Sims and Wu (2020, *JME*)

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1 Overview

This note describes Sims and Wu (2020): "Evaluating Central Banks' Tool Kit: Past, Present, and Future" (forthcoming in *Journal of Monetary Economics*).

The paper itself evaluates three unconditional policy tools: quantitative easing (QE), negative interest rate policy (NIRP), and forward guidance (FG). For the purposes of this note, I am going to focus solely on the QE part, which is where most of the paper's quantitative results lie. When studying quantitative easing, we will impose a zero lower bound (ZLB) on the short-term policy rate, which is an interest rate on reserves. In the equilibrium of the model I describe here, the interest rate on reserves equals the interest rate on short-term deposits. The ZLB is thus imposed on both. To implement NIRP, Sims and Wu (2020) suppose that the (net) deposit rate is bound from below by zero – i.e. households will not accept a negative nominal return (though the model itself abstracts from cash, though the existence of cash forms the basis for the logic of the zero lower bound in the first place, so assuming households will not accept a negative nominal return is a bit of a reduced-form shortcut). But they allow the interest rate on reserves (the policy rate) to go negative. They allow for negative policy rates by imposing a reserve requirement on banks – with a negative nominal return, banks in the model would not want to hold reserves, so to implement such a policy the central bank just requires the banks to hold the reserves at negative interest. In this way, a negative rate works like a tax on banks, which in and of itself is contractionary because the banks are balance sheet constrained, as we will see. But the expansionary effect of negative interest rates comes through a forward guidance type channel. Because the notional (i.e. desired) policy rate follows an inertial Taylor rule, moving the interest rate on reserves into negative territory – even if the economically relevant short-term interest rate cannot go negative – nevertheless signals lower future deposits rates once the ZLB period has ended. This is a stimulative forward guidance channel. In principle NIRP can be expansionary (the FG channel dominates) or contractionary (the "banking channel" dominates), but will be weaker the bigger is the central bank's balance sheet (because then the "tax" on banks is larger, and the contractionary channel is more important). In terms of conventional forward guidance, Sims and Wu (2020) allow for a "credibility parameter" that is a reduced-form way to capture the imperfect credibility associated with signaling future policy rates.

As I said, I'm going to focus on the QE part. The core parts of the model are sort of a hybrid between Carlstrom, Fuerst, and Paustian (2017) and Gertler and Karadi (2011, 2013). Similarly to Carlstrom, Fuerst, and Paustian (2017), firms are required to finance some of investment by issuing long-term bonds. These long-term bonds take the form of perpetuities with decaying coupon payments. Long-term bonds can only be held by intermediaries – markets are segmented in the sense that household cannot directly hold these securities. Financial intermediaries take the form in Gertler and Karadi (2013). Intermediaries stochastically die with some probability; this is tantamount to making them extra impatient. They can hold long-term bonds issued by firms or long-term bonds issued by the government. They can also hold interest-bearing reserves, which is a feature not present in either of these previous papers. They fund themselves with deposits and net worth. Net worth is accumulated by retained earnings, where given the existence of excess returns it makes sense for intermediaries to not pay out dividends until they die. New intermediaries are given a small amount of "startup funds." Intermediaries face a limited enforcement constraint. At the end of a period, they can default and abscond with some of their assets. We assume that they can keep a fraction of private bonds, a smaller fraction of government bonds, and no reserves (i.e. reserves are completely recoverable by creditors in the event of default). This balance sheet constraint effectively limits how much leverage these intermediaries can take on, and results in excess returns in equilibrium.

In the model, quantitative easing involves the central bank purchasing either private or government long-term bonds, financed via the creation of reserves. The way to think about how QE works in the model is that by changing the *composition* of assets intermediaries hold the central bank can change the *tightness* of their balance sheet constraint. So think about a QE shock as the central bank swapping out bonds (either public or private) in exchange for reserves. Because reserves are perfectly recoverable in the event of default, but other assets are not, this asset swap eases the balance sheet constraint facing the intermediaries. This allows them to buy more privately issued bonds. This pushes up the bond price and lowers the excess return. This in turn makes firms more willing to engage in capital investment, and thereby stimulates aggregate demand.

At the end of the day, the effects of exogenous QE shocks in Sims and Wu (2020) are fairly similar to Carlstrom, Fuerst, and Paustian (2017) and Gertler and Karadi (2011, 2013). The real innovation in the paper is to model *endogenous* QE as a *substitute* for conventional policy (adjustment of the short-term policy rate) at the ZLB. Sims and Wu (2020) postulate a "Taylor rule" type version of a QE rule that reacts to inflation and output, but *only turns on* when conventional policy is unavailable due to a binding ZLB. They show that this simple rule can nearly perfectly recrate the responses to a variety of different shocks without taking the ZLB into account. The broader message is that QE may be a very effective substitute for conventional monetary policy. This has implications for how costly the ZLB is (or isn't) and whether policies should be undertaken to reduce the frequency of future ZLB episodes (e.g. raising the inflation target).

2 Model

There are several agents in the model – a representative household; a labor market that includes a competitive labor packer that transforms differentiated labor from unions into labor available for production, where unions in turn purchase labor from the household; a capital goods producing firm; a representative wholesale firm; a continuum of retail firms, who purchase and repackage wholesale output for sale to a final good firm; a fiscal authority; and a monetary authority. The subsections below lay out the problems and optimality conditions for each type of agent.

2.1 Household

There is a representative household with preferences over consumption and labor given by:

$$U_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \ln\left(C_t - bC_{t-1}\right) - \chi \frac{L_t^{1+\eta}}{1+\eta} \right\}$$

Households consume and save through nominal deposits, D_t. They earn income from supply labor to labor unions at nominal wage MRS_t. They receive dividends from ownership in non-financial firms as well as the equity leftover from remaining intermediaries. Each period, households make a *fixed* real equity transfusion to newly born intermediaries. This is given by X. They also pay a lump sum tax to the government. The flow budget constraint in nominal terms is:

$$P_t C_t + D_t \leq MRS_t L_t + R_{t-1}^D D_{t-1} + DIV_t - P_t X - P_t T_t \tag{1}$$

A Lagrangian is:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \ln\left(C_t - bC_{t-1}\right) - \chi \frac{L_t^{1+\eta}}{1+\eta} + \lambda_t^n \left[MRS_t L_t + R_{t-1}^D D_{t-1} + DIV_t - P_t X - P_t T_t - P_t C_t - D_t \right] \Bigg\}$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = \frac{1}{C_t - bC_{t-1}} - \lambda_t^n P_t - \beta b \, \mathbb{E}_t \, \frac{1}{C_{t+1} - bC_t}$$

$$\frac{\partial \mathbb{L}}{\partial L_t} = -\chi L_t^{\eta} + \lambda_t^n MRS_t$$

$$\frac{\partial \mathbb{L}}{\partial D_t} = -\lambda_t^n + \beta \, \mathbb{E}_t \, \lambda_{t+1}^n R_t^d$$

Define $\mu_t = P_t \lambda_t^n$ as the *real* marginal utility of consumption. Further define the real stochastic discount factor as:

$$\Lambda_{t-1,t} = \frac{\beta \mu_t}{\mu_{t-1}} \tag{2}$$

Using this notation and setting the above to zero, we get:

$$\mu_t = \frac{1}{C_t - bC_{t-1}} - \beta b \, \mathbb{E}_t \frac{1}{C_{t+1} - bC_t} \tag{3}$$

$$\chi L_t^{\eta} = \mu_t mrs_t \tag{4}$$

$$1 = R_t^d \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{-1} \tag{5}$$

Where $mrs_t = MRS_t/P_t$ is the real wage relevant for the household and $\Pi_t = P_t/P_{t-1}$ is the gross inflation rate.

2.2 Labor Market

There are two layers to the labor market. There are a unit measure of labor unions, index by $h \in [0,1]$, who purchase labor from households and repackage for resale to a labor packer at $W_t(h)$. Then a competitive labor packer combines union labor into a final labor input.

Work backwards. The labor packer transforms union labor, $L_{d,t}(h)$, into final labor available for production via a CES technology:

$$L_{d,t} = \left(\int_0^1 L_{d,t}(h)^{\frac{\epsilon_w - 1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{6}$$

The labor packer sells final labor input, $L_{d,t}$, to production firms at nominal wage W_t. It purchases union labor at $W_t(h)$. The labor packer is competitive and earns no profit in equilibrium. Its problem is to pick each $L_{d,t}(h)$ to maximize:

$$\max_{L_{d,t}(h)} \; W_t \left(\int_0^1 L_{d,t}(h)^{\frac{\epsilon_w - 1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - W_t(h) L_{d,t}(h)$$

Optimization gives rise to a standard downward-sloping demand curve for labor and an aggregate wage index:

$$L_{d,t}(h) = \left(\frac{W_t(h)}{W_t} \right)^{-\epsilon_w} L_{d,t} \tag{7}$$

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(h)^{1-\epsilon_w} dh \tag{8}$$

The labor unions simply repackage labor purchases from households for sale to the labor packer: $L_{d,t}(h) = L_t(h)$. Labor is purchased from the household at MRS_t and sold to the packer at $W_t(h)$. Nominal profit is:

$$DIV_{L,t}(h) = W_t(h) L_{d,t}(h) - MRS_t L_{d,t}(h)$$

Unions are subject to a Calvo wage rigidity: each period, there is a $1-\phi_w$ probability they can adjust a nominal wage. I'm going to abstract from indexation, which the paper allows for. Plugging in the demand function, flow nominal dividends are:

$$DIV_{L,t}(h) = W_t(h)^{1-\epsilon_w} W_t^{\epsilon_w} L_{d,t} - MRS_t W_t(h)^{-\epsilon_w} W_t^{\epsilon_w} L_{d,t}$$

Divide by P_t to write this in real terms, and write the non-union specific terms in real terms as well (i.e. $w_t = W_t/P_t$ and $mrs_t = MRS_t/P_t$):

$$div_{L,t}(h) = W_t(h)^{1-\epsilon_w} w_t^{\epsilon_w} P_t^{\epsilon_w - 1} L_{d,t} - mrs_t P_t^{\epsilon_w} W_t(h)^{-\epsilon_w} w_t^{\epsilon_w} L_{d,t}$$

Consider the problem of a union getting to choose a new $W_t(h)$ in period t. It chooses this to maximize the PDV of real dividends, where discounting is by the real SDF of the household as well as the probability a chosen wage is still relevant in the future:

$$\max_{W_t^*} \; \mathbb{E}_t \sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j} \left\{ W_t(h)^{1-\epsilon_w} w_{t+j}^{\epsilon_w} P_{t+j}^{\epsilon_w - 1} L_{d,t+j} - mrs_{t+j} P_{t+j}^{\epsilon_w} W_t(h)^{-\epsilon_w} w_{t+j}^{\epsilon_w} L_{d,t+j} \right\}$$

The FOC is:

$$(1-\epsilon_w) W_t(h)^{-\epsilon_w} \, \mathbb{E}_t \sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j} w_{t+j}^{\epsilon_w} P_{t+j}^{\epsilon_w - 1} L_{d,t+j} + \epsilon_w W_t(h)^{-\epsilon_w - 1} \, \mathbb{E}_t \sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j} mrs_{t+j} P_{t+j}^{\epsilon_w} w_{t+j}^{\epsilon_w} L_{d,t+j} = 0$$

Setting equal to zero, we have:

$$W_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\mathbb{E}_t \sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j} mrs_{t+j} P_{t+j}^{\epsilon_w} w_{t+j}^{\epsilon_w} L_{d,t+j}}{\mathbb{E}_t \sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j} w_{t+j}^{\epsilon_w} P_{t+j}^{\epsilon_w - 1} L_{d,t+j}}$$

I have replaced $W_t(h)$ with $W_t^{\#}$ because nothing on the right hand depends on h; all updating unions choose the same wage regardless of history. We can write this recursively, defining the numerator and denominator as:

$$F_{1,t} = mrs_t P_t^{\epsilon_w} w_t^{\epsilon_w} L_{d,t} + \phi_w \, \mathbb{E}_t \, \Lambda_{t,t+1} F_{1,t+1}$$

$$F_{2,t} = P_t^{\epsilon_w - 1} w_t^{\epsilon_w} L_{d,t} + \phi_w \, \mathbb{E}_t \, \Lambda_{t,t+1} F_{2,t+1}$$

Define $f_{1,t} = F_{1,t}/P_t^{\epsilon_w}$ and $f_{2,t} = F_{2,t}/P_t^{\epsilon_w - 1}$. We get:

$$f_{1,t} = mrs_t w_t^{\epsilon_w} L_{d,t} + \phi_w \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_w} f_{1,t+1} \tag{9}$$

$$f_{2,t} = w_t^{\epsilon_w} L_{d,t} + \phi_w \, \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_w - 1} f_{2,t+1} \tag{10}$$

Since $F_{1,t}/F_{2,t} = (f_{1,t}/f_{2,t})P_t$, we can write the reset real wage, $w_t^{\#} = W_t^{\#}/P_t$, as:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{f_{1,t}}{f_{2,t}} \tag{11}$$

2.3 Investment Goods Producer

New capital, $\widehat{I}_t$, is produced using unconsumed output, I_t. It is sold to firms at P_t^K. The production function is:

$$\widehat{I}_t = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \tag{12}$$

$S(\cdot)$ is an investment adjustment cost as in Christiano, Eichenbaum, and Evans (2005). Nominal profit is:

$$DIV_{k,t} = P_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - P_t I_t$$

Or, in real terms, with $p_t^k = P_t^k / P_t$:

$$div_{k,t} = p_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - I_t$$

The objective is to pick I_t to maximize the PDV of real profit, where discounting is by the stochastic discount factor. The problem is:

$$\max_{I_t} \; \mathbb{E}_t \sum_{j=0}^{\infty} \Lambda_{t,t+1} \left\{ p_{t+j}^k \left[1 - S\left(\frac{I_{t+j}}{I_{t+j-1}}\right)\right] I_{t+j} - I_{t+j} \right\}$$

The FOC is:

$$p_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}}\right] - 1 + \mathbb{E}_t \, \Lambda_{t,t+1} p_{t+1}^k S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 = 0$$

Or:

$$1 = p_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}}\right] + \mathbb{E}_t \, \Lambda_{t,t+1} p_{t+1}^k S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \tag{13}$$

2.4 Goods Production

There are three layers to production. A final good firm purchases retail outputs, where there are a continuum of retailers indexed by $f \in [0,1]$, at $P_t(f)$ and resells at P_t. The production technology is CES:

$$Y_t = \left(\int_0^1 Y_t(f)^{\frac{\epsilon_p - 1}{\epsilon_p}} df \right)^{\frac{\epsilon_p}{\epsilon_p - 1}}$$

The problem is:

$$\max_{Y_t(f)} \; P_t \left(\int_0^1 Y_t(f)^{\frac{\epsilon_p - 1}{\epsilon_p}} df \right)^{\frac{\epsilon_p}{\epsilon_p - 1}} - P_t(f) Y_t(f)$$

Optimization gives a standard downward-sloping demand for each retail output and an aggregate price index.

$$Y_t(f) = \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon_p} Y_t$$

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(f)^{1-\epsilon_p} df$$

The final good firm earns no profit.

