Graduate Macro Theory II: A New Keynesian Model with Both Price and Wage Stickiness

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1 Introduction

This set of notes augments the basic NK model to include nominal wage rigidity. Wage rigidity is introduced in an analogous way to price rigidity via the Calvo (1983) staggered pricing assumption, which facilitates aggregation. As with price-setting, to get wage-setting we need to introduce some kind of monopoly power in wage-setting. To do this we assume that households supply differentiated labor. This imperfect substitutability between types of labor gives them some market power, and allows us to think about the consequences of wage stickiness.

2 Production

The production side of the economy is basically identical to what we had in the basic New Keynesian model, and as such we discuss it first. Production is split into two sectors: a representative competitive final goods firm, and a continuum of monopolistically competitive intermediate goods firms who have pricing power but are subject to price stickiness via the Calvo (1983) assumption.

2.1 Final Goods Sector

The final output good is a CES aggregate of a continuum of intermediates:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} \right)^{\frac{\epsilon_p}{\epsilon_p - 1}} \tag{1}$$

Here $\epsilon_p > 1$. I index it by p because we'll have a similar parameter at play when it comes to wage stickiness. Profit maximization by the final goods firm yields a downward-sloping demand curve for each intermediate:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t \tag{2}$$

This says that the relative demand for the j^{th} intermediate is a function of its relative price, with ϵ_p the price elasticity of demand. The price index (derived from the definition of nominal output as the sum of prices times quantities of intermediates) can be seen to be:

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon_p} dj \right)^{\frac{1}{1-\epsilon_p}} \tag{3}$$

2.2 Intermediate Producers

A typical intermediate producers produces output according to a constant returns to scale technology in labor, with a common productivity shock, A_t:

$$Y_t(j) = A_t N_t(j) \tag{4}$$

Intermediate producers face a common wage. They are not freely able to adjust price so as to maximize profit each period, but will always act to minimize cost. The cost minimization problem is to minimize total cost subject to the constraint of producing enough to meet demand:

$$\min_{N_t(j)} \quad W_t N_t(j)$$

s.t.

$$A_t N_t(j) \geq \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t$$

A Lagrangian is:

$$\mathcal{L} = -W_t N_t(j) + \varphi_t(j) \left(A_t N_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t \right)$$

The FOC is:

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} = 0 \Leftrightarrow W_t = \varphi_t(j) A_t$$

Or:

$$\varphi_t = \frac{W_t}{A_t} \tag{5}$$

Here I have dropped the j reference: marginal cost (φ_t) is equal to the wage divided by productivity, both of which are common to all intermediate goods firms.

Real flow profit for intermediate producer j is:

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j)$$

From (5), we know $W_t = \varphi_t A_t$. Plugging this into the expression for profits, we get:

$$\Pi_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j)$$

Where I have defined $mc_t \equiv \frac{\varphi_t}{P_t}$ as real marginal cost.

Firms are not freely able to adjust price each period. In particular, each period there is a fixed probability of $1 - \phi_p$ that a firm can adjust its price. This means that the probability a firm will be stuck with a price one period is ϕ_p, for two periods is ϕ_p^2, and so on. Consider the pricing problem of a firm given the opportunity to adjust its price in a given period. Since there is a chance that the firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic. Firms will discount profits s periods into the future by $\widetilde{M}_{t+s}\phi_p^s$, where $\widetilde{M}_{t+s} = \beta^s \frac{u'(C_{t+s})}{u'(C_t)}$ is the stochastic discount factor. The dynamic problem can be written:

$$\max_{P_t(j)} \quad E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{u'(C_{t+s})}{u'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} \right)$$

Here I have imposed that output will equal demand. Multiplying out, we get:

$$\max_{P_t(j)} \quad E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s \frac{u'(C_{t+s})}{u'(C_t)} \left(P_t(j)^{1-\epsilon_p} P_{t+s}^{\epsilon_p - 1} Y_{t+s} - mc_{t+s} P_t(j)^{-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} \right)$$

The first order condition can be written:

$$(1-\epsilon_p) P_t(j)^{-\epsilon_p} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s u'(C_{t+s}) P_{t+s}^{\epsilon_p - 1} Y_{t+s} + \epsilon_p P_t(j)^{-\epsilon_p - 1} E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s} = 0$$

