagram below:



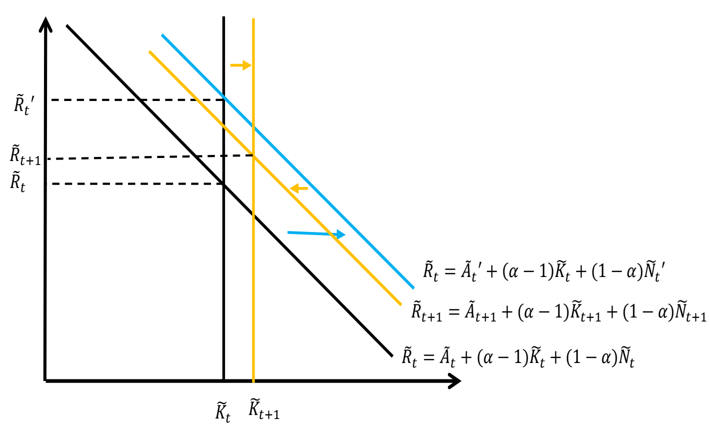
The figure below qualitatively shows the impulse response functions of both the real wage and labor input to a permanent increase in productivity. The wage definitely jumps up on impact and continues to rise thereafter. The impact effect on employment is ambiguous; for plausible parameterizations it goes up, as I have shown here. The dynamics of $\widetilde{N}_t$ subsequent to impact are also ambiguous per the diagram above; below I show it where labor hours come back down, and eventually revert to where they were before. This means that in the long run, the substitution effect of a higher real wage (which says to work more) exactly offsets the income effect (which says to work less). The preferences I have used above are consistent with labor hours being stationary in response to a permanent increase in productivity, and hence with labor hours return to where they started before the shock.

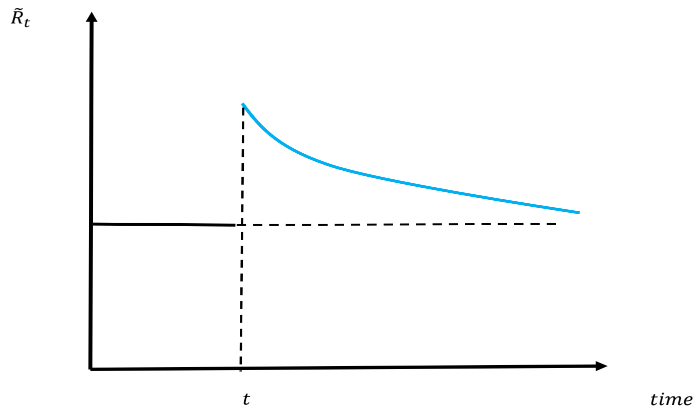


Next, go to the picture for capital demand. In the period of the shock, higher productivity and higher employment (what I have shown here) shift the capital demand curve to the right. Within period, the capital supply curve is fixed. Hence, $\widetilde{R}_t$ must rise. If labor hours were to decline on impact, it is apparently conceivable that the capital demand curve could shift in, with $\widetilde{R}_t$ falling. This turns out to be impossible for the production function we are using. We can write the expression for the rental rate as $R_t = \alpha \frac{Y_t}{K_t}$ – in other words, for a Cobb-Douglas production function, the marginal products (factor prices) are proportional to the average products (ratios of output to the input), with the weights the Cobb-Douglas shares. As I will argue below, output must go up after an increase in A_t, even if N_t declines. In any event, we also know that $\widetilde{R}_{t+1}$ must go up, since consumption must be expected to grow along the saddle path.

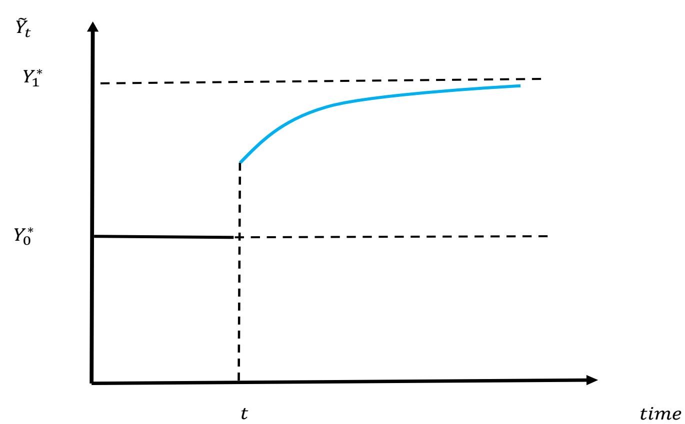


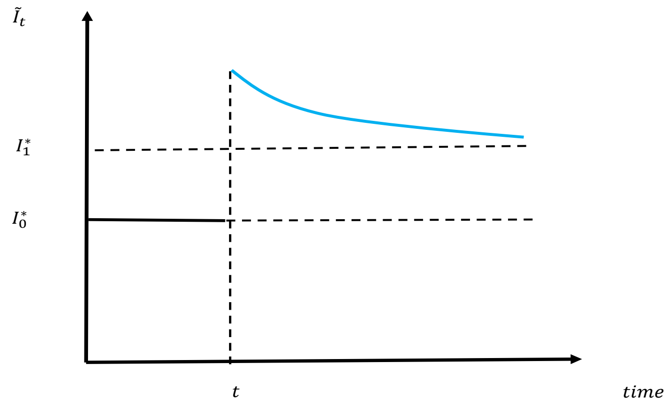
The dynamic behavior of the rental rate is considered below. We actually know that the rental rate must return to its original steady state after an increase in A_t. Why is this? From the Euler equation, if consumption is constant, $\widetilde{R}_t$ must be zero. Since consumption eventually goes to a new steady state in which it is constant, the rental rate must return to its original steady state. What happens qualitatively over time is that as we accumulate capital along the saddle path, the supply of capital increases. This works to drive down the equilibrium rental rate. In addition, if labor hours are returning to where they started, this depresses the demand for capital, which also drives down the equilibrium rental rate. Both of these effects mean that we would expect $\widetilde{R}_t$ to jump up immediately but then start declining, eventually returning to where it started. This is shown below:





