Online Appendix to Accompany: "State-Dependent Fiscal Multipliers: Calvo vs. Rotemberg"*

Eric Sims
University of Notre Dame
& NBER

Jonathan Wolff
Miami University

July 20, 2017

Abstract

This online appendix accompanies "State-Dependent Fiscal Multipliers: Calvo vs. Rotemberg." Section 1 provides some numerical details concerning the computation of impulse responses and fiscal multipliers. Section 2 shows results using a second order approximation and compares them to the third order approximation used in the paper. Section 3 studies robustness to alternative specifications of the monetary policy rule.

1 Numerical Details

Let $\mathbf{s}_t$ be a $n_s \times 1$ vector of state variables and let $\mathbf{x}_t$ be a $n_x \times 1$ vector of non-predetermined variables. Let ϵ_t be a $n_k \times 1$ vector of shocks. To economize on notation, define $\mathbf{y}_t = \begin{bmatrix} \mathbf{s}_t \\ \mathbf{x}_t \end{bmatrix}$ as the full $(n_s + n_x) \times 1$ vector of state and non-predetermined variables. The solution to the model is a non-linear mapping, $\mathbf{h}(\cdot)$, from period $t-1$ values of the states and period t values of the shocks into the full period t vector of variables:

$$\mathbf{y}_t = \mathbf{h}\left(\mathbf{s}_{t-1}, \epsilon_t\right) \tag{1}$$

Letting variables with tildes denote deviations from the steady state, the third order approximation takes the form:

$$\begin{aligned}\widetilde{\mathbf{y}}_t \approx \mathbf{A}_0 + \mathbf{A}_1\widetilde{\mathbf{s}}_{t-1} + \mathbf{A}_2\epsilon_t + \mathbf{A}_3\left(\widetilde{\mathbf{s}}_{t-1}\otimes\widetilde{\mathbf{s}}_{t-1}\right) + \mathbf{A}_4\left(\epsilon_t\otimes\epsilon_t\right) + \mathbf{A}_5\left(\widetilde{\mathbf{s}}_{t-1}\otimes\epsilon_t\right)\\ + \mathbf{A}_6\left(\widetilde{\mathbf{s}}_{t-1}\otimes\widetilde{\mathbf{s}}_{t-1}\otimes\widetilde{\mathbf{s}}_{t-1}\right) + \mathbf{A}_7\left(\epsilon_t\otimes\epsilon_t\otimes\epsilon_t\right) + \mathbf{A}_8\left(\widetilde{\mathbf{s}}_{t-1}\otimes\widetilde{\mathbf{s}}_{t-1}\otimes\epsilon_t\right) + \mathbf{A}_9\left(\widetilde{\mathbf{s}}_{t-1}\otimes\epsilon_t\otimes\epsilon_t\right)\end{aligned} \tag{2}$$

The coefficient matrixes $\mathbf{A}_i$, $i = 1, \ldots, 9$, refer to different order derivatives of the function $\mathbf{h}(\cdot)$ evaluated in the non-stochastic steady state (i.e. the point of approximation is $\mathbf{s}_{t-1} = \mathbf{s}^*$ and $\epsilon_t = 0$). Implicitly, the $\frac{1}{1!}$, $\frac{1}{2!}$ and $\frac{1}{3!}$ terms are incorporated into these matrixes. The matrix $\mathbf{A}_0$ is a "shift term" that arises due to the variance of the shocks in a higher order approximation. In a first order solution, $\mathbf{A}_i$, $i = 3, \ldots, 9$, are matrixes of zeros, and $\mathbf{A}_0$ is a vector of zeros. In a second order solution, $\mathbf{A}_i$, $i = 6, \ldots, 9$ are matrixes of zeros. As is well-known, the solution has the properties

*Author contact info Sims: esims1@nd.edu; Wolff: wolffjs@miamioh.edu

that $\mathbf{A}_1$ and $\mathbf{A}_2$ are the same whether a first, second, or third order approximation is employed, while $\mathbf{A}_3$, $\mathbf{A}_4$, and $\mathbf{A}_5$ are the same whether a second or third order solution is used.