Simplifying:

$$P_t(j) = \frac{\epsilon_p}{\epsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s}}{E_t \sum_{s=0}^{\infty} (\beta\phi_p)^s u'(C_{t+s}) P_{t+s}^{\epsilon_p - 1} Y_{t+s}}$$

First, note that since nothing on the right hand side depends on j, all updating firms will update to the same reset price, call it $P_t^{\#}$. We can write the expression more compactly as:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} \tag{6}$$

Here:

$$X_{1,t} = u'(C_t) mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta E_t X_{1,t+1} \tag{7}$$

$$X_{2,t} = u'(C_t)P_t^{\epsilon_p - 1}Y_t + \phi_p \beta E_t X_{2,t+1} \tag{8}$$

If $\phi_p = 0$, then the right hand side would reduce to $mc_t P_t = \varphi_t$. In this case, the optimal price would be a fixed markup, $\frac{\epsilon_p}{\epsilon_p - 1}$, over nominal marginal cost, φ_t.

3 Households

The new action related to wage stickiness is on the household side. To introduce wage stickiness in an analogous way to price stickiness, we need households to supply differentiated labor input, which gives them some pricing power in setting their own wage. In a similar way to the final goods firm, we introduce the concept of a labor "packer" (or union, if you like) which combines different types of labor into a composite labor good that it then leases to firms at wage rate W_t. We first consider the problem of the competitive labor packing firm, and then the problem of the household.

3.1 Labor Packer

Total labor input is equal to:

$$N_t = \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{9}$$

Here $\epsilon_w > 1$, and l indexes the differentiated labor inputs, which populate the unit interval. The profit maximization problem of the competitive labor packer is:

$$\max_{N_t(l)} \quad W_t \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_t(l) N_t(l) dl$$

The first order condition for the choice of labor of variety l is:

$$W_t \frac{\epsilon_w}{\epsilon_w - 1} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1} - 1} \frac{\epsilon_w - 1}{\epsilon_w} N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w} - 1} = W_t(l)$$

This can be simplified somewhat:

$$N_t(l)^{-\frac{1}{\epsilon_w}} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{1}{\epsilon_w - 1}} = \frac{W_t(l)}{W_t}$$

Or:

$$N_t(l) \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{-\frac{\epsilon_w}{\epsilon_w - 1}} = \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w}$$

Or:

$$N_t(l) = \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t \tag{10}$$

In a way exactly analogous to intermediate goods, the relative demand for labor of type l is a function of its relative wage, with elasticity ϵ_w. We can derive an aggregate wage index in a similar way to above, by defining:

$$W_t N_t = \int_0^1 W_t(l) N_t(l) dl = \int_0^1 W_t(l)^{1-\epsilon_w} W_t^{\epsilon_w} N_t dl$$

Or:

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(l)^{1-\epsilon_w} dl$$

So:

$$W_t = \left(\int_0^1 W_t(l)^{1-\epsilon_w} dl \right)^{\frac{1}{1-\epsilon_w}} \tag{11}$$

3.2 Households

Households are heterogenous and are indexed by $l \in (0, 1)$, supplying differentiated labor input to the labor packer above. I'm going to assume that preferences are additively separable in consumption and labor, which turns out to be somewhat important. If wages are subject to frictions like the Calvo (1983) pricing friction, households will charge different wages, meaning they will work different hours, meaning they will have different incomes and therefore different consumption and bond-holding decision. Erceg, Henderson, and Levin (2000, *JME*) show that if there exist state contingent claims that insure households against idiosyncratic wage risk, and if preferences are separable in consumption and leisure, households will be identical in their choice of consumption and bond-holdings, and will only differ in the wage they charge and labor supply. As such, in the notation below, I will suppress dependence on l for consumption and bonds, but leave it for wages and labor input. I also abstract from money altogether, noting that I could include real balances as a separable argument in the utility function without any effects on the rest of the model.