Retail firms purchase wholesale output at $P_{w,t}$. They simply repackage wholesale output: $Y_t(f) = Y_{w,t}(f)$, and then sell it to the final goods firm at $P_t(f)$. This is analogous to the labor union. Nominal profit for retailers is:

$$DIV_{R,t}(f) = P_t(f) Y_t(f) - P_{w,t} Y_{w,t}(f)$$

Plugging in the demand function:

$$DIV_{R,t}(f) = P_t(f)^{1-\epsilon_p} P_t^{\epsilon_p} Y_t - P_{w,t} P_t(f)^{-\epsilon_p} P_t^{\epsilon_p} Y_t$$

In real terms, where $p_{w,t} = P_{w,t}/P_t$ is the real wholesale price, we have:

$$div_{R,t}(f) = P_t(f)^{1-\epsilon_p} P_t^{\epsilon_p - 1} Y_t - p_{w,t} P_t(f)^{-\epsilon_p} P_t^{\epsilon_p} Y_t$$

Retailers can only adjust their price with probability $1 - \phi_p$. This makes their price-setting problem dynamic. A retailer with the opportunity to adjust will choose $P_t(f)$ to maximize the PDV of real profits, where discounting is the by stochastic discount factor as well as the probability that a price chosen today will remain in effect in the future. The problem is:

$$\max_{P_t(f)} \; \mathbb{E}_t \sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j} \left\{ P_t(f)^{1-\epsilon_p} P_{t+j}^{\epsilon_p - 1} Y_{t+j} - p_{w,t+j} P_t(f)^{-\epsilon_p} P_{t+j}^{\epsilon_p} Y_{t+j} \right\}$$

The FOC is:

$$(1-\epsilon_p)P_t(f)^{-\epsilon_p}\,\mathbb{E}_t\sum_{j=0}^{\infty}\phi_p^j\Lambda_{t,t+j}P_{t+j}^{\epsilon_p-1}Y_{t+j}+\epsilon_pP_t(f)^{-\epsilon_p-1}\,\mathbb{E}_t\sum_{j=0}^{\infty}\phi_p^j\Lambda_{t,t+j}p_{w,t+j}P_{t+j}^{\epsilon_p}Y_{t+j}=0$$

Which may be written:

$$P_t^{\#}=\frac{\epsilon_p}{\epsilon_p-1}\frac{\mathbb{E}_t\sum_{j=0}^{\infty}\phi_p^j\Lambda_{t,t+j}p_{w,t+j}P_{t+j}^{\epsilon_p}Y_{t+j}}{\mathbb{E}_t\sum_{j=0}^{\infty}\phi_p^j\Lambda_{t,t+j}P_{t+j}^{\epsilon_p-1}Y_{t+j}}$$

Since nothing on the RHS depends on f, we see that all updating retailers choose the same price, $P_t^{\#}$. We can write the numerator and denominator recursively as:

$$X_{1,t}=p_{w,t}P_t^{\epsilon_p}Y_t+\phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}X_{1,t+1}$$

$$X_{2,t}=P_t^{\epsilon_p-1}Y_t+\phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}X_{2,t+1}$$

Define $x_{1,t}=X_{1,t}/P_t^{\epsilon_p}$ and $x_{2,t}=X_{2,t}/P_t^{\epsilon_p-1}$. We therefore have:

$$x_{1,t}=p_{w,t}Y_t+\phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_p}x_{1,t+1} \tag{14}$$

$$x_{2,t}=Y_t+\phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_p-1}x_{1,t+1} \tag{15}$$

Since $X_{1,t}/X_{2,t}=(x_{1,t}/x_{2,t})P_t$, we can define $\Pi_t^{\#}$ as the relative reset price and write the pricing condition:

$$\Pi_t^{\#}=\frac{\epsilon_p}{\epsilon_p-1}\frac{x_{1,t}}{x_{2,t}} \tag{16}$$

There is a representative wholesale firm. It produces output using capital that it accumulates and labor purchased from the labor packer. Its production function is:

$$Y_{w,t}=A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha} \tag{17}$$

u_t is capital utilization. In nominal terms, the wholesaler's profit is:

$$DIV_{w,t}=P_{w,t}A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha}-W_tL_{d,t}-P_t^k\widehat{I}_t-F_{w,t-1}+Q_t(F_{w,t}-\kappa F_{w,t-1})$$

The wholesaler has outstanding coupon liabilities on long bonds of $F_{w,t-1}$. It can issue new long bonds for Q_t, where $Q_t(F_{w,t}-\kappa F_{w,t-1})$ is the value of new bond issuance.

The wholesale firm is subject to a standard law of motion for physical capital:

$$K_{t+1}=\widehat{I}_t+(1-\delta(u_t))K_t \tag{18}$$

$\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2$ is utilization adjustment cost. The wholesale firm is also subject to an investment in advance constraint:

$$P_t^k \widehat{I}_t \leq Q_t(F_{w,t} - \kappa F_{w,t-1}) \tag{19}$$

In other words, (19) says that nominal expenditure on new investment cannot exceed issuance of new bonds.

Write dividends and the loan in advance constraint in real terms:

$$div_{w,t} = p_{w,t} A_t (u_t K_t)^\alpha L_{d,t}^{1-\alpha} - w_t L_{d,t} - p_t^k \widehat{I}_t - \frac{F_{w,t-1}}{P_t} + Q_t \left(\frac{F_{w,t}}{P_t} - \kappa \frac{F_{w,t-1}}{P_t} \right)$$

$$\psi p_t^k \widehat{I}_t \leq Q_t \left(\frac{F_{w,t}}{P_t} - \kappa \frac{F_{w,t-1}}{P_t} \right)$$

ψ is a fraction of a investment that must be financed by debt; $\psi = 1$ would correspond to Carlstrom, Fuerst, and Paustian (2017). Profits are discounted by the household's real SDF. A Lagrangina is:

$$\mathbb{L}_t = \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \Bigg\{ p_{w,t} A_t (u_t K_t)^\alpha L_{d,t}^{1-\alpha} - w_t L_{d,t} - p_t^k \widehat{I}_t - \frac{F_{w,t-1}}{P_t} + Q_t \left(\frac{F_{w,t}}{P_t} - \kappa \frac{F_{w,t-1}}{P_t} \right)$$
$$+ \nu_{1,t} \left(\widehat{I}_t + (1 - \delta(u_t)) K_t - K_{t+1} \right) + \nu_{2,t} \left(Q_t \left(\frac{F_{w,t}}{P_t} - \kappa \frac{F_{w,t-1}}{P_t} \right) - \psi p_t^k \widehat{I}_t \right) \Bigg\}$$

The derivatives of the Lagrangian are:

$$\frac{\partial \mathbb{L}}{\partial L_{d,t}} = (1-\alpha) p_{w,t} A_t (u_t K_t)^\alpha L_{d,t}^{-\alpha} - w_t$$

$$\frac{\partial \mathbb{L}}{\partial u_t} = \alpha p_{w,t} A_t (u_t K_t)^{\alpha-1} K_t L_{d,t}^{1-\alpha} - \nu_{1,t} \delta'(u_t) K_t$$

$$\frac{\partial \mathbb{L}}{\partial \widehat{I}_t} = -p_t^k + \nu_{1,t} - \nu_{2,t} \psi p_t^k$$

$$\frac{\partial \mathbb{L}}{\partial K_{t+1}} = -\nu_{1,t} + \mathbb{E}_t \Lambda_{t,t+1} \left[\alpha p_{w,t+1} A_{t+1} (u_{t+1} K_{t+1})^{\alpha-1} u_{t+1} L_{d,t+1}^{1-\alpha} + \nu_{1,t+1}(1 - \delta(u_{t+1})) \right]$$

$$\frac{\partial \mathbb{L}}{\partial F_{w,t}} = \frac{Q_t}{P_t} + \nu_{2,t} \frac{Q_t}{P_t} - \mathbb{E}_t \Lambda_{t,t+1} \left[\frac{1}{P_{t+1}} + \kappa \frac{Q_{t+1}}{P_{t+1}} + \nu_{2,t+1} \kappa \frac{Q_{t+1}}{P_{t+1}} \right]$$

Setting equal to zero, the first three become:

$$w_t = (1-\alpha) p_{w,t} A_t (u_t K_t)^\alpha L_{d,t}^{-\alpha} \tag{20}$$

$$\nu_{1,t} \delta'(u_t) = \alpha p_{w,t} A_t (u_t K_t)^{\alpha-1} L_{d,t}^{1-\alpha} \tag{21}$$

$$(1 + \psi \nu_{2,t}) p_t^k = \nu_{1,t} \tag{22}$$

Then for the two dynamic Euler equations, we have:

$$\nu_{1,t} = \mathbb{E}_t \, \Lambda_{t,t+1} \left[\alpha p_{w,t+1} A_{t+1} (u_{t+1} K_{t+1})^{\alpha-1} u_{t+1} L_{d,t+1}^{1-\alpha} + \nu_{1,t+1}(1 - \delta(u_{t+1})) \right] \quad (23)$$

$$(1 + \nu_{2,t}) Q_t = \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{-1} \left[1 + (1 + \nu_{2,t+1}) \kappa Q_{t+1} \right] \quad (24)$$

Note that discounting in (23) is by the *real* stochastic discount factor, whereas discounting in (24) is by the *nominal* stochastic discount factor, $\Lambda_{t,t+1}\Pi_{t+1}^{-1}$. This is because capital is a real asset whereas long-bonds are nominal.

Introduce two auxiliary variables. Let $M_{1,t} = 1 + \nu_{2,t}$ and $M_{2,t} = 1 + \psi \nu_{2,t}$. We can then write the FOC for investment as:

$$\nu_{1,t} = p_t^k M_{2,t} \quad (25)$$

We can then eliminate the multiplier in the utilization FOC:

$$p_t^k M_{2,t} \delta'(u_t) = \alpha p_{w,t} A_t (u_t K_t)^{\alpha-1} L_{d,t}^{1-\alpha} \quad (26)$$

We can then also write the dynamic Euler equations as:

$$p_t^k M_{2,t} = \mathbb{E}_t \, \Lambda_{t,t+1} \left[\alpha p_{w,t+1} A_{t+1} (u_{t+1} K_{t+1})^{\alpha-1} u_{t+1} L_{d,t+1}^{1-\alpha} + (1 - \delta(u_{t+1})) p_{t+1}^k M_{2,t+1} \right] \quad (27)$$

$$Q_t M_{1,t} = \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{-1} \left[1 + \kappa Q_{t+1} M_{1,t+1} \right] \quad (28)$$

Where we have:

$$\frac{M_{1,t} - 1}{M_{2,t} - 1} = \psi^{-1} \quad (29)$$

Before proceeding, it is again useful to note the distortion we are introducing relative to a more standard model. This distortion is captured by $\nu_{2,t}$, where $\nu_{2,t} > 0$ means $M_{1,t} > 1$ and $M_{2,t} > 1$, and therefore distorts the FOC for utilization, the dynamic Euler equation for capital, and the dynamic Euler equation for long bonds. Without this distortion, these FOC would look standard.

2.5 Monetary Policy

The central bank sets the *notional* or *desired* interest rate on reserves, R_t^{tr}, according to a Taylor rule:

$$\ln R_t^{tr} = (1 - \rho_r) \ln R^{tr} + \rho_r \ln R_{t-1}^{tr} + (1 - \rho_r) \left[\phi_\pi (\ln \Pi_t - \ln \Pi) + \phi_y (\ln Y_t - \ln Y_{t-1}) \right] + s_r \varepsilon_{r,t} \quad (30)$$

The actual interest rate on reserves is assumed be subject to a zero lower bound (I am not considering negative interest rate policy, NIRP, as Sims and Wu do in the paper):

$$R_t^{re} = \max\left\{1, R_t^{tr}\right\} \tag{31}$$

The central bank has a balance sheet where the size is completely up to the central bank. We assume that the central bank can hold either private investment bonds (loosely, think about these as mortgage-backed securities, MBS) or long-term government bonds (loosely, long-term Treasuries). It finances this with reserves (the model is cashless, so there is no currency in circulation).

$$Q_t F_{cb,t} + Q_{B,t} B_{cb,t} = RE_t \tag{32}$$

We have to assume something about how various aspects of the balance sheet adjust. For now, just assume that real bond holdings follow exogenous processes:

$$f_{cb,t} = (1-\rho_f) f_{cb} + \rho_f f_{cb,t-1} + s_f \varepsilon_{f,t} \tag{33}$$

$$b_{cb,t} = (1-\rho_b) b_{cb} + \rho_b b_{cb,t-1} + s_b \varepsilon_{b,t} \tag{34}$$

The balance sheet constraint in real terms is:

$$Q_t f_{cb,t} + Q_{B,t} b_{cb,t} = re_t \tag{35}$$

Given $f_{cb,t}$ and $b_{cb,t}$, re_t automatically adjusts to make the balance sheet hold.

The central bank earns income on its assets and pays interest on its liabilities (reserves). In particular, it earns revenue:

$$P_t T_{cb,t} = (1+\kappa Q_t) F_{cb,t-1} + (1+\kappa Q_{B,t}) B_{cb,t-1} - R_{t-1}^{re} RE_{t-1}$$

This can be written:

$$P_t T_{cb,t} = \frac{1+\kappa Q_t}{Q_{t-1}} Q_{t-1} F_{cb,t-1} + \frac{1+\kappa Q_{B,t}}{Q_{B,t-1}} Q_{B,t-1} B_{cb,t-1} - R_{t-1}^{re} RE_{t-1}$$

But then using the balance sheet condition to sub out reserves, and defining $R_t^F = \frac{1+\kappa Q_t}{Q_{t-1}}$ and $R_t^B = \frac{1+\kappa Q_{B,t}}{Q_{B,t-1}}$, we have:

$$P_t T_{cb,t} = \left(R_t^F - R_{t-1}^{re}\right) Q_{t-1} F_{cb,t-1} + \left(R_t^B - R_{t-1}^{re}\right) Q_{B,t-1} B_{cb,t-1}$$

Or, in real terms:

$$T_{cb,t} = \left(R_t^F - R_{t-1}^{re}\right) \Pi_t^{-1} Q_{t-1} f_{cb,t-1} + \left(R_t^B - R_{t-1}^{re}\right) \Pi_t^{-1} Q_{B,t-1} b_{cb,t-1} \tag{36}$$

In other words, the central bank earns spreads over the of cost funds on its asset holdings. This is remitted to the government each period, so that the central bank maintains no equity.

2.6 Fiscal Policy

The fiscal authority consumes, G_t, taxes the household, T_t, and issues debt, $B_{G,t}$. It also receives a lump sum transfer each period from the central bank, $T_{cb,t}$. Its flow budget constraint is:

$$P_t G_t + B_{G,t-1} = P_t T_t + P_t T_{cb,t} + Q_{B,t}(B_{G,t} - \kappa B_{G,t-1}) \tag{37}$$

Government debt has the same structure as private investment bonds, with coupon payouts decaying at κ. Government bonds trade at $Q_{B,t}$, which is not necessarily equal to Q_t (unlike in Carlstrom, Fuerst, and Paustian 2017). In this model, Ricardian Equivalence does not hold. So we have to make some assumptions on the path of government debt (i.e. it is not innocuous as in a standard model). I am going to assume that real government debt, where $b_{G,t} = \frac{B_t}{P_t}$, follows an exogenous AR(1) process. Lump sum taxes will then automatically adjust to make the government's budget constraint hold. We don't need to keep track of it. We assume that government spending itself follows an AR(1) in the log:

$$\ln G_t = (1 - \rho_G) \ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{38}$$

2.7 Financial Intermediaries

There are a fixed mass of intermediaries indexed by i. Intermediaries hold long-term private issued bonds and government bonds as well as bank reserves; and they finance themselves with their own equity as well as deposits. The balance sheet of a typical intermediary in nominal terms is:

$$Q_t F_{i,t} + Q_{B,t} B_{i,t} + RE_{i,t} = D_{i,t} + N_{i,t} \tag{39}$$

Each period, an exogenous fraction $1 - \sigma$ of intermediaries stochastically die. Upon death, they simply return their net worth to the household. The household replaces the dying intermediaries with the same number of new intermediaries, given these new intermediaries start-up net worth of X (distributed among all the new intermediaries evenly).

As long as it can earn excess returns (which it will given the constraints we shall introduce), it behooves an intermediary to not pay any dividends – it just wants to accumulate net worth until it stochastically exits. Net worth can be shown to evolve according to:

$$N_{i,t} = \left(R_t^F - R_{t-1}^d\right) Q_{t-1} F_{i,t-1} + \left(R_t^B - R_{t-1}^d\right) Q_{B,t-1} B_{i,t-1} + \left(R_{t-1}^{re} - R_{t-1}^d\right) RE_{i,t-1} + R_{t-1}^d N_{i,t-1} \tag{40}$$

If the intermediary earned no excess returns (i.e. none of the spreads were greater than zero), net worth would just grow at the cost of funds, the deposit rate. At this point the intermediary would be indifferent about accumulating net worth or paying it back to its owners (i.e. the household). But with excess returns, the intermediary is better off accumulating net worth to take advantage of lending spreads. As we shall see, the stochastic death assumption effectively makes intermediaries extra impatient and prevents them from accumulating enough net worth to overcome the limited enforcement constraint that we shall introduce below.

Consider an intermediary in period t. It needs to choose its balance sheet variables. Its objective is to maximize the expected value of *terminal net worth* – as noted above, the intermediary is just going to keep accumulating until it dies. Conditional on know it will survive from t into $t+1$, there is $1-\sigma$ probability that it dies in $t+1$. There is a $\sigma(1-\sigma)$ probability of exit in $t+2$ (i.e. a $1-\sigma$ probability of surviving past $t+1$, and a σ probability of exist in $t+2$. And so on. Accordingly, an intermediary's value function is:

$$V_{i,t} = \max \ (1-\sigma)\,\mathbb{E}_t \sum_{j=1}^{\infty} \sigma^{j-1} \Lambda_{t,t+j} n_{i,t+j} \tag{41}$$

Where $\Lambda_{t,t+j}$ is the household's stochastic discount factor and $n_{i,t} = N_{i,t}/P_t$ is real net worth. At the end of period t, before $t+1$, an intermediary can abscond with some of its assets and default. In particular, an intermediary can take $\theta_t Q_t f_{i,t}$ and $\theta_t \Delta Q_{B,t} b_{i,t}$, where $0 \leq \Delta \leq 1$. θ_t is also between zero and one, but is time-varying. We will consider this to be a credit shock. Equivalently, depositors (i.e. the household) can recover $1-\theta_t$ of private bonds and $1-\theta_t\Delta$ of government bonds. This enforcement constraint is relatively "tighter" for private bonds – it is "easier" for an intermediary to abscond with these relative to government bonds as long as $\Delta < 1$. The intermediary cannot abscond with reserves; these are perfectly recoverable by creditors in the event of default. The left hand side of (41) is the enterprise value of being an intermediary (i.e. the value of continuing). This enforcement constraint can be written:

$$V_{i,t} \geq \theta_t (Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t}) \tag{42}$$

Constraint (42) says that creditors will only allow intermediaries to borrow up until the point where it is not optimal for them to default.