What about the behavior of output and investment? We know that output will jump immediately. Note that this must happen even if N_t declines (which is not how I drew the pictures above but which is nevertheless conceivable). Why is this? We know that I_t must increase immediately, since along the saddle path capital must be increasing. For labor hours to decline on impact, it would have to be the case that consumption increases on impact, which means that the inward shift of the labor supply curve "dominates" the outward shift of the labor demand curve above. But if consumption and investment both increase, then output must increase. What about if consumption declines on impact? If consumption declines on impact, then labor hours definitely must go up (because the labor supply curve would shift out, not in). But if labor hours go up and the capital stock is fixed, then output must go up. Output ought to continue to rise over time as capital is accumulated and we transition towards the new steady state. Dynamically, we would expect investment to "overshoot" its long run steady state response because the capital stock will grow fastest immediately after the shock. These impulse response diagrams are shown below:





To summarize, following a permanent increase in TFP, we have an ambiguous initial jump in consumption (though likely up), an increase in the real wage, an ambiguous change in employment, an increase in output, an increase in investment, and an increase in the real interest rate. After these impact effects, we follow the dynamics of the phase diagram. In particular, consumption and the capital stock will grow (which means the real interest rate and investment will stay high). The real wage will continue to grow – this is because, as consumption grows, the labor supply curve will continue to shift in, and as capital grows, the labor demand curve will continue to shift out. Employment may continue to increase or decrease, but with the preferences I have assumed it will come back down to where it began. Output will go up and continue to rise as capital accumulates, and investment will also rise.

Next, consider a transitory change in $\widetilde{A}_t$. As in the neoclassical model with fixed labor, we would expect a transitory change in productivity to be associated with the same direction of jump for consumption immediately as in the case of a permanent shock, though the dynamics will be different. For the sake of simplicity, and in accord with what will happen with reasonable parameterizations, assume that consumption jumps up in response in response to the permanent productivity shock.

To make ideas as stark as possible, suppose that the change in A_t is only one period. In other words, it only lasts today and the expected value of A_{t+1} is unchanged. In the phase diagram, this will *approximately* do nothing. In reality consumption must jump a little, ride unstable dynamics for one period, and then end up on the original saddle path, from which point it must be expected to return to its initial steady state. Since the change is only "in effect" for a very short period of time, we can approximate the jump in consumption as being zero. Thus, what approximately happens in the phase diagram is nothing at all.

Given that consumption doesn't jump, next go to the labor market. Labor demand immediately shifts out, and the outward shift is the same as in the case of a permanent change in productivity. Labor demand only depends on current productivity, and hence the persistence of the shock does not factor in at all. But since consumption doesn't change, labor supply doesn't shift. Hence, we observe that both hours and the real wage go up. Importantly, relative to the case of the permanent shock, hours rise by more (in the permanent case the effect on hours was ambiguous) and the real wage rises by less. This is because there is effectively no wealth effect on labor supply, which there is in the case of a permanent shock.

Once we know what happens to employment, then we know what happens to output. Since employment goes up by more in the case of the purely transitory shock, we can see that output rises by more on impact to a productivity shock when it is transitory than when it is permanent. Because output goes up by more and consumption is approximately unaffected, we would expect investment to go up by more on impact in response to a transitory change in productivity than when the change in productivity is permanent.

Finally, go to the demand for capital curve. The demand for capital depends just on current conditions; hence it shifts out. It is important to note that it actually shifts out by more than in the case of a permanent technology shock, the reason being that employment increases by more here, so the marginal product of capital increases by more. This means that R_t ought to increase by more in response to a transitory productivity change.

What happens to r_t? In reality, consumption must jump a little bit today, but then return to its original steady state eventually. This means that consumption must actually be expected to fall over time, which means that r_t must go down.

The two cases thus far considered are somewhat knife edge. Most of the time we are interested in looking at what happens in response to persistent – but transitory – shocks. The above exercises are helpful because they provided "bounding results". The more persistent the shock is (i.e. the bigger is ρ), the more the results look like the permanent case. The less persistent the shock (i.e. the smaller is ρ), the more the results like the purely transitory case.

We can thus make the following qualitative statements that can be verified quantitatively by numerically solving the model:

1. The more persistent the increase in productivity, the more consumption increases on impact (or falls by less). In the limiting case where the change in TFP is just one period, consumption will approximately not react.

2. The more persistent the increase in productivity, the less hours react and the more real wages increase.

3. The more persistent the increase in productivity, the less output reacts

4. The more persistent the increase in productivity, the less investment reacts and the real interest rate increases by more (or falls by less).