The impulse response of the vector of variables $\widetilde{\mathbf{y}}_t$ to a one standard deviation innovation to the i^{th} exogenous shock is defined as:

$$\mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1},\ \epsilon_t(i) = s_i\right] - \mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1}\right],\ \ h = 0, \ldots, H \tag{3}$$

The impulse response is defined as the change in conditional expectations of the future paths of variables conditional on (i) a one standard deviation shock to the variable of interest and (ii) the initial state vector. Because of the non-linear nature of the model and because of the lack of certainty equivalence in a higher order approximation, the conditional expectations in (3) must be computed via simulation. Our process for computing these responses numerically is as follows:

(i) Start with an initial value of the state vector, $\widetilde{\mathbf{s}}_{t-1}$.

(ii) Draw a random sample of dimension $(H+1) \times n_k$ of random shocks. Rows correspond to time periods (first row is period t, second is $t+1$, on to $t+H$). Columns correspond to the k different exogenous shocks.

(iii) Given the assumed initial state vector, the draw of shocks, and the coefficients of the approximated policy function, compute values of $\widetilde{\mathbf{y}}_{t+h}$ for $h = 0, \ldots H$.

(iv) Repeat step (ii)-(iii) N times. Average $\widetilde{\mathbf{y}}_{t+h}$, for each $h = 0, \ldots, H$, over the N different replications. This yields $\mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1}\right]$.

(v) Repeat steps (ii)-(iv), using the same draw of shocks, but add s_i to the first observation of the i^{th} shock at each iteration. This yields $\mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1},\ \epsilon_t(i) = s_i\right]$

(vi) The impulse response is then the difference between $\mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1},\ \epsilon_t(i) = s_i\right]$ and $\mathbb{E}\left[\widetilde{\mathbf{y}}_{t+h} \mid \widetilde{\mathbf{s}}_{t-1}\right]$.

The numerical procedure outlined above is based on Koop, Pesaran and Potter (1996). There will be a different impulse response function for each different value of $\widetilde{\mathbf{s}}_{t-1}$. We first compute impulse responses where the initial state vector is the non-stochastic steady state, so $\widetilde{\mathbf{s}}_{t-1} = \mathbf{0}$. To examine state-dependence more generally, we proceed as follows:

(a) Draw one large vector of exogenous shocks of dimension $n_k \times T$. The different columns correspond to different time periods – the first column is period t, the second $t+1$, the j^{th} column is $t+j$, on to $t+T-1$.

(b) Assume an initial value of the state vector of the non-stochastic steady state, $\widetilde{\mathbf{s}}_{t-1} = \mathbf{0}$. Use the simulated shocks and coefficients of the policy function (2) to create simulated values of the full vector of endogenous variables, $\widetilde{\mathbf{y}}_{t+j}$, $j = 0, \ldots, T$.

(c) Drop the first b periods as a burn-in to limit the influence of the assumed initial position. Collect the simulated values of the state vector for remaining periods, $\widetilde{\mathbf{s}}_{t+b+j}$, $j = 0, \ldots, T-b$.

(d) Use each of the remaining simulated state vectors as starting values for the states, and then repeat steps (i)-(vi) at each simulated state. This produces a distribution of impulse responses across different states

The fiscal multiplier is defined as the impact response of output divided by the impact response to government spending. From (d) above, we therefore obtain a distribution of fiscal multipliers. Statistics from these distributions are presented in the tables of the paper. For the calculations in the paper, we use $H = 0$ (since we focus on multipliers on impact, we need only to calculate impulse responses on impact), $N = 150$, $T = 10,100$, and $b = 100$ when monetary policy is characterized by a Taylor rule. To speed up our computations, we reduce $N = 20$ when considering the pegged interest rates. Our results are not very sensitive to the value of N. When doing the interest rate peg responses, we have to increase H to include the number of periods over which the interest rate is to be pegged.

2 Second vs. Third Order Approximation

In the paper, we solve both variants of the NK model using a third order approximation. In this section, we present results similar to those presented in Table 2 of the paper for a second order solution to both the Calvo and Rotemberg variants of the model. Results are summarized in Table 1.