Letting $\lambda_{i,t}$ denote the multiplier on the enforcement constraint, we have a Lagrangian in the recursive formulation of the value function:

$$\mathbb{L} = (1+\lambda_{i,t}) \left[(1-\sigma)\,\mathbb{E}_t\, \Lambda_{t,t+1} n_{i,t+1} + \sigma\,\mathbb{E}_t\, \Lambda_{t,t+1} V_{i,t+1} \right] - \lambda_{i,t}\theta_t \left(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t}\right)$$

Plugging in the evolution of real net worth, (40) divided through by P_t and writing all quantities in real terms, we have:

$$\mathbb{L} = (1+\lambda_{i,t})\,\mathbb{E}_t\left\{(1-\sigma)\Lambda_{t,t+1}\Big[\left(R^F_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t f_{i,t} + \left(R^B_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_{B,t}b_{i,t} + \right.$$

$$\left.\left(R^{re}_t-R^d_t\right)\Pi^{-1}_{t+1}re_{i,t} + R^d_t\Pi^{-1}_{t+1}n_{i,t}\Big] + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}V_{i,t+1}\right\} - \lambda_{i,t}\theta_t\left(Q_t f_{i,t} + \Delta Q_{B,t}b_{i,t}\right)$$

The derivatives of the Lagrangian are:

$$\frac{\partial\mathbb{L}}{\partial f_{i,t}} = (1+\lambda_{i,t})\left\{\mathbb{E}_t(1-\sigma)\Lambda_{t,t+1}\left(R^F_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\frac{\partial n_{i,t+1}}{\partial f_{i,t}}\right\} - \lambda_{i,t}\theta_t Q_t$$

$$\frac{\partial\mathbb{L}}{\partial b_{i,t}} = (1+\lambda_{i,t})\left\{\mathbb{E}_t(1-\sigma)\Lambda_{t,t+1}\left(R^B_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\frac{\partial n_{i,t+1}}{\partial b_{i,t}}\right\} - \lambda_{i,t}\theta_t\Delta Q_{B,t}$$

$$\frac{\partial\mathbb{L}}{\partial re_{i,t}} = (1+\lambda_{i,t})\left\{\mathbb{E}_t(1-\sigma)\Lambda_{t,t+1}\left(R^{re}_t-R^d_t\right)\Pi^{-1}_{t+1} + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\frac{\partial n_{i,t+1}}{\partial re_{i,t}}\right\}$$

Note that:

$$\frac{\partial n_{i,t+1}}{\partial f_{i,t}} = \left(R^F_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t$$

$$\frac{\partial n_{i,t+1}}{\partial b_{i,t}} = \left(R^B_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_{B,t}$$

$$\frac{\partial n_{i,t+1}}{\partial re_{i,t}} = \left(R^{re}_t-R^d_t\right)\Pi^{-1}_{t+1}$$

Plug these in and set to zero. We get:

$$(1+\lambda_{i,t})\,\mathbb{E}_t\left\{(1-\sigma)\Lambda_{t,t+1}\left(R^F_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\left(R^F_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_t\right\} = \lambda_{i,t}\theta_t Q_t$$

$$(1+\lambda_{i,t})\,\mathbb{E}_t\left\{(1-\sigma)\Lambda_{t,t+1}\left(R^B_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_{B,t} + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\left(R^B_{t+1}-R^d_t\right)\Pi^{-1}_{t+1}Q_{B,t}\right\} = \lambda_{i,t}\theta_t\Delta Q_{B,t}$$

$$(1+\lambda_{i,t})\,\mathbb{E}_t\left\{(1-\sigma)\Lambda_{t,t+1}\left(R^{re}_t-R^d_t\right)\Pi^{-1}_{t+1} + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}\frac{\partial V_{i,t+1}}{\partial n_{i,t+1}}\left(R^{re}_t-R^d_t\right)\Pi^{-1}_{t+1}\right\} = 0$$

Define:

$$\Omega_{i,t+1} = 1 - \sigma + \sigma \frac{\partial V_{i,t+1}}{\partial n_{i,t+1}} \tag{43}$$

We can then write these FOC as:

$$\mathbb{E}_t \, \Lambda_{t,t+1} \left(R^F_{t+1} - R^d_t \right) \Pi^{-1}_{t+1} \Omega_{i,t+1} = \frac{\lambda_{i,t}}{1+\lambda_{i,t}} \theta_t$$

$$\mathbb{E}_t \, \Lambda_{t,t+1} \left(R^B_{t+1} - R^d_t \right) \Pi^{-1}_{t+1} \Omega_{i,t+1} = \frac{\lambda_{i,t}}{1+\lambda_{i,t}} \Delta\theta_t$$

$$\mathbb{E}_t \, \Lambda_{t,t+1} \left(R^{re}_t - R^d_t \right) \Pi^{-1}_{t+1} \Omega_{i,t+1} = 0$$

Since R^{re}_t and R^d_t are known at the time expectations are formed, we can pull them out of the expectations operator and conclude:

$$R^{re}_t = R^d_t \tag{44}$$

For other part, we need to show that nothing depends on i for aggregation. *Guess* that the value function is linear in net worth:

$$V_{i,t} = a_t n_{i,t} \tag{45}$$

When the constraint binds, given this guess we have:

$$a_t n_{i,t} = \theta_t \left(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t} \right)$$

Define $\phi_{i,t}$ as a modified leverage ratio:

$$\phi_{i,t} = \frac{Q_t f_{i,t} + \Delta Q_{b,t} b_{i,t}}{n_{i,t}}$$

But this would imply: $a_t = \theta_t \phi_t$. Since we are guessing a_t doesn't vary with i, then neither can ϕ_t. So we have:

$$a_t = \phi_t \theta_t$$

If this is the case, then:

$$\Omega_t = 1 - \sigma + \sigma \phi_t \theta_t \tag{46}$$

Now write down the law of motion for net worth, led forward one period:

$$n_{i,t+1} = \Pi^{-1}_{t+1} \left[\left(R^F_{t+1} - R^d_t \right) Q_t f_{i,t} + \left(R^B_{t+1} - R^d_t \right) Q_{B,t} b_{i,t} + \left(R^{re}_t - R^d_t \right) re_{i,t} + R^d_t n_{i,t} \right]$$

Multiply both sides by $\Lambda_{t,t+1}\Omega_{t+1}$:

$$\Lambda_{t,t+1}\Omega_{t+1}n_{i,t+1} = \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left[\left(R_{t+1}^{F} - R_t^d\right)Q_t f_{i,t} + \left(R_{t+1}^{B} - R_t^d\right)Q_{B,t}b_{i,t} + \left(R_t^{re} - R_t^d\right)re_{i,t} + R_t^d n_{i,t}\right]$$

Now take expectations of both sides:

$$\begin{aligned}\mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}n_{i,t+1} = {} & \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{F} - R_t^d\right)Q_t f_{i,t} + \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{B} - R_t^d\right)Q_{B,t}b_{i,t} \\ & + \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_t^{re} - R_t^d\right)re_{i,t} + \mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1}R_t^d n_{i,t}\end{aligned}$$

Now where is this getting us? From above, we have:

$$\mathbb{E}_t\,\Lambda_{t,t+1}\left(R_{t+1}^{B} - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{i,t+1} = \Delta\,\mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{F} - R_t^d\right)$$

So plug this in. We get:

$$\begin{aligned}\mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}n_{i,t+1} = {} & \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{F} - R_t^d\right)Q_t f_{i,t} + \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{F} - R_t^d\right)\Delta Q_{B,t}b_{i,t} \\ & + \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_t^{re} - R_t^d\right)re_{i,t} + \mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1}R_t^d n_{i,t}\end{aligned}$$

We also know that $R_t^{re} = R_t^d$ from the FOC. Hence, we can write:

$$\mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}n_{i,t+1} = \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^{F} - R_t^d\right)\phi_t n_{i,t} + \mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1}R_t^d n_{i,t}$$

Now go back to the value function. We have:

$$V_{i,t} = (1-\sigma)\,\mathbb{E}_t\,\Lambda_{t,t+1}n_{i,t+1} + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}V_{i,t+1}$$

Now plug in our guess of the value function:

$$a_t n_{i,t} = (1-\sigma)\,\mathbb{E}_t\,\Lambda_{t,t+1}n_{i,t+1} + \sigma\,\mathbb{E}_t\,\Lambda_{t,t+1}a_t n_{i,t+1}$$

But this is:

$$a_t n_{i,t} = \mathbb{E}_t\,\Lambda_{t,t+1}n_{i,t+1}(1-\sigma+\sigma a_{t+1})$$

Which is:

$$a_t n_{i,t} = \mathbb{E}_t\,\Lambda_{t,t+1}n_{i,t+1}\Omega_{t+1}$$

But from above we know what $\mathbb{E}_t\,\Lambda_{t,t+1}n_{i,t+1}\Omega_{t+1}$ is:

$$a_t n_{i,t} = \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)\phi_t n_{i,t} + \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1} R_t^d n_{i,t}$$

The $n_{i,t}$ cancel out:

$$a_t = \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)\phi_t + \mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1} R_t^d$$

But given our guess, we have $a_t = \phi_t\theta_t$. So we have:

$$\phi_t\theta_t = \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)\phi_t + \mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1} R_t^d$$

Therefore:

$$\phi_t\left[\theta_t - \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)\right] = \mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1} R_t^d$$

So:

$$\phi_t = \frac{\mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1} R_t^d}{\theta_t - \mathbb{E}_t\, \Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)} \tag{47}$$

(47) is consistent with our guess – ϕ_t does not depend on anything firm specific, and hence neither does a_t. But then this means Ω_t really does not depend on anything firm specific, which then from the FOC means that $\lambda_{i,t} = \lambda_t$ and is the same across firms. The FOC taking all this into account may be written:

$$\mathbb{E}_t\, \Lambda_{t,t+1}\left(R_{t+1}^F - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = \frac{\lambda_t}{1+\lambda_t}\theta_t \tag{48}$$

$$\mathbb{E}_t\, \Lambda_{t,t+1}\left(R_{t+1}^B - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = \frac{\lambda_t}{1+\lambda_t}\Delta\theta_t \tag{49}$$

$$\mathbb{E}_t\, \Lambda_{t,t+1}\left(R_t^{re} - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = 0 \tag{50}$$

Before proceeding, note that we can combine (48) with (47) to write:

$$\phi_t = \frac{\mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1} R_t^d}{\theta_t - \frac{\lambda_t}{1+\lambda_t}\theta_t}$$

But this is:

$$\theta_t\phi_t = (1+\lambda_t)\,\mathbb{E}_t\, \Lambda_{t,t+1} R_t^d \Pi_{t+1}^{-1}\Omega_{t+1} \tag{51}$$

Suppose that the enforcement constraint were never binding. Then we would have $\lambda_t = 0$. So we could write (51) as:

$$\theta_t\phi_t = \mathbb{E}_t\, \Lambda_{t,t+1}\Pi_{t+1}^{-1} R_t^d\left(1 - \sigma + \sigma\theta_{t+1}\phi_{t+1}\right)$$

But then we can guess and verify that $\theta_t \phi_t = 1$ is a solution at all times, because:

$$1 = \mathbb{E}_t \, \Lambda_{t,t+1} \Pi_{t+1}^{-1} R_t^d$$

Which is just the household's first order condition for bonds. So, if $\lambda_t = 0$, then we have $a_t = \phi_t \theta_t = 1$. This makes sense – if the intermediaries are not constrained, then there are no excess returns to holding long bonds. Then net worth is just as valuable "inside" the firm as outside of it, i.e. $\frac{\partial V_{i,t}}{\partial n_{i,t}} = 1$. But if $\lambda_t > 0$, then we know that $\theta_t \phi_t > 1$ – i.e. net worth is worth more inside the firm than outside of it, because only inside the FI can long bonds be held and excess returns achieved.

2.8 Aggregation

Aggregate inflation evolves according to:

$$1 = (1 - \phi_p) \left(\Pi_t^{\#} \right)^{1-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p - 1} \tag{52}$$

Similarly, the aggregate real wage obeys:

$$w_t^{1-\epsilon_w} = (1 - \phi_w) \left(w_t^{\#} \right)^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{53}$$

To get the aggregate production function, integrate across retailers:

$$\int_0^1 Y_t(f) df = Y_t \int_0^1 \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon_p} df$$

Recall that retailers just repackage wholesale output. Hence, aggregate demand for retail output, $\int_0^1 Y_t(f) df$, just equals wholesale output, $Y_{w,t}$. So we have:

$$Y_{w,t} = Y_t v_t^p \tag{54}$$

$v_t^p = \int_0^1 \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon_p} df$ is a measure of price dispersion that may be written recursively using properties of Calvo pricing:

$$v_t^p = (1 - \phi_p) \left(\Pi_t^{\#} \right)^{-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p} v_{t-1}^p \tag{55}$$

Similarly, integrate demand the demand for union labor across unions:

$$\int_0^1 L_{d,t}(h) dh = L_{d,t} \int_0^1 \left(\frac{w_t(h)}{w_t} \right)^{-\epsilon_w} dh$$

Unions purchase labor from the household. Aggregate union labor demand, $\int_0^1 L_{d,t}(h) dh$, equals household labor supply, L_t. So we have:

$$L_t = L_{d,t} v_t^w \tag{56}$$

$v_t^w = \int_0^1 \left(\frac{w_t(h)}{w_t}\right)^{-\epsilon_w} dh$ is a measure of wage dispersion. It throws a wedge between household labor supply and labor that gets used in production. Using properties of Calvo wage-setting, this satisfies:

$$v_t^w = (1-\phi_w)\left(\frac{w_t^{\#}}{w_t}\right)^{-\epsilon_w} + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}(h)}{W_t}\right)^{-\epsilon_w} dh$$

Note because it is a ratio, I can (and am intentionally) using either the real wage, w_t or $w_t^{\#}$, and the nominal wage, W_t. But we can write:

$$v_t^w = (1-\phi_w)\left(\frac{w_t^{\#}}{w_t}\right)^{-\epsilon_w} + W_t^{\epsilon_w} W_{t-1}^{-\epsilon_w} \int_{1-\phi_w}^1 \left(\frac{W_{t-1}(h)}{W_{t-1}}\right)^{-\epsilon_w} dh$$

But this can be written:

$$v_t^w = (1-\phi_w)\left(\frac{w_t^{\#}}{w_t}\right)^{-\epsilon_w} + \left(\frac{w_t}{w_{t-1}}\right)^{\epsilon_w} \Pi_t^{\epsilon_w} \int_{1-\phi_w}^1 \left(\frac{W_{t-1}(h)}{W_{t-1}}\right)^{-\epsilon_w} dh$$

But via a law of large numbers, $\int_{1-\phi_w}^1 \left(\frac{W_{t-1}(h)}{W_{t-1}}\right)^{-\epsilon_w} dh = \phi_w v_{t-1}^w$. So we have:

$$v_t^w = (1-\phi_w)\left(\frac{w_t^{\#}}{w_t}\right)^{-\epsilon_w} + \phi_w \left(\frac{w_t}{w_{t-1}}\right)^{\epsilon_w} \Pi_t^{\epsilon_w} v_{t-1}^w \tag{57}$$

The FI balance sheet condition is linear in FI-specific variables. So it simply sums up to the same aggregate condition. Market-clearing for long-bonds requires that bonds issued by the wholesale firm and government, respectively, are either held by the central bank or the financial intermediaries:

$$f_{w,t} = f_t + f_{cb,t} \tag{58}$$

$$b_{G,t} = b_t + b_{cb,t} \tag{59}$$

Aggregate net worth evolves as follows. A fraction σ of intermediaries survive from $t-1$ to t. The typical such intermediary has real net worth:

$$n_{i,t} = P_t^{-1}\left[\left(R_t^F - R_{t-1}^d\right) Q_{t-1} F_{i,t-1} + \left(R_t^B - R_{t-1}^d\right) Q_{B,t-1} B_{i,t-1} + \left(R_{t-1}^{re} - R_{t-1}^d\right) RE_{i,t-1} + R_{t-1}^d N_{i,t-1}\right]$$

Each of the bond/net worth terms inside brackets needs to be divided by P_{t-1} to put in real terms. So, multiplying and dividing by P_{t-1}, we get:

$$n_{i,t} = \Pi_t^{-1}\left[\left(R_t^F - R_{t-1}^d\right) Q_{t-1} f_{i,t-1} + \left(R_t^B - R_{t-1}^d\right) Q_{B,t-1} b_{i,t-1} + \left(R_{t-1}^{re} - R_{t-1}^d\right) re_{i,t-1} + R_{t-1}^d n_{i,t-1}\right]$$

This is just linear in all variables the FI can choose. So we can sum this across FIs. Because those that die are randomly chosen, via a law of large numbers, the sum of surviving-FI variables (e.g. $n_{i,t}$) is just proportional the aggregate via σ. Newly borne intermediaries are given, in aggregate, X of real start-up net worth. Hence, aggregate real net worth evolves as a convex-combination of these:

$$n_t = \sigma \Pi_t^{-1} \left[\left(R_t^F - R_{t-1}^d \right) Q_{t-1} f_{t-1} + \left(R_t^B - R_{t-1}^d \right) Q_{B,t-1} b_{t-1} + \left(R_{t-1}^{re} - R_{t-1}^d \right) re_{t-1} + R_{t-1}^d n_{t-1} \right] + X \tag{60}$$

Before proceeding, it is worth pointing something out about a sort of unmodeled friction here. This unmodeled friction is that we are not allowing the household to choose the new equity transferred to intermediaries, X. Given that the intermediaries earn excess returns on long bonds that the household cannot directly access, it would be optimal for the households to transfer more equity to intermediaries each period than they do. We are assuming, implicitly, that something stops this from happening.