The household problem is:

$$\max_{C_t, N_t(l), W_t(l), B_{t+1}} \quad E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t(l)^{1+\eta}}{1+\eta} \right)$$

s.t.

$$P_t C_t + B_{t+1} \leq W_t(l) N_t(l) + \Pi_t + (1 + i_{t-1}) B_t$$

$$N_t(l) = \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t$$

P_t is the nominal price of goods, Π_t is nominal profit distributed from firms, B_t is the nominal stock of bonds which pay off in period t, which pay the nominal interest rate known in period

$t-1$. Imposing that labor supply exactly equal demand, which allows me to switch notation from choosing $N_t(l)$ to instead choosing $W_t(l)$, a Lagrangian is:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{\left(\left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t \right)^{1+\eta}}{1+\eta} + \lambda_t \left(W_t(l) \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t) + \Pi_t + (1+i_{t-1})B_t - P_t C_t - B_{t+1} \right) \right)$$

Let's take the FOC with respect to C_t and B_{t+1}.

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Leftrightarrow C_t^{-\sigma} = P_t \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \Leftrightarrow -\lambda_t + \beta E_t \lambda_{t+1}(1+i_t)$$

Combining these, we get:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma}(1+i_t)\frac{P_t}{P_{t+1}} \tag{12}$$

This is the standard Euler equation for bonds.

Now, let's think about wage setting. In writing the Lagrangian, I have eliminated $N_t(l)$ as a choice variable, instead writing the problem as choosing $W_t(l)$. As with prices, assume that households are not freely able to choose their wage each period. In particular, each period they face the probability $1-\phi_w$ of being able to adjust their wage. With probability ϕ_w they are stuck with a wage for one period, ϕ_w^2 for two periods, and so on. Before proceeding, let's re-write the problem in terms of choosing the real wage instead of the nominal wage. The reason we may want to do this is that, depending on the monetary policy rule, inflation could be non-stationary, which would make nominal wages non-stationary, but real wages stationary. Define the real wage a household charges as:

$$w_t(l) = \frac{W_t(l)}{P_t}$$

And similarly for the aggregate real wage:

$$w_t = \frac{W_t}{P_t}$$

Since both of these real wages are divided by the same price level, the relative demand for labor of variety l can be written either in terms of the ratio of nominal wages or the ratio of real wages, as these are equivalent.

Now, let's consider the problem of a household who can update its nominal wage in period t. The probability that nominal wage will still be operative in period $t+s$ is ϕ_w^s. The real wage a household charges in period $t+s$ if it is stuck with the nominal wage it choose in period t is:

$$w_{t+s}(l) = \frac{W_t(l)}{P_{t+s}}$$

This can be written in terms of the period t real wage as:

$$w_{t+s}(l) = \frac{W_t(l)}{P_t}\frac{P_t}{P_{t+s}}$$

Define $\Pi_{t,t+s} = \frac{P_{t+s}}{P_t}$ as the gross inflation between t and $t+s$. This is just equal to the product of period-over-period gross inflation. Define $\pi_t = \frac{P_t}{P_{t-1}} - 1$ as the period-over-period net inflation, we have:

$$\Pi_{t,t+s} = \prod_{m=1}^{s}(1+\pi_{t+m}) = \frac{P_{t+1}}{P_t} \times \frac{P_{t+2}}{P_{t+1}} \times \cdots \times \frac{P_{t+s}}{P_{t+s-1}} = \frac{P_{t+s}}{P_t}$$

This means that the real wage a household with a stuck nominal wage will charge in period $t+s$ can be written:

$$w_{t+s}(l) = w_t(l)\Pi_{t,t+s}^{-1}$$

Where w_t is the real wage chosen in period t.

Now, when choosing $w_t(l)$, households will discount the future not just by β^s but by ϕ_w^s as well, since the latter is the probability that a household will be stuck with that wage in period $t+s$. Reproducing just the parts of the Lagrangian that related to the choice of labor, we have:

$$\widetilde{\mathcal{L}} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left(-\psi \frac{\left(\frac{w_t(l)\Pi_{t,t+s}^{-1}}{w_{t+s}}\right)^{-\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{1+\eta} + \lambda_{t+s}P_{t+s}\left(w_t(l)\Pi_{t,t+s}^{-1}\left(\frac{w_t(l)\Pi_{t,t+s}^{-1}}{w_{t+s}}\right)^{-\epsilon_w} N_{t+s}\right)\right)$$