Table 1: **State-Dependence of Multipliers, Second Order Approximation**

	Calvo	Rotemberg
Steady State	0.8408	0.8423
Mean	0.8410	0.8434
Standard Deviation	0.0049	0.0186
Minimum	0.8228	0.7728
Maximum	0.8597	0.9161
First Order Approx	0.8399	0.8399

Note: This table shows statistics for the multipliers in both the Calvo and Rotemberg variants of the NK model solved under a second order approximation.

There is little discernible difference between the results presented above and those shown under a third order approximation in Table 2 of the paper. What differences do exist are intuitive. As one might expect, the differences between the multipliers evaluated in steady state, as well as the mean multipliers across simulations, are smaller across the Calvo and Rotemberg models in a second order approximation compared to the third order approximation. Moreover, the standard deviations of multipliers across simulations are slightly smaller in the second order approximation for both versions of the model compared to the third order approximation. Nevertheless, these differences are very small.

3 Alternative Monetary Policy Rule Specifications

The assumed monetary policy rule in the paper is:

$$i_t = (1-\rho_i)i^* + \rho_i i_{t-1} + (1-\rho_i)\left[\phi_\pi\left(\pi_t - \pi^*\right) + \phi_y\left(\ln Y_t - \ln Y_t^f\right)\right] + s_i\varepsilon_{i,t} \tag{4}$$

As a baseline, we set $\rho_i = 0.8$, $\phi_\pi = 1.5$, and $\phi_y = 0$. In this section, we show results (for both variants of the model) for three alternative parameterizations of the policy rule: (1) no smoothing and no response to the output gap ($\rho_i = 0$, $\phi_\pi = 1.5$, and $\phi_y = 0$; (2) no smoothing and a positive response to the output gap (ρ_i = 0, ϕ_π = 1.5, and ϕ_y = 0.5); and (3) smoothing with a positive response to the output gap (ρ_i = 0.8, ϕ_π = 1.5, and ϕ_y = 0.5).

Table 2: **State-Dependence of Multipliers, Alternative Policy Rule Parameterizations**

		Calvo	Rotemberg
(1) $\rho_i = 0$, $\phi_\pi = 1.5$, $\phi_y = 0$			
	Steady State	0.7532	0.7555
	Mean	0.7534	0.7587
	Standard Deviation	0.0088	0.0268
	Minimum	0.7164	0.6912
	Maximum	0.7826	0.8869
	First Order Approx	0.7526	0.7526
(2) $\rho_i = 0$, $\phi_\pi = 1.5$, $\phi_y = 0.5$			
	Steady State	0.7382	0.7373
	Mean	0.7385	0.7371
	Standard Deviation	0.0046	0.0127
	Minimum	0.7217	0.6822
	Maximum	0.7544	0.7786
	First Order Approx	0.7393	0.7393
(2) $\rho_i = 0.8$, $\phi_\pi = 1.5$, $\phi_y = 0.5$			
	Steady State	0.7960	0.7936
	Mean	0.7957	0.7950
	Standard Deviation	0.0025	0.0099
	Minimum	0.7852	0.7571
	Maximum	0.8042	0.8270
	First Order Approx	0.7949	0.7949

Note: This table shows statistics for the multipliers in both the Calvo and Rotemberg variants of the NK model. Monetary policy is characterized by the different parameterizations of the Taylor rule as indicated in the left column.

Results are summarized in Table 2. Focusing on mean multipliers, the results are fairly intuitive. In both variants of the model, the mean multipliers are larger the more interest rate smoothing there is and are smaller when policy reacts to the output gap in addition to inflation. Multipliers are more volatile across states when there is no interest smoothing, and are less volatile the larger is the response to the output gap. In all specifications the multiplier in the Rotemberg model is significantly more volatile across states in comparison to the Calvo model – the ratio of standard deviations of multipliers in the Rotemberg model to the Calvo model is between 3 and 4 (compared to a relative standard deviation of about 4 in our baseline parameterization).

References

Koop, G., M. H. Pesaran and S. M. Potter, "Impulse Response Analysis in Nonlinear Multivariate Models," *Journal of Econometrics* 74 (1996).