The limited enforcement constraint will bind in the region of the steady state we are interest in. This requires that $V_{i,t} = a_t n_{i,t} = \phi_t \theta_t \geq \theta \left(Q_t f_{i,t} + \Delta Q_{B,t} b_{i,t} \right)$. Summing across intermediaries gives the aggregate version of the constraint:

$$\phi_t = \frac{Q_t f_t + \Delta Q_{B,t} b_t}{n_t} \tag{61}$$

The aggregate resource constraint is messy. First of all, profit remitted investment firms is straightforward:

$$div_{k,t} = p_t^k \widehat{I}_t - I_t \tag{62}$$

Nominal dividends from a typical labor union are:

$$DIV_{L,t}(h) = W_t(h)^{1-\epsilon_w} W_t^{\epsilon_w} L_{d,t} - MRS_t \left(\frac{W_t(h)}{W_t} \right)^{-\epsilon_w} L_{d,t}$$

Integrate across unions to get the aggregate:

$$DIV_{L,t} = W_t^{\epsilon_w} L_{d,t} \int_0^1 W_t(h)^{1-\epsilon_w} dh - MRS_t L_{d,t} \int_0^1 \left(\frac{W_t(h)}{W_t} \right)^{-\epsilon_w} dh$$

The first integral on the RHS is $W_t^{1-\epsilon_w}$ (i.e. the definition of the price index), while the second integral is wage dispersion. So we get:

$$DIV_{L,t} = W_t L_{d,t} - MRS_t L_{d,t} v_t^w$$

But then dividing by P_t to put in real terms, and noting that $L_{d,t} v_t^w = L_t$ from above, we have:

$$div_{L,t} = w_t L_{d,t} - mrs_t L_t \tag{63}$$

Similarly, dividends from a typical retain firm are:

$$DIV_{R,t}(f) = P_t(f)^{1-\epsilon_p} P_t^{\epsilon_p} Y_t - P_{w,t} \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon_w} Y_t$$

Integrate across f to get the aggregate:

$$DIV_{R,t} = P_t^{\epsilon_p} Y_t \int_0^1 P_t(f)^{1-\epsilon_p} df - P_{w,t} Y_t \int_0^1 \left(\frac{P_t(f)}{P_t} \right)^{-\epsilon_w} df$$

But, similarly to above, the first integral is $P_t^{1-\epsilon_p}$, and the second is v_t^p. Then noting that $Y_t v_t^p = Y_{w,t}$, and dividing by P_t, we get real aggregate dividends from retail firms:

$$div_{R,t} = Y_t - p_{w,t} Y_{w,t} \tag{64}$$

The real dividend from the wholesale firm is:

$$div_{w,t} = p_{w,t} Y_{w,t} - w_t L_{d,t} - p_t^k \widehat{I}_t + Q_t f_{w,t} - \frac{1}{P_t} \left(1 + \kappa Q_t \right) F_{w,t-1}$$

Which may be written:

$$div_{w,t} = p_{w,t} Y_{w,t} - w_t L_{d,t} - p_t^k \widehat{I}_t + Q_t f_{w,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1} \tag{65}$$

The last part follows from multiplying and dividing the $F_{w,t-1}$ term by Q_{t-1} and P_{t-1}.

If we sum up dividends across the investment firm, labor unions, retail firms, and the wholesale firm, we get total non-financial firm dividends, i.e. sum (62-(65):

$$div_{NF,t} = p_t^k \widehat{I}_t - I_t + w_t L_{d,t} - mrs_t L_t + Y_t - p_{w,t} Y_{w,t} + \\ p_{w,t} Y_{w,t} - w_t L_{d,t} - p_t^k \widehat{I}_t + Q_t f_{w,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1}$$

Some stuff cancels, leaving:

$$div_{NF,t} = Y_t - I_t - mrs_t L_t + Q_t f_{w,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1} \tag{66}$$

$1 - \sigma$ of intermediaries exit each period and return their net worth to the household. This is just:

$$div_{FI,t} = (1-\sigma) \Pi_t^{-1} \left[R_t^F Q_{t-1} f_{t-1} + R_t^B Q_{B,t-1} b_{t-1} + R_{t-1}^{re} re_{t-1} - R_{t-1}^d d_{t-1} \right] \tag{67}$$

Total dividends are then the sum of dividends from non-financial and financial firms: $div_t = div_{NF,t} + div_{FI,t}$. Plug this into the household's budget constraint written in real terms:

$$C_t + d_t - R^d_{t-1}\Pi_t^{-1}d_{t-1} - mrs_t L_t + X + T_t =$$
$$Y_t - I_t - mrs_t L_t + Q_t f_{w,t} - R^F_t \Pi_t^{-1} Q_{t-1} f_{w,t-1} + (1-\sigma)\Pi_t^{-1}\left[R^F_t Q_{t-1} f_{t-1} + R^B_t Q_{B,t-1} b_{t-1} + R^{re}_{t-1} re_{t-1} - R^d_{t-1} d_{t-1}\right]$$

The $mrs_t L_t$ cancel, so we can write:

$$C_t + I_t + d_t - R^d_{t-1}\Pi_t^{-1}d_{t-1} + X + T_t =$$
$$Y_t + Q_t f_{w,t} - R^F_t \Pi_t^{-1} Q_{t-1} f_{w,t-1} + (1-\sigma)\Pi_t^{-1}\left[R^F_t Q_{t-1} f_{t-1} + R^B_t Q_{B,t-1} b_{t-1} + R^{re}_{t-1} re_{t-1} - R^d_{t-1} d_{t-1}\right]$$

Now, where is this getting us? From above, we know that X saatisfies:

$$X = n_t - \sigma\Pi_t^{-1}\left[R^F_t Q_{t-1} f_{t-1} + R^B_t Q_{B,t-1} b_{t-1} + R^{re}_{t-1} re_{t-1} - R^d_{t-1} d_{t-1}\right]$$

But then if we plug this in to the household's constraint, we get:

$$C_t + I_t + d_t - R^d_{t-1}\Pi_t^{-1}d_{t-1} + n_t + T_t =$$
$$Y_t + Q_t f_{w,t} - R^F_t \Pi_t^{-1} Q_{t-1} f_{w,t-1} + \Pi_t^{-1}\left[R^F_t Q_{t-1} f_{t-1} + R^B_t Q_{B,t-1} b_{t-1} + R^{re}_{t-1} re_{t-1} - R^d_{t-1} d_{t-1}\right]$$

Which simplifies somewhat to:

$$C_t + I_t + d_t + n_t + T_t =$$
$$Y_t + Q_t f_{w,t} - R^F_t \Pi_t^{-1} Q_{t-1} f_{w,t-1} + \Pi_t^{-1}\left[R^F_t Q_{t-1} f_{t-1} + R^B_t Q_{B,t-1} b_{t-1} + R^{re}_{t-1} re_{t-1}\right]$$

Now, note that we can write the government's budget constraint, in real terms, as:

$$T_t = G_t + (1 + \kappa Q_{B,t})\frac{B_{G,t-1}}{P_t} - T_{cb,t} - Q_{B,t} b_{G,t}$$

Which can be written:

$$T_t = G_t + R^B_t \Pi_t^{-1} Q_{B,t-1} b_{G,t-1} - Q_{b,t} b_{G,t} - T_{cb,t}$$

The central bank's transfer is:

$$T_{cb,t} = R^F_t \Pi_t^{-1} Q_{t-1} f_{cb,t-1} + R^B_t \Pi_t^{-1} Q_{B,t-1} b_{cb,t-1} - R^{re}_{t-1}\Pi_t^{-1} re_{t-1}$$

Combining these, we get:

$$T_t = G_t + R_t^B \Pi_t^{-1} Q_{B,t-1} b_{G,t-1} - Q_{b,t} b_{G,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{cb,t-1} - R_t^B \Pi_t^{-1} Q_{B,t-1} b_{cb,t-1} + R_{t-1}^{re} \Pi_t^{-1} re_{t-1}$$

Now plug this into the household's budget constraint:

$$\begin{aligned} C_t + I_t + G_t + d_t + n_t + G_t + R_t^B \Pi_t^{-1} Q_{B,t-1} b_{G,t-1} - Q_{b,t} b_{G,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{cb,t-1} - R_t^B \Pi_t^{-1} Q_{B,t-1} b_{cb,t-1} + R_{t-1}^{re} re_{t-1} \\ = Y_t + Q_t f_{w,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1} + \Pi_t^{-1} \left[R_t^F Q_{t-1} f_{t-1} + R_t^B Q_{B,t-1} b_{t-1} + R_{t-1}^{re} re_{t-1} \right] \end{aligned}$$

Which can be written further:

$$\begin{aligned} C_t + I_t + G_t + d_t + n_t + G_t + R_t^B \Pi_t^{-1} Q_{B,t-1} b_{G,t-1} - Q_{b,t} b_{G,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{cb,t-1} - R_t^B \Pi_t^{-1} Q_{B,t-1} b_{cb,t-1} \\ = Y_t + Q_t f_{w,t} - R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1} + \Pi_t^{-1} \left[R_t^F Q_{t-1} f_{t-1} + R_t^B Q_{B,t-1} b_{t-1} \right] \end{aligned}$$

We can write this further by re-arranging terms:

$$\begin{aligned} C_t + I_t + G_t + d_t + n_t - Q_{b,t} b_{G,t} - Q_t f_{w,t} + R_t^B \Pi_t^{-1} Q_{B,t-1} b_{G,t-1} + R_t^F \Pi_t^{-1} Q_{t-1} f_{w,t-1} = \\ Y_t + \Pi_t^{-1} R_t^F Q_{t-1} (f_{t-1} + f_{cb,t-1}) + \Pi_t^{-1} R_t^B Q_{B,t-1} (b_{t-1} + b_{cb,t-1}) \end{aligned}$$

But since market-clearing requires $f_{w,t} = f_t + f_{cb,t}$ and $b_{G,t} = b_t + b_{cb,t}$, several terms drop out, leaving:

$$C_t + I_t + G_t + d_t + n_t - Q_{B,t} b_{G,t} - Q_t f_{w,t} = Y_t$$

But now plug in the balance sheet condition of the intermediaries:

$$C_t + I_t + G_t + d_t + (Q_t f_t + Q_{B,t} b_t + re_t - d_t) = Y_t + Q_{B,t} b_{G,t} + Q_t f_{w,t}$$

The d_ts cancel, and we get:

$$C_t + I_t + G_t + Q_t (f_t - f_{w,t}) + Q_{B,t} (b_t - b_{G,t}) + re_t = Y_t$$

But market-clearing requires that $f_t - f_{w,t} = -f_{cb,t}$ and $b_t - b_{G,t} = -b_{cb,t}$. Hence:

$$C_t + I_t + G_t - Q_t f_{cb,t} - Q_{B,t} b_{cb,t} + re_t = Y_t$$

But $re_t = Q_t f_{cb,t} + Q_{B,t} b_{cb,t}$. So we get a standard resource constraint:

$$Y_t = C_t + I_t + G_t \tag{68}$$

Aggregate productivity follows an AR(1) in the log:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{69}$$

Similarly, the credit shock follows an AR(1):

$$\ln \theta_t = (1-\rho_\theta)\ln\theta + \rho_\theta \ln\theta_{t-1} + s_\theta \varepsilon_{\theta,t} \tag{70}$$

3 All Equilibrium Conditions in One Place

- Household:

$$\Lambda_{t-1,t} = \frac{\beta\mu_t}{\mu_{t-1}} \tag{71}$$

$$\mu_t = \frac{1}{C_t - bC_{t-1}} - \beta b\,\mathbb{E}_t\,\frac{1}{C_{t+1}-bC_t} \tag{72}$$

$$\chi L_t^{\eta} = \mu_t mrs_t \tag{73}$$

$$1 = R_t^d\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1} \tag{74}$$

- Labor union:

$$w_t^{\#} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{f_{1,t}}{f_{2,t}} \tag{75}$$

$$f_{1,t} = mrs_t w_t^{\epsilon_w} L_{d,t} + \phi_w\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_w} f_{1,t+1} \tag{76}$$

$$f_{2,t} = w_t^{\epsilon_w} L_{d,t} + \phi_w\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_w - 1} f_{2,t+1} \tag{77}$$

- Investment Firm:

$$\widehat{I}_t = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \tag{78}$$

$$1 = p_t^k\left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right] + \mathbb{E}_t\,\Lambda_{t,t+1}p_{t+1}^k S'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \tag{79}$$

- Retail firm:

$$\Pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}\frac{x_{1,t}}{x_{2,t}} \tag{80}$$

$$x_{1,t} = p_{w,t}Y_t + \phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_p} x_{1,t+1} \tag{81}$$

$$x_{2,t} = Y_t + \phi_p\,\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{\epsilon_p - 1} x_{1,t+1} \tag{82}$$

- Wholesale firm:

$$w_t = (1-\alpha)p_{w,t}A_t(u_tK_t)^{\alpha}L_{d,t}^{-\alpha} \tag{83}$$

$$p_t^k M_{2,t}\delta'(u_t) = \alpha p_{w,t}A_t(u_tK_t)^{\alpha-1}L_{d,t}^{1-\alpha} \tag{84}$$

$$p_t^k M_{2,t} = \mathbb{E}_t\,\Lambda_{t,t+1}\left[\alpha p_{w,t+1}A_{t+1}(u_{t+1}K_{t+1})^{\alpha-1}u_{t+1}L_{d,t+1}^{1-\alpha} + (1-\delta(u_{t+1}))p_{t+1}^k M_{2,t+1}\right] \tag{85}$$

$$Q_tM_{1,t} = \mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1}\left[1+\kappa Q_{t+1}M_{1,t+1}\right] \tag{86}$$

$$\frac{M_{1,t}-1}{M_{2,t}-1} = \psi^{-1} \tag{87}$$

$$Y_{w,t} = A_t(u_tK_t)^{\alpha}L_{d,t}^{1-\alpha} \tag{88}$$

$$K_{t+1} = \widehat{I}_t + (1-\delta(u_t))K_t \tag{89}$$

$$\psi p_t^k\widehat{I}_t = Q_t\left(f_{w,t} - \kappa\Pi_t^{-1}f_{w,t-1}\right) \tag{90}$$

- Financial intermediary:

$$\mathbb{E}_t\,\Lambda_{t,t+1}\left(R_{t+1}^F - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = \frac{\lambda_t}{1+\lambda_t}\theta_t \tag{91}$$

$$\mathbb{E}_t\,\Lambda_{t,t+1}\left(R_{t+1}^B - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = \frac{\lambda_t}{1+\lambda_t}\Delta\theta_t \tag{92}$$

$$\mathbb{E}_t\,\Lambda_{t,t+1}\left(R_t^{re} - R_t^d\right)\Pi_{t+1}^{-1}\Omega_{t+1} = 0 \tag{93}$$

$$\Omega_t = 1-\sigma+\sigma\phi_t\theta_t \tag{94}$$

$$\phi_t = \frac{\mathbb{E}_t\,\Lambda_{t,t+1}\Pi_{t+1}^{-1}\Omega_{t+1}R_t^d}{\theta_t - \mathbb{E}_t\,\Lambda_{t,t+1}\Omega_{t+1}\Pi_{t+1}^{-1}\left(R_{t+1}^F - R_t^d\right)} \tag{95}$$

$$Q_tf_t + Q_{B,t}b_t + re_t = d_t + n_t \tag{96}$$

$$\phi_t = \frac{Q_tf_t + \Delta Q_{B,t}b_t}{n_t} \tag{97}$$

$$n_t = \sigma\Pi_t^{-1}\left[\left(R_t^F - R_{t-1}^d\right)Q_{t-1}f_{t-1} + \left(R_t^B - R_{t-1}^d\right)Q_{B,t-1}b_{t-1} + \left(R_{t-1}^{re} - R_{t-1}^d\right)re_{t-1} + R_{t-1}^d n_{t-1}\right] + X \tag{98}$$

- Central Bank

$$\ln R_t^{tr} = (1-\rho_r)\ln R^{tr} + \rho_r\ln R_{t-1}^{tr} + (1-\rho_r)\left[\phi_\pi(\ln\Pi_t - \ln\Pi) + \phi_y(\ln Y_t - \ln Y_{t-1})\right] + s_r\varepsilon_{r,t} \tag{99}$$

$$R_t^{re} = \max\left\{1, R_t^{tr}\right\} \tag{100}$$

$$f_{cb,t} = (1-\rho_f)f_{cb} + \rho_f f_{cb,t-1} + s_f\varepsilon_{f,t} \tag{101}$$

$$b_{cb,t} = (1-\rho_b)b_{cb} + \rho_b b_{cb,t-1} + s_b\varepsilon_{b,t} \tag{102}$$

$$Q_t f_{cb,t} + Q_{B,t} b_{cb,t} = re_t \tag{103}$$

- Aggregate conditions:

$$1 = (1-\phi_p)\left(\Pi_t^{\#}\right)^{1-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p - 1} \tag{104}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{105}$$

$$Y_{w,t} = Y_t v_t^p \tag{106}$$

$$v_t^p = (1-\phi_p)\left(\Pi_t^{\#}\right)^{-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p} v_{t-1}^p \tag{107}$$

$$L_t = L_{d,t} v_t^w \tag{108}$$

$$v_t^w = (1-\phi_w)\left(\frac{w_t^{\#}}{w_t}\right)^{-\epsilon_w} + \phi_w \left(\frac{w_t}{w_{t-1}}\right)^{\epsilon_w} \Pi_t^{\epsilon_w} v_{t-1}^w \tag{109}$$

$$f_{w,t} = f_t + f_{cb,t} \tag{110}$$

$$b_{G,t} = b_t + b_{cb,t} \tag{111}$$

$$Y_t = C_t + I_t + G_t \tag{112}$$

$$R_t^F = \frac{1+\kappa Q_t}{Q_{t-1}} \tag{113}$$

$$R_t^B = \frac{1+\kappa Q_{B,t}}{Q_{B,t-1}} \tag{114}$$

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{115}$$

$$\ln \theta_t = (1-\rho_\theta)\ln\theta + \rho_\theta \ln\theta_{t-1} + s_\theta \varepsilon_{\theta,t} \tag{116}$$

$$\ln G_t = (1-\rho_G)\ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{117}$$

$$\ln b_{G,t} = (1-\rho_B)\ln b_G + \rho_B \ln b_{G,t-1} + s_B \varepsilon_{B,t} \tag{118}$$

The list of endogenous variables is $\Big\{Y_t, C_t, I_t, G_t, L_t, L_{d,t}, \widehat{I}_t, Y_{w,t}, K_t, u_t, w_t, w_t^{\#}, mrs_t, \Pi_t, \Pi_t^{\#}, p_t^k,$ $p_{w,t}, f_{1,t}, f_{2,t}, x_{1,t}, x_{2,t}, v_t^p, v_t^w, Q_t, Q_{B,t}, R_t^F, R_t^B, R_t^d, R_t^{re}, R_t^{tr}, M_{1,t}, M_{2,t}, \mu_t, \Lambda_{t-1,t}, f_{w,t}, f_t, b_t, re_t, d_t, n_t$ $\lambda_t, \phi_t, \Omega_t, f_{cb,t}, b_{cb,t}, b_{G,t}, A_t, \theta_t\Big\}$. This is 48 equations and 48 variables. The exogenous shocks are $\varepsilon_{A,t}$, $\varepsilon_{\theta,t}$, $\varepsilon_{G,t}$, $\varepsilon_{B,t}$, $\varepsilon_{f,t}$, $\varepsilon_{b,t}$, and $\varepsilon_{r,t}$.

4 Steady State

We are going to focus on a zero inflation steady state. This means that $\Pi = 1$, so $\Pi^{\#} = 1$, $v^p = 1$, $v^w = 1$, and $w^{\#} = w$. $v^w = 1$ means that $L = L_d$ and $Y_w = Y$. Similarly, since the investment adjustment cost is irrelevant in the steady state, we have $\widehat{I} = I$. I will also normalize the model such that $L = 1$. I am also going to pick parameters to have steady state utilization be 1. The utilization adjustment cost is:

$$\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2 \tag{119}$$

Focusing first on the household block, we get:

$$\Lambda = \beta \tag{120}$$

Which implies:

$$R^d = R^{re} = R^{tr} = \beta^{-1} \tag{121}$$

I am going to target two spreads: sp_F is the private lending spread, $R^F - R^d$¡ and $sp_B = R^B - R^d$ is the government lending spread. I will choose $sp_F = 1.03^{1/4}$ and $sp_B = 1.01^{1/4}$, so that I am targeting steady state spreads of 300 and 100 basis points, respectively, at an annual frequency. This then gives me:

$$R^F = sp_F R^d \tag{122}$$

$$R^B = sp_B R^d \tag{123}$$

But this then gives us steady state long bond prices as functions of κ, which I set to be $1 - 40^{-1}$:

$$Q = \left(R^F - \kappa\right)^{-1} \tag{124}$$

$$Q_B = \left(R^B - \kappa\right)^{-1} \tag{125}$$

But now I can solve for M_1 since:

$$QM_1 = \beta(1 + \kappa QM_1)$$

Which implies:

$$M_1 = \frac{\beta}{Q(1 - \beta\kappa)} \tag{126}$$

But then we have M_2:

$$M_2 = 1 + \psi(M_1 - 1) \tag{127}$$

As in Sims and Wu (2020), I pick $\psi = 0.81$.

For both price and wage-setting, I am going to assume that $\epsilon_p = \epsilon_w = 11$. From the price-setting and wage setting conditions, I then get:

$$p_w = \frac{\epsilon_p - 1}{\epsilon_p} \tag{128}$$

$$mrs = \frac{\epsilon_w - 1}{\epsilon_w} w \tag{129}$$

(128) is steady state real marginal cost; equivalently, this is the inverse steady state markup of price over marginal cost. (129) tells us the wage the household receives is a markdown over the wage charged to the wholesale firm; the difference is captured by unions.

Given that I am normalizing $L_d = u = 1$, I can now solve for steady state capital from the capital Euler equation. First, note from the FOC for investment that $p^k = 1$. We then have:

$$M_2 = \beta \left[\alpha p_w K^{\alpha-1} + (1-\delta_0) M_2\right]$$

Which implies:

$$K = \left(\frac{\alpha p_w}{M_2 \left(\frac{1}{\beta} - (1-\delta_0)\right)} \right)^{\frac{1}{1-\alpha}} \tag{130}$$

It is useful to look at (130) and point out different distortions matter. First, $p_w < 1$, owing to monopoly power in price-setting, lowers steady state capital. Second, $M_2 > 1$, which comes about because of positive interest rate spreads making the loan in advance constraint binding for the wholesale firm, also results in too little steady state capital relative to what would be efficient.

Once I have K, I have $Y = Y_w$ as well as $I = \widehat{I}$ and $w = w^{\#}$:

$$Y = K^{\alpha} \tag{131}$$

$$I = \delta_0 K \tag{132}$$

$$w = (1-\alpha) p_w K^{\alpha} \tag{133}$$

Note that $\delta'(u) = \delta_1$. We need to pick δ_1 to be consistent with our normalization; δ_0 and δ_2 are free parameters. In particular, from the FOC for utilization, we must have:

$$\delta_1 = \frac{\alpha p_w K^{\alpha-1}}{M_2} \tag{134}$$

Let's assume that in steady state $G/Y = g$ (e.g. $g = 0.2$). Then we can solve for steady state consumption as:

$$C = (1-g)Y - I \tag{135}$$

But then we can solve for μ:

$$\mu = \frac{1}{C} \frac{1-\beta b}{1-b} \tag{136}$$

As noted above, we know that:

$$mrs = \frac{\epsilon_w - 1}{\epsilon_w} w \tag{137}$$

But then we can solve for χ to be consistent with the normalization of $L = 1$:

$$\chi = \mu mrs \tag{138}$$

We can now figure out how much debt the wholesale firm must float in steady state:

$$f_w = \frac{\psi I}{Q(1-\kappa)} \tag{139}$$

Let's suppose that the total size of the central bank's balance sheet is some fraction of output, say $bcs = 0.1Y$. This tells us steady state reserves, since that is the steady state balance sheet size. Suppose that some other fraction, $bcbGs$ of the central bank's balance sheet is held in government bonds. Let $bcbGs = 0.9$. This then gives us steady state central bank government debt holdings:

$$b_{cb} = \frac{bcbGs \times re}{Q_B} \tag{140}$$

But then we can determine central bank holdings of private bonds via the central bank budget constraint:

$$f_{cb} = \frac{re - Q_B b_{cb}}{Q} \tag{141}$$

Which then from the adding up constraint tells us how much private debt FIs must hold:

$$f = f_w - f_{cb} \tag{142}$$

Now suppose that the outstanding value of government debt is some fraction of GDP, by (e.g. 0.5). So we have:

$$b_G = \frac{by \times Y}{Q_B} \tag{143}$$

But then from the market-clearing constraint, we have government bonds held by the FI:

$$b = b_G - b_{cb} \tag{144}$$

Let's then target a total leverage ratio, lev, where lev is the ratio of total assets to net worth. I will use $lev = 5$. This implies a steady state value of net worth:

$$n = \frac{Qf + Q_B b + re}{lev} \tag{145}$$

We can then get steady state deposits from the FIs balance sheet condition:

$$d = Qf + Q_B b + re - n \tag{146}$$

Note that there is a restriction implied on Δ, which is the relative recoverability of government bonds to private bonds. From the FOC from the FI problem, in steady state we have:

$$\Delta = \frac{R^B - R^d}{R^F - R^d} \tag{147}$$

In other words, (147) tells us that Δ governs the relative spread between private and government bonds.

But now we can also get the steady state value of the modified leverage ratio, ϕ, given Δ:

$$\phi = \frac{Qf + \Delta Q_B b}{n} \tag{148}$$

Now we can get θ from the FOC giving us. This is more complicated than it looks because ϕ and θ show up in Ω. We have:

$$\phi\left(\theta - \beta(1-\sigma+\sigma\phi\theta)(R^F - R^d)\right) = 1 - \sigma + \sigma\phi\theta$$

I'm going to set $\sigma = 0.95$. This parameter governs how long FIs are expected to live. The above is now one equation in one unknown, θ. Multiply the LHS through:

$$\phi\theta - \phi\beta(1-\sigma)(R^F - R^d) - \beta\sigma\phi^2\theta(R^F - R^d) = 1 - \sigma + \sigma\phi\theta$$

Isolate the terms involving θ on the LHS:

$$\phi\theta - \beta\sigma\phi^2\theta(R^F - R^d) - \sigma\phi\theta = 1 - \sigma + \phi\beta(1-\sigma)(R^F - R^d)$$

Solving for θ:

$$\theta = \frac{1 - \sigma + \phi\beta(1-\sigma)(R^F - R^d)}{(1-\sigma)\phi - \beta\sigma\phi^2(R^F - R^d)} \tag{149}$$

Now, there is something useful to notice here. In particular, if there is no lending spread, then we get $\theta = 1/\phi$. This is useful because we know the firm's value function is proportional to net worth via $a = \theta\phi$. With no spread, then $a = 1$, which tells us that net worth is as valuable inside an FI as not. But with spreads, net worth is more valuable inside the firm than out.

We can then solve for the equity transfer given everything else we have found:

$$X = n - \sigma\left[(R^F - R^d)Qf + (R^B - R^d)Q_B b + R^d n\right]$$

We can finally solve for the steady state value of the multiplier on the limited enforcement constraint for the FIs:

$$\beta(R^F - R^d)(1 - \sigma + \sigma\phi\theta) = \frac{\lambda}{1+\lambda}\theta$$

So:

$$\lambda = \left(\frac{\theta}{\beta(R^F - R^d)(1 - \sigma + \sigma\phi\theta)} - 1 \right)^{-1} \tag{150}$$

Note if there is no spread, then the first term inside the parentheses goes to infinity, so $\lambda \to 0$. The bigger is the spread, the bigger is λ (i.e. the tighter is the constraint).

5 Calibration and Impulse Responses

I calibrate the model loosely following Sims and Wu (2020). I describe the parameterization here. I set $\beta = 0.995$, which implies a steady state real interest rate of 2 percent annualized. $\kappa = 1 - 40^{-1}$, implying a 10 year duration on corporate and government bonds. I set $\psi = 0.81$ – firms must finance 80 percent of their investment via issuing debt. $\epsilon_p = \epsilon_w = 11$, implying steady state price and wage markups of 10 percent. I set $\alpha = 1/3$. I set $\delta_0 = 0.025$ (this is steady state capital depreciation) and $\delta_2 = 0.01$, which implies rather volatile capital utilization. δ_1 is fixed to be consistent with the normalization of $u = 1$. I set the government spending share of output to $g = 0.2$ in steady state. The habit formation parameter is $b = 0.8$. The inverse Frisch elasticity, η, is 1. χ is chosen to be consistent with the normalization that $L = 1$ in steady state.

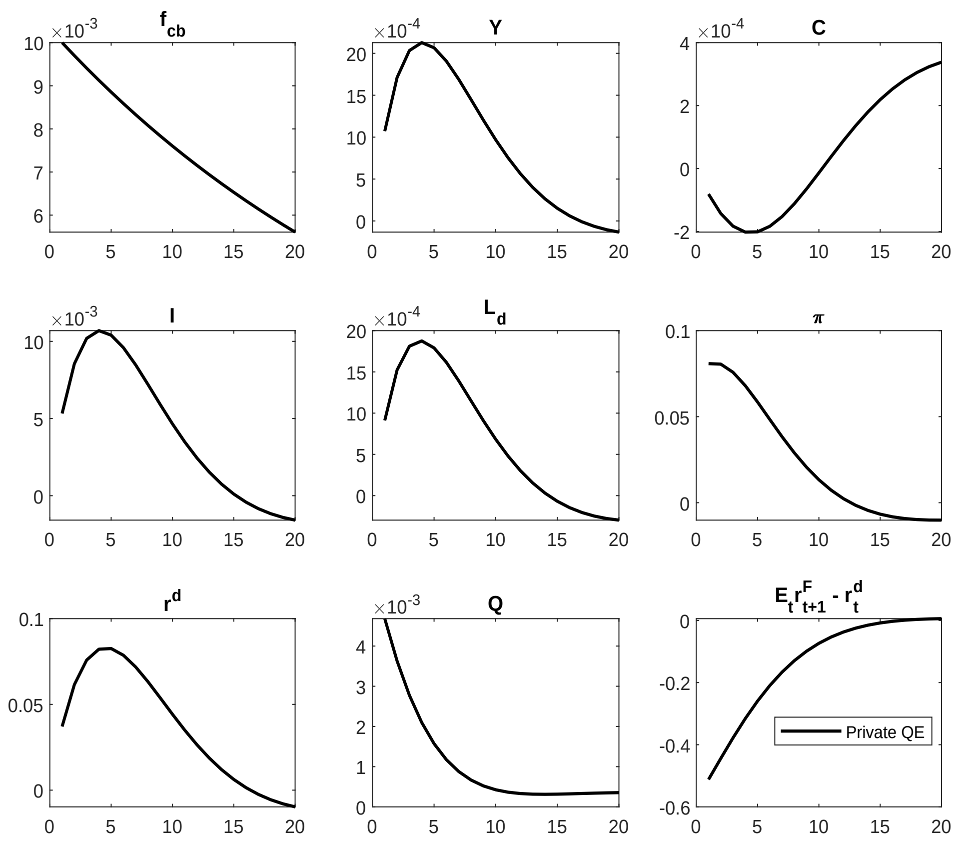
For financial variables, I set $\sigma = 0.95$. I target a total leverage ratio (the ratio of all assets to net worth in steady state, not the modified leverage ratio ϕ) to be 5. I assume that the central bank's steady state balance sheet is 10 percent of output, and that 90 percent of its assets are government bonds (so only a small fraction are corporate bonds). I assume that the steady state debt-GDP ratio for the fiscal authority is 50 percent. I target a corporate bond spread of 3 percent annualized, and a government bond spread of 1 percent annualized. Altogether, these targets imply values of X, θ, and Δ. In particular, I get $X = 0.0442$, $\theta = 0.6555$, and $\Delta = 0.33$. Concretely, this means that in default an intermediary may abscond with about two-thirds of its private assets and a little more than 20 percent of its government bonds. To put X into perspective, steady state net worth of intermediaries is $n = 3.75$. So the new equity infusion to new intermediaries is only about 1 percent of total equity.

I set the price and wage stickiness parameters to $\phi_p = \phi_w = 0.75$. This implies average four quarter durations between price/wage changes. The investment adjustment cost function is: $S\left(I_t/I_{t-1} - 1\right) = \frac{\psi_k}{2}\left(I_t/I_{t-1} - 1\right)^2$. I set $\psi_k = 2$. The parameters of the Taylor rule are $\rho_r = 0.8$, $\phi_\pi = 1.5$, and $\phi_y = 0.15$.

It remains to parameterize the shock prices. The shock standard deviations matter for unconditional moments but impulse responses are just scaled versions of the shock sizes. Consequently, I'm not going to focus here on trying to get the shock sizes correct to match any particular unconditional moments; rather I'm going to focus on impulse responses and how the model works. To be transparent, I just set all the shock standard deviations to 0.01. I'm going to set the AR(1) terms on government spending, government bonds, and productivity to be $\rho_G = \rho_B = \rho_A = 0.90$. I'm going to set the AR(1) on the credit shock variable to $\rho_\theta = 0.95$. Finally, I set the AR(1) parameter on exogenous private and government bond purchases by the central bank to $\rho_f = \rho_b = 0.97$. This is a good bit higher than what Sims and Wu (2020) use. The impulse responses I show are of logs of variables – so we can interpret units of things like output, consumption, and investment in percentage tersm. For inflation and interest, plotting the responses of logged gross rates gives the net rates (hence lowercase letters). Interest rates and inflation rate responses are multiplied by 400, to express them in annualized percentage terms.

First, let's look at the impulse responses to a private QE shock, by which I mean an exogenous increase in central bank corporate bond holdings. The responses are shown below.

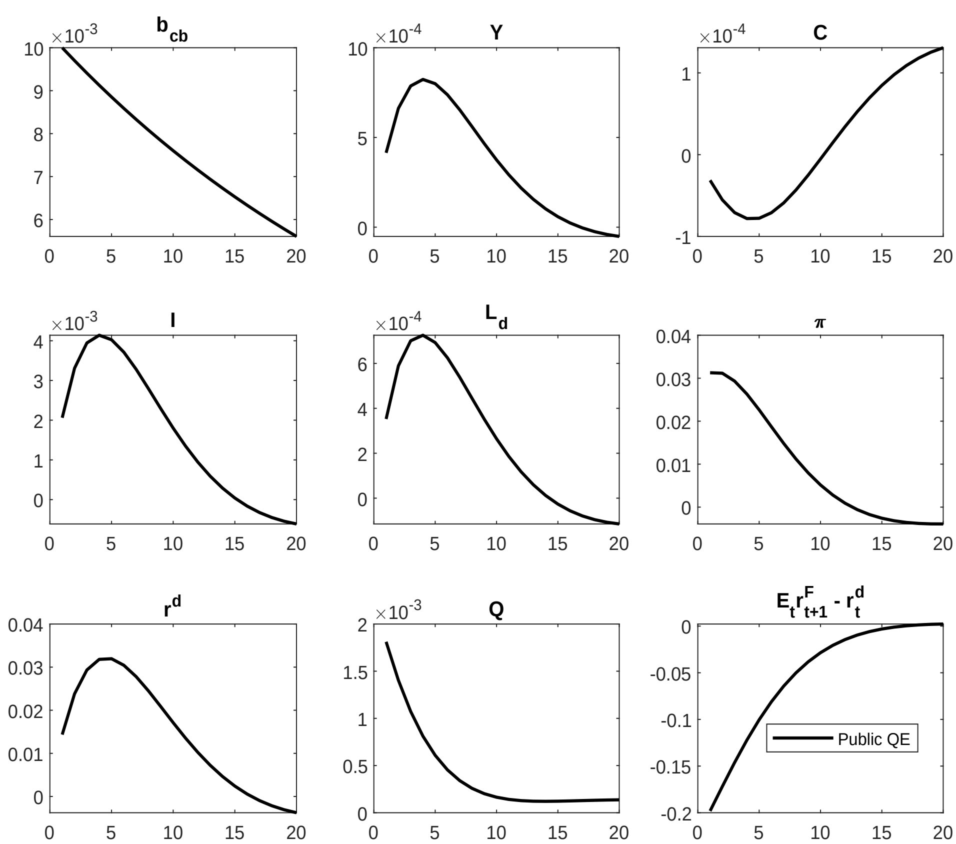
Figure 1: IRFs to Private QE Shock



The private QE shock causes output, investment, and labor input to rise. It also increases bond prices, Q_t, and pushes down the excess return on corporate bonds, $\mathbb{E}_t\, r^f_{t+1} - r^d_t$. Note that

r_t^F initially shoots way up – the immediate, surprise increase in Q_t massively increases the holding period return for those who held long bonds initially – but I'm plotting $\mathbb{E}_t\, r_{t+1}^F$, which is what is relevant for the decision of firms deciding how much debt to issue. Inflation rises – this shock is stimulated aggregate demand. The rise in inflation causes the central bank to raise the policy rate somewhat. Consumption initially declines and then rises. Effectively, the shock makes investment more attractive than consumption, so in general equilibrium consumption declines somewhat before turning positive. The intuition for these effects is pretty simple. Banks are balance sheet constrained. Purchasing corporate long bonds frees up space on the balance sheet for banks to buy more – indeed, it doesn't matter whether the central bank buys the bonds from the banks (as they do in practice) or directly from the wholesale firm. This pushes up the price of long bonds, which makes it easier for the wholesale firm to issue more debt given the loan in advance constraint it faces. The increase in investment drives aggregate demand up.

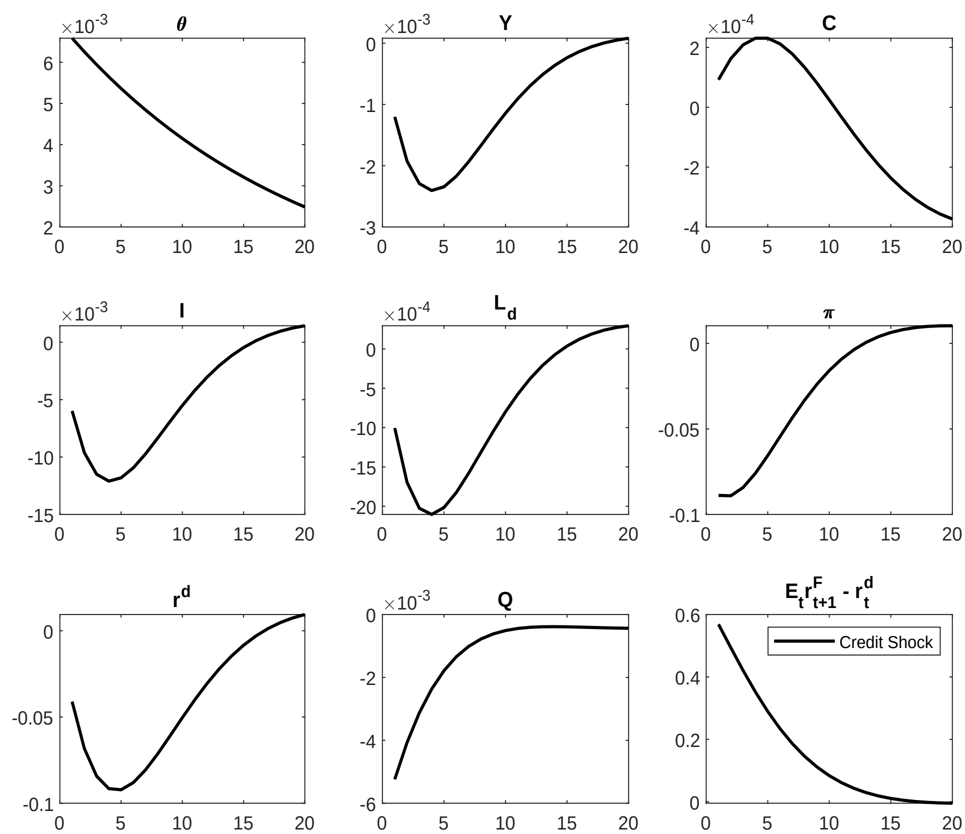
Figure 2: IRFs to Public QE Shock



Next consider a shock to public QE (i.e. central bank purchases of government bonds, financed via the creation of reserves). These responses are shown in the figure above. These are basically *the same* as the responses to the private QE shock, but *smaller*. They are smaller because buying the same amount of corporate bonds eases the banks' balance sheet constraints by less – by roughly a factor of Δ, which is about one-third. This means that QE focused on government bonds works in the same way as private bonds, but is less effective.

Next, consider the responses to the credit shock. These responses are shown below. These basically the inverse of a private or public QE shock. A credit shock exogenously tightens intermediary balance sheet constraints, which results in less lending and hence less investment and aggregate demand. A QE shock is kind of an inverse credit shock – it loosens intermediary balance sheet constraints. This forms the basis of the result, in for example Carlstrom, Fuerst, and Paustian (2017) or Sims and Wu (2019), that QE can completely undo the effects of credit shocks without changing the policy interest rate.

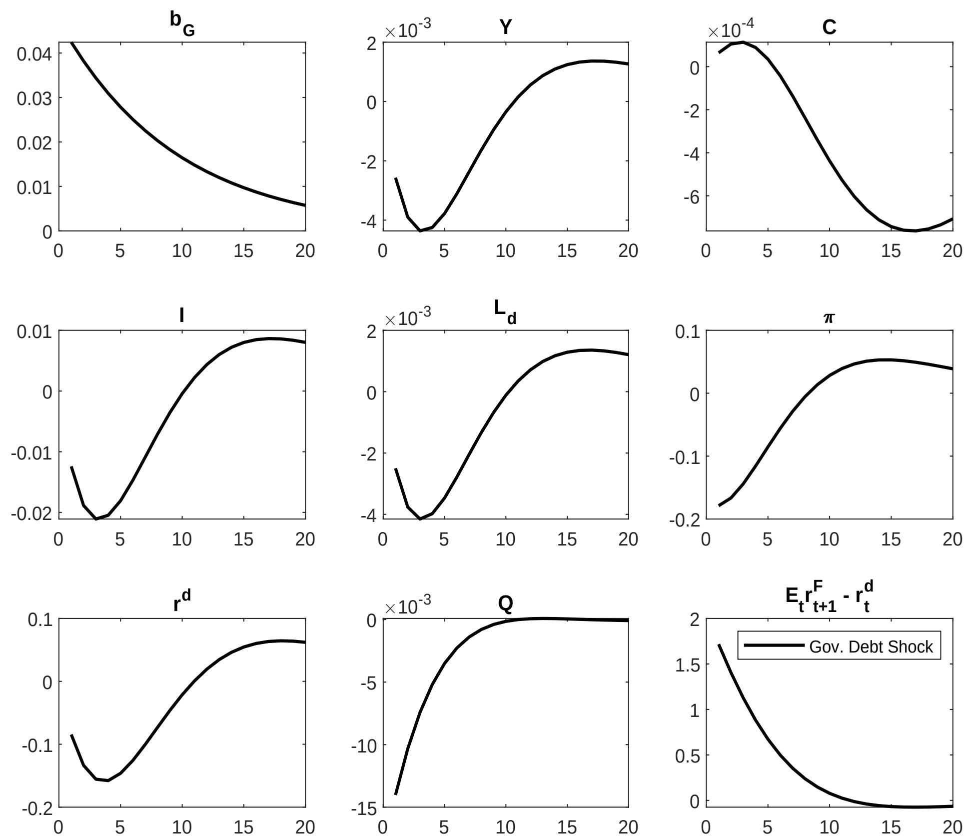
Figure 3: IRFs to Credit Shock



Next, I show impulse responses to an exogenous increase in government bonds issued by the

Treasury. In a standard model where Ricardian Equivalence holds, this would be completely irrelevant. But here issuing government debt is basically like a tightening of credit shocks or a negative QE shock. Since financial intermediaries must hold government bonds, issuing more of them *tightens* their balance sheet constraint and *crowds out* intermediary purchases of private debt. The corresponding lower bond price makes it more expensive for the wholesale firm to invest, and so aggregate demand contracts.

Figure 4: IRFs to Gov. Debt Shock



An important an interesting point here: public QE, private QE, a credit shock, and a government debt shock are all basically the same thing (albeit with different signs). In a standard model with frictionless financial markets, none of these shocks would matter. In this model, they all matter because they affect the extent to which intermediaries are balance-sheet constrained. Shocks that exogenously tighten this constraint are contractionary, and shocks which loosen the constraint are expansionary.

Next, I show response to the "non-financial shocks" in the model – government spending, productivity, and the conventional monetary shock. These look fairly similar to what one would get in a standard model. We can think about the "amplification" or "dampening" effects as being captured by the behavior of the excess return. The financial friction dampens the response to the government spending shock. Firms want to do less investment, which lowers the price of long bonds. But this tightens the balance sheet constraint of intermediaries, so we see the excess return rising, so on net output reacts less than it would without the financial friction. We see the reverse for the government spending shocks. Firms want to increase investment; this puts upward pressure on long bond prices. But this eases firm balance sheet constraints, which results in an increase in the supply of credit and a lower excess return. So, on net, output reacts more. A similar force is at play for the monetary shock (though the sign is reversed). Less demand for long bonds from firms doing less investment puts downward pressure on prices, but this tightens firm balance sheets, and results in an increase in excess returns, which makes the monetary shock more contractionary.

Figure 5: IRFs to Gov. Spending Shock

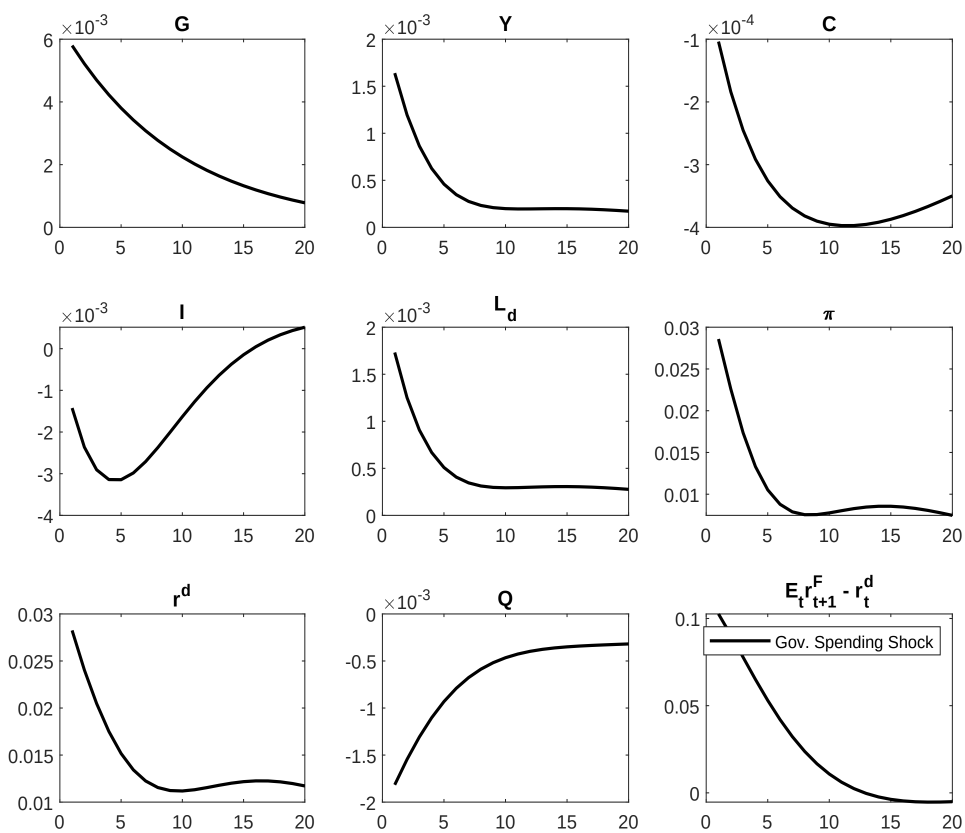


Figure 6: IRFs to Productivity Shock

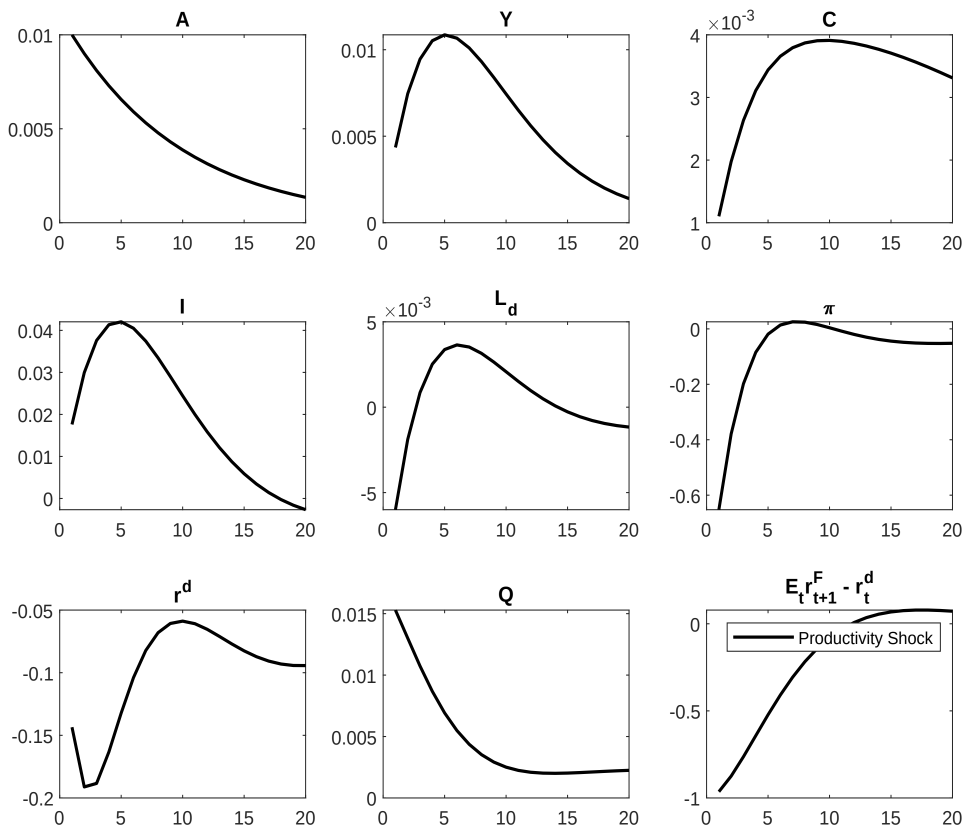
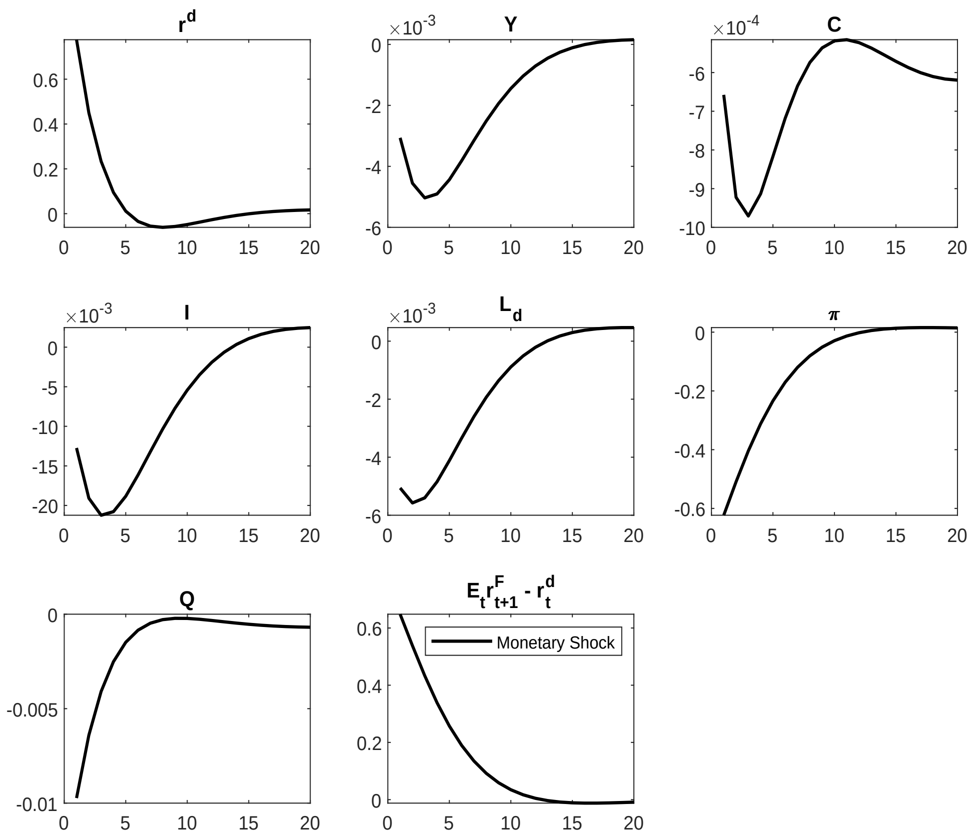


Figure 7: IRFs to Monetary Shock



6 The ZLB

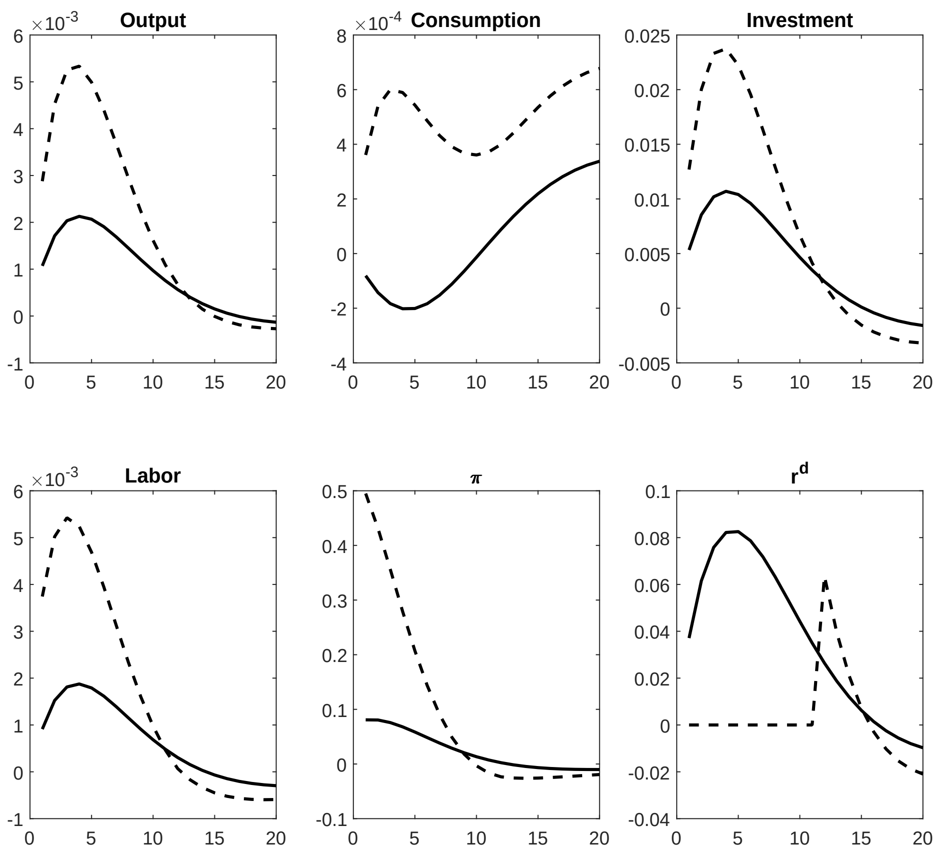
In practice, QE in the United States and other countries was not deployed to combat credit shocks per se, but rather to deal with monetary paralysis at the ZLB. In this section, I solve the model taking the ZLB into account.

I implement a ZLB constraint using “occbin,” which is a Dynare toolki developed in Guerrieri and Iacoviello (2015). The setup is quite general, and Iacoviello has pretty good documentation and codes available on his website. I will not go into much detail here. The basic idea is fairly standard. The base model is as we described above. There is a constraint that sometimes binds, in this case the ZLB constraint, where we do not allow the policy rate (the interest rate on reserves, which in equilibrium equals the deposit rate) to go below 1 in gross terms (or 0 in net terms). The algorithm solves and simulates two models. The first is the base model as described here. We then include a constraint condition, which is that whenever the *notional* interest rate, R_t^{tr}, goes below 1 (again, this in gross terms), we switch to an alternative model. In the alternative model, we fix the gross rate on reserves at 1, $R_t^{re} = 1$. If you tried to solve that model on its own, you would get indeterminacy – we know that an interest rate peg results in equilibrium indeterminacy. But since the economy initially goes back to the steady state which is away from the ZLB, the determinacy properties of the model are based on the base model. There are alternative ways to accomplish the same thing. For example, one can rig the code to fix the policy rate for a desired number of periods, after which time we go back to obeying the Taylor rule. Alternatively, one can include "monetary news shocks" to sterilize the effects of other shocks on the policy rate for a desired length of time. These different approaches will generate similar answers (though in some cases there are important differences that can arise when the interest rate peg length is deterministic vs. stochastic, see Carlstrom, Fuerst, and Paustian 2014).

To generate impulse responses using the occbin procedure, you need to pick some sequence of shocks to drive the economy to the constraint – i.e. to push the notional interest rate below unity in gross terms. Starting from the non-stochastic steady state, what I do is assume a sequence of large, positive credit shocks for 9 periods. Conditional on these shocks, I generate a simulation of the endogenous variables obeying the occasionally binding constraint. In practice, I generate a shock sequence that causes the ZLB to bind for about 10 periods, or 2.5 years. Then I do another simulation with the same sequence of shocks, but then in the 10th period add in a shock to one of the other exogenous variables. I repeat the simulation, again obeying the ZLB constraint. Then I take the difference between the simulation with the additional exogenous shock and the simulation without it. That is the impulse response function.

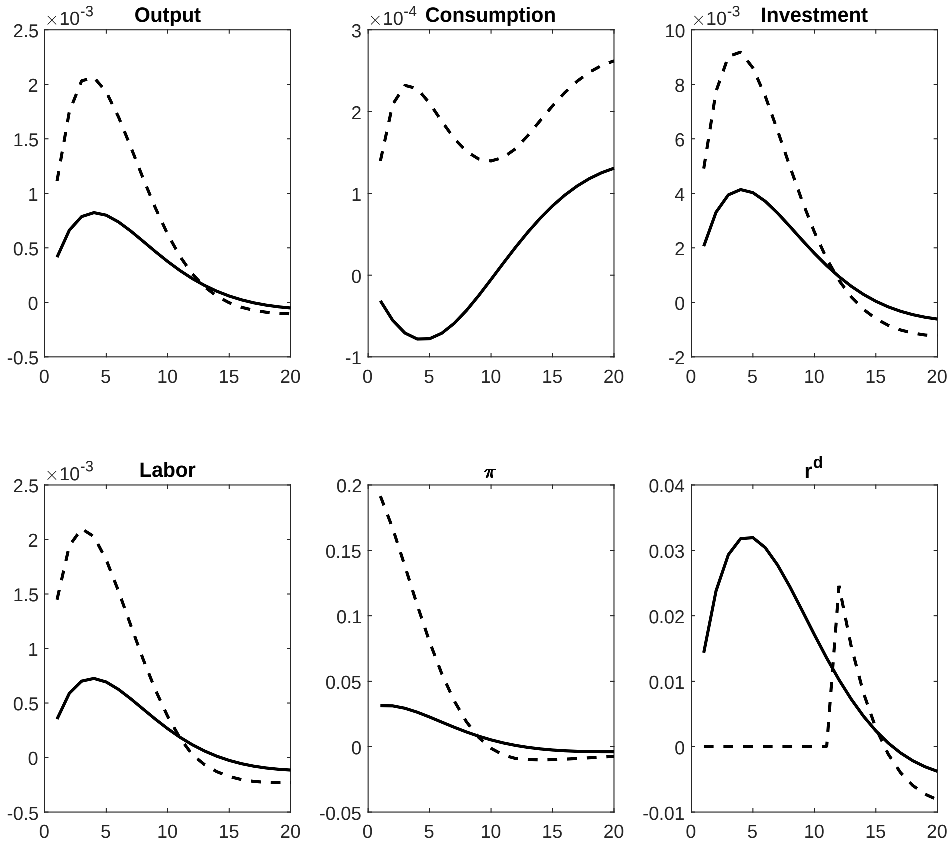
The figures below show impulse responses to a subset of the exogenous shocks in the model. Solid lines just repeat the responses I produced above. Dashed lines are responses obeying the ZLB. You can see in these simulations that the ZLB binds for about 10 periods.

Figure 8: IRFs to Private QE Shock, ZLB



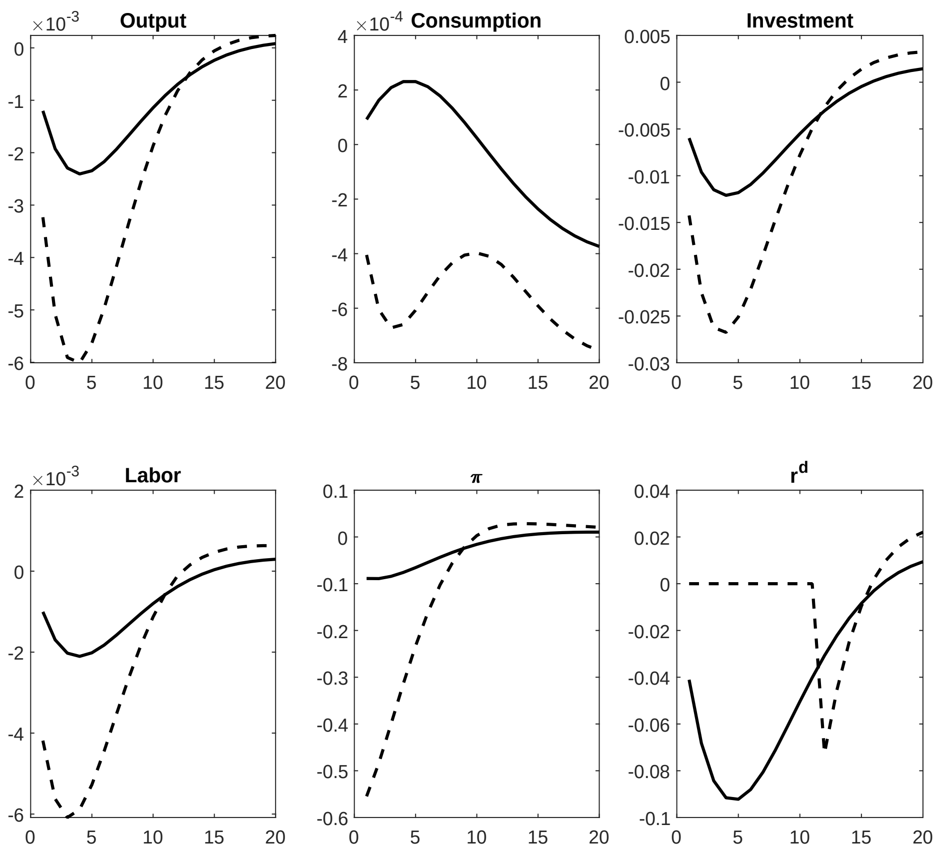
Consider first the effects a private QE shock. As shown above, this stimulates output even if the Taylor reacts so as to partially sterilize it. These effects are *magnified* at the ZLB. Output, consumption, investment, and labor input all react more during the periods the ZLB is active. Inflation also rises substantially more. This is sort of a general rule – the ZLB *amplifies* the effects of demand shocks (shocks which move output and inflation in the same direction). Under active monetary policy, higher inflation triggers a higher nominal rate, which partially sterilizes the effects on aggregate demand. At the ZLB, the inability of the nominal rate to adjust allows demand to expand more. But then there is a feedback effect: more aggregate demand means higher inflation, which with a fixed nominal rate, means even lower short-term real interest rates, and even more demand. The basic conclusion here is that QE is more powerful when the policy rate is stuck at the ZLB compared to normal times. The next figure shows IRFs to the public QE shock. Once again, these are basically the same as the private QE shock, only smaller. But it is still the case that the ZLB amplifies the effects of the QE shock.

Figure 9: IRFs to Public QE Shock, ZLB



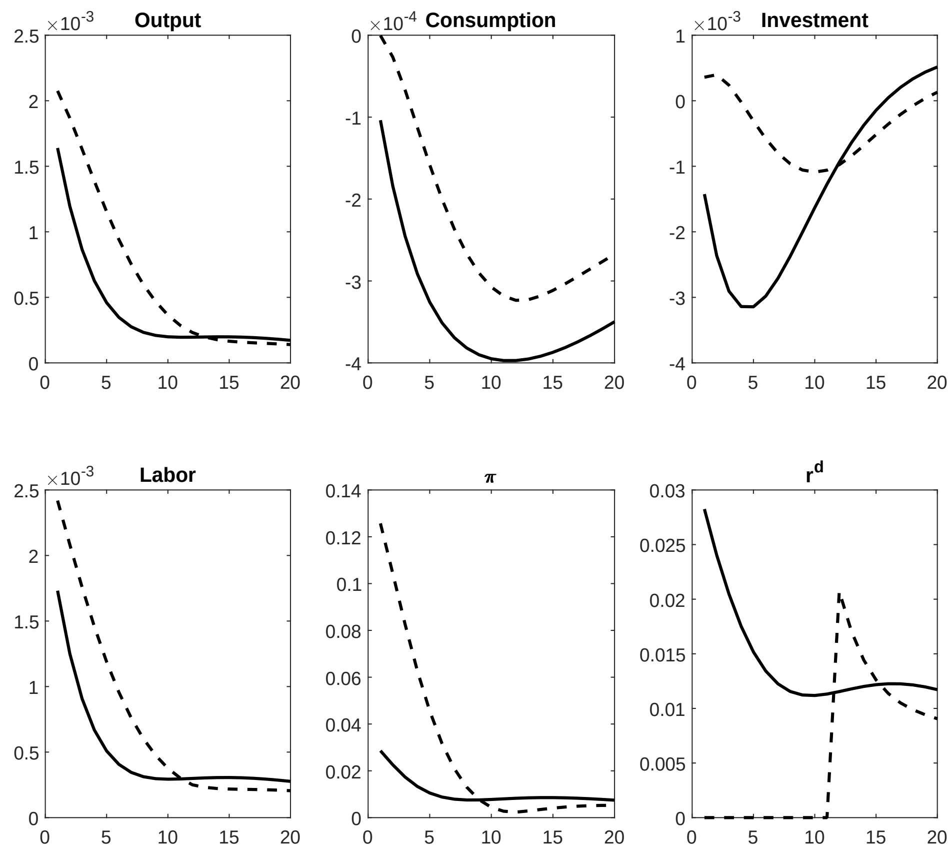
Next, I show impulse responses to the credit shock. Because this is an aggregate demand shock, the ZLB amplifies the effects. In ordinary times, the central bank following a Taylor rule would cut interest rates, which would work to essentially "soften the blow" of the shock. The inability to do so with inflation falling results in higher real rates, which further dampens demand (and hence inflation. The amplification effects here are pretty big – at peak, the output response to the financial shock is more than twice as big at the ZLB compared to normal times.

Figure 10: IRFs to Credit Shock, ZLB



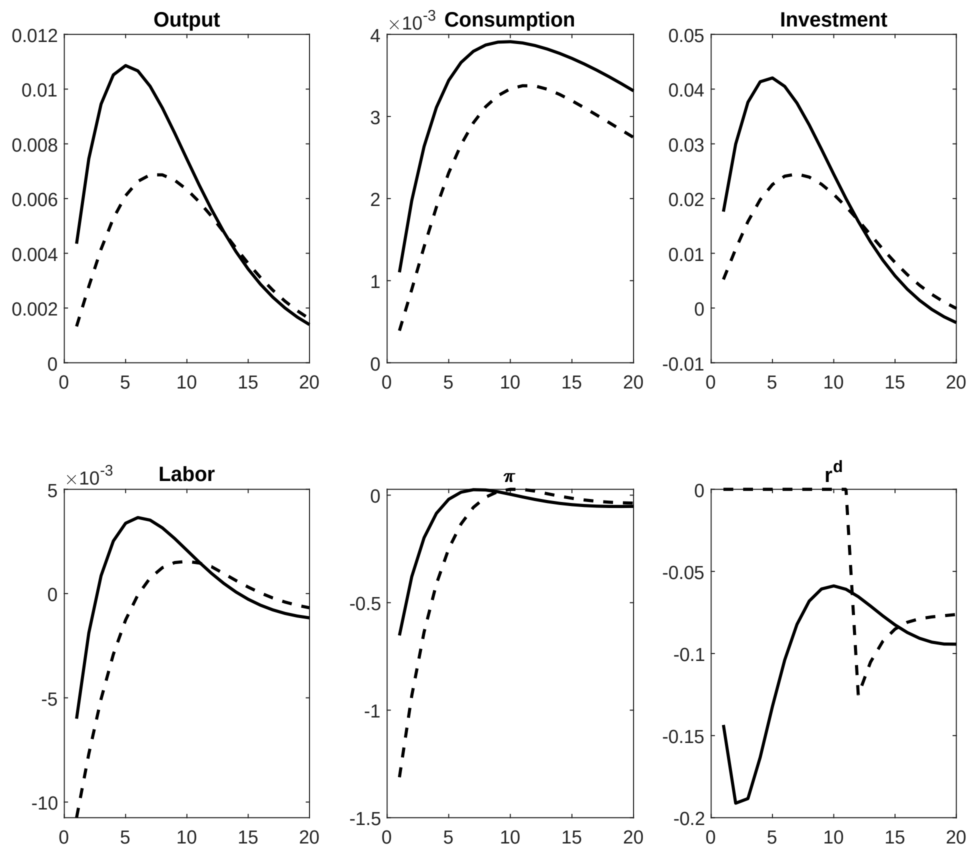
Next, I show impulse responses to a government spending shock. Here again, the ZLB amplifies the output effects, although the amplification is not that large. In particular, the government spending multiplier remains below 1 even with a ZLB last 2.5 years – I can see this by noting that both consumption and investment still decline (albeit not as much) at the ZLB.

Figure 11: IRFs to Gov. Spending Shock, ZLB



Finally, I show responses to the productivity shock. Here the effects of the ZLB are flipped. Output expands *less* when the policy rate is stuck at zero in comparison to normal times. The basic intuition is the behavior of inflation. When inflation falls but the policy rate can't react, the real short-term rate goes up. This chokes off demand, resulting in output and other variables reacting by less.

Figure 12: IRFs to Productivity Shock, ZLB



7 Endogenous QE at the ZLB

As noted above, QE was originally deployed as a *substitute* for conventional policy at the ZLB. How good of a substitute it is it?

The principal innovation of Sims and Wu (2020 is to study *endogenous* quantitative easing in response to other shocks, and ask the question how well this can substitute for conventional policy. In turn, this helps inform us about how costly the ZLB is (or isn't). In the baseline model above, I assumed that QE (whether public or private) follows an exogenous process. I shall continue to assume this so long as the ZLB is not binding. But when the ZLB binds, an *endogenous* component to the QE rule kicks in. Focusing for ease of exposition on private QE, suppose that QE obeys the following rule:

$$f_{cb,t} = (1-\rho_f)f_{cb} + \rho_f f_{cb,t-1} + s_f \varepsilon_{f,t} \quad \text{if } R_t^{tr} > 1$$
$$f_{cb,t} = (1-\rho_f)f_{cb} + \rho_f f_{cb,t-1} - \Psi_f(1-\rho_f)\left[\phi_\pi \ln \Pi_t + \phi_y(\ln Y_t - \ln Y_{t-1})\right] + s_f \varepsilon_{f,t} \quad \text{if } R_t^{tr} \leq 1$$

The idea here is fairly simple. When the notional Taylor rule rate is zero or negative in net terms, so $R_t^{tr} \leq 1$, an endogenous component to QE "kicks on" that looks qualitatively like the reaction in the basic Taylor rule. This is given by the term $-\Psi_f(1-\rho_f)\left[\phi_\pi \ln \Pi_t + \phi_y(\ln Y_t - \ln Y_{t-1})\right]$. The target variables are the same as the Taylor rule, and the ϕ_π and ϕ_y are the same as well. There is a negative sign outside – this reflects that purchasing bonds is equivalent to cutting the policy rate, so the QE rule during the ZLB needs to react the opposite way from how the standard Taylor rule would.

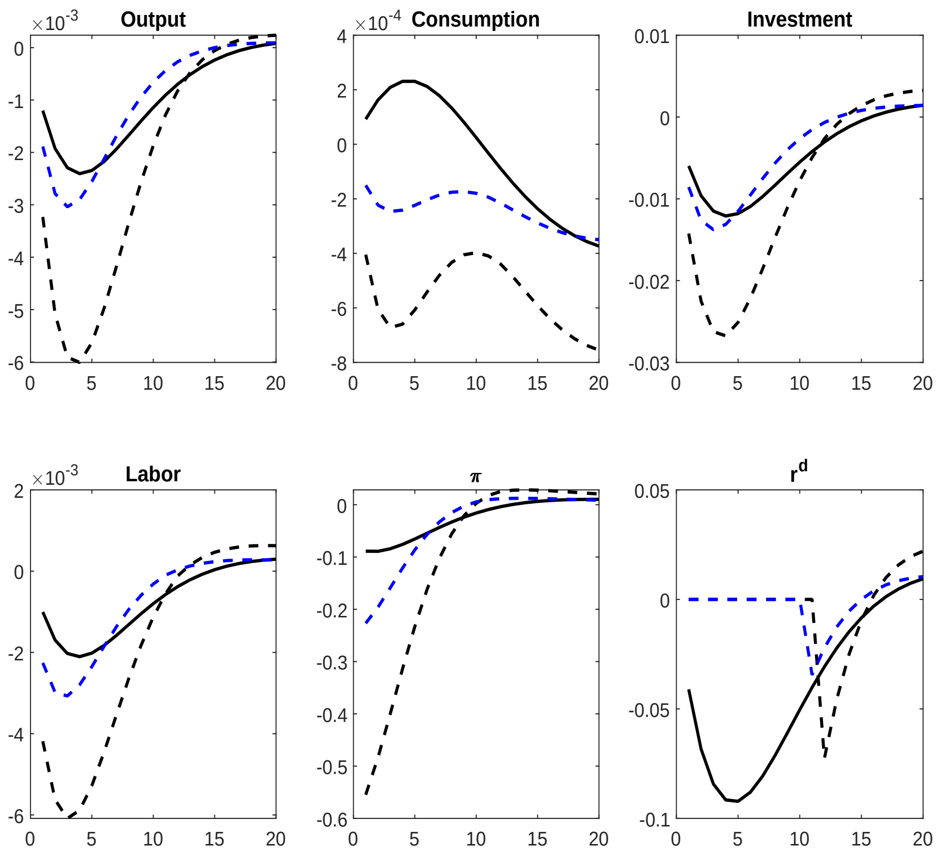
For the exercises that follow, I use a value of $\Psi_f = 47$. Sims and Wu (2020) use $\Psi_f = 7$. What accounts for the difference? Sims and Wu (2020) use $\rho_f = 0.8$, but for the purposes of this note I am using $\rho_f = 0.97$. $\Psi(1-\rho_f) = 1.4$ in both cases. Where does this conversion factor come from? As discussed in Sims and Wu (2020), it is how an exogenous QE shock needs to be scaled to generate roughly the same output response as to a conventional policy shock of 100 basis points (annualized).

To be clear, in my solution, the endogenous QE rule only turns on when the ZLB binds. Once the ZLB lifts, we go back to the base regime, and central bank bond-holdings just follow the exogenous AR(1) process (and hence eventually return back to zero).

In the figures below, I show responses of selected variables (i) ignoring the ZLB with exogenous central bank bond-holdings (solid lines), (ii) imposing the ZLB but no endogenous central bank bond-holdings (dashed black lines), and (iii) imposing the ZLB but allowing endogenous QE to turn on as in the rule specified below (dashed blue lines).

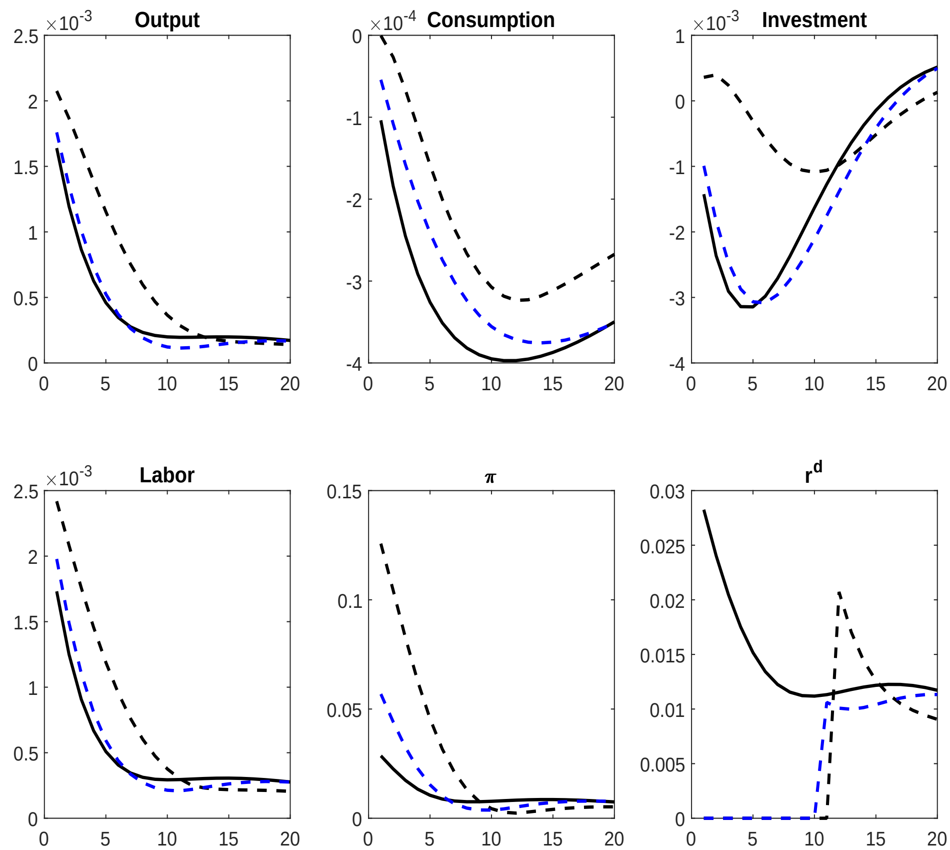
Consider first the credit shock. For output and investment, the dashed blue lines are very close to the solid black lines. There is much less output loss compared to the case of the ZLB and doing nothing at all. The inflation, labor, and consumption responses are also closer to the base case than the ZLB with exogenous QE case, though here the differences are more noticeable. This is something that we will see for all three shocks. Endogenous QE does a good job stabilizing output and investment, but not as good of a job getting consumption to mimic its patterns absent a ZLB. The reason for this is fairly straightforward. Investment depends on the yield on the long term bond, and this is what QE is impacting. Investment accounts for most of the volatility in output. Consumption, in contrast, depends on the short-term rate via the Euler equation, and this is not being (directly) affected via endogenous QE. But, overall, we can observe that QE seems to be a very effective substitute for conventional policy at the ZLB, at least condition on the credit shock.

Figure 13: IRFs to Credit Shock, ZLB with Endogenous QE



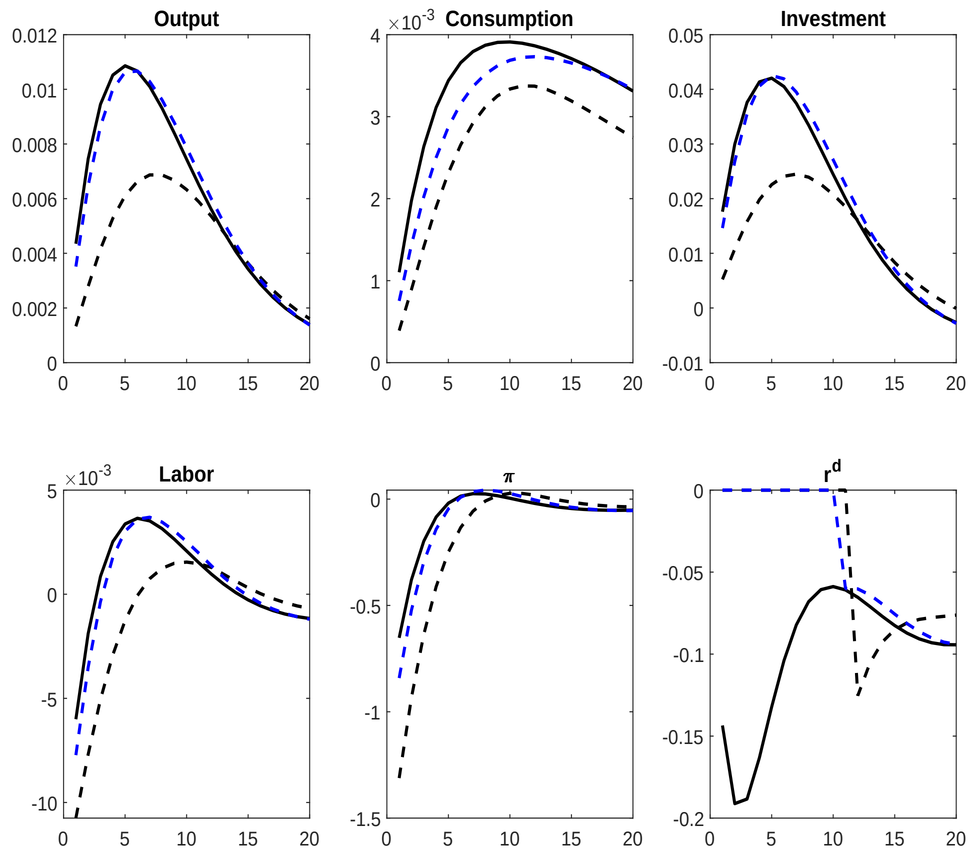
What about for other shocks? It might not be particularly surprising that QE is a good substitute for conventional policy conditional on credit shocks. As discussed above, and as discussed for example in Carlstrom, Fuerst, and Paustian (2017), QE can be used to completely sterilize credit shocks without changing the policy rate. So it's not surprising that this simple rule, restricted to look like the reaction part of a conventional Taylor rule, does pretty well conditioning on a credit shock. What about other shocks? Below are responses to a government spending shock. Here we observe that the dashed blue lines are very close to the solid black lines, and represent a significant improvement over the dashed black lines at the ZLB with no endogenous QE. So, here again, we would conclude that QE is an effective substitute for conventional policy.

Figure 14: IRFs to Gov. Spending Shock, ZLB with Endogenous QE



Finally, consider a productivity shock. The IRFs are shown below. Yet again, the dashed blue lines are very close to the solid black lines. It's quite clear that endogenous QE is an effective substitute for conventional policy.

Figure 15: IRFs to Productivity Shock, ZLB with Endogenous QE



Although Sims and Wu (2020) also discuss negative interest rate policy and forward guidance, really their paper is mostly about QE and in particularly endogenous QE. The results above echo their main point: QE can be a very effective substitute for conventional short-term policy rate adjustments at the ZLB. Among other things, this calls into question the overall cost of the ZLB.