Note that the multiplier, λ_{t+s}, gets multiplied by P_{t+s} because I'm writing the wage in real terms here (so I'm de-facto multiplying and dividing by P_{t+s}). By multiplying out, this can be re-written:

$$\widetilde{\mathcal{L}} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left(-\psi \frac{w_t(l)^{-\epsilon_w(1+\eta)} w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{1+\eta} + \lambda_{t+s}P_{t+s}\left(w_t(l)^{1-\epsilon_w} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s}\right)\right)$$

The first order condition is:

$$\frac{\partial \widetilde{\mathcal{L}}}{\partial w_t(l)} = \epsilon_w w_t(l)^{-\epsilon_w(1+\eta)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta} + \dots$$

$$\dots (1-\epsilon_w) w_t(l)^{-\epsilon_w} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s} = 0$$

Simplifying:

$$\epsilon_w w_t(l)^{-\epsilon_w(1+\eta)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta} = (\epsilon_w - 1) w_t(l)^{-\epsilon_w} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s}$$

Or:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s}} \tag{13}$$

Above, I have gotten rid of the dependence on the l index, because everything on the right hand side is independent of l, meaning that all updating households will update to the same wage, which I call $w_t^{\#}$ or the reset wage. This can be written more compactly as:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{H_{1,t}}{H_{2,t}} \tag{14}$$

Where:

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta\phi_w E_t (1+\pi_{t+1})^{\epsilon_w(1+\eta)} H_{1,t+1} \tag{15}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w E_t (1+\pi_{t+1})^{\epsilon_w - 1} H_{2,t+1} \tag{16}$$

These lines follow because $\Pi_{t,t} = 1$, and $\Pi_{t,t+1} = (1+\pi_{t+1})$, so the $\Pi_{t,t+s}$ is effectively like an additional part of the discount factor, and $\lambda_t P_t = C_t^{-\sigma}$.

3.3 What if wages were flexible?

This FOC for labor input looks complicated, and in particular looks different than a "normal" static FOC for labor. To see that it's not so crazy, consider the case of wage flexibility, in which $\phi_w = 0$. Then the FOC would break down to:

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta}}{C_t^{-\sigma} w_t^{\epsilon_w} N_t}$$

If $\phi_w = 0$, then all firms update, so the reset wage is equal to the actual real wage: $w_t^{\#} = w_t$. This means:

$$w_t^{1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\psi N_t^{\eta} w_t^{\epsilon_w \eta}}{C_t^{-\sigma}}$$

Or:

$$w_t = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\psi N_t^{\eta}}{C_t^{-\sigma}}$$

Since $\epsilon_w > 1$, we have $\frac{\epsilon_w}{\epsilon_w - 1} > 1$. So what this says is that wage is a markup over the marginal rate of substitution between labor and consumption ($\frac{\psi N_t^{\eta}}{C_t^{-\sigma}}$ is the MRS). If $\epsilon_w \to \infty$, this would be *exactly* the FOC that we had in the flexible wage case.

4 Equilibrium and Aggregation

Assume that the central bank sets interest rates according to a Taylor Rule. In the Taylor rule I target only inflation, but it would be straightforward to also target output, the output gap, or output growth. As long as households get utility from real balances in an additively separable way, this will determine the price level and we can ignore money:

$$i_t = (1 - \rho_i) i + \rho_i i_{t-1} + \phi_\pi (\pi_t - \pi) + \varepsilon_{i,t} \tag{17}$$

Where again variables without time subscripts denote steady state values. $\varepsilon_{i,t}$ is a monetary policy shock. Productivity follows an AR(1) in the log:

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \tag{18}$$

In equilibrium, bond-holding is always zero: $B_t = 0$. Using this, the household budget constraint can be written in real terms:

$$C_t = \frac{W_t(l)}{P_t} N_t(l) + \frac{\Pi_t}{P_t} \tag{19}$$

Integrating over l:

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} N_t(l) dl + \frac{\Pi_t}{P_t}$$

Real dividends received by the household are just the sum of real profits from intermediate goods firms:

$$\frac{\Pi_t}{P_t} = \int_0^1 \left(\frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj$$

This can be written:

$$\frac{\Pi_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) - w_t \int_0^1 N_t(j) dj$$

Where above I used the definition that $w_t \equiv \frac{W_t}{P_t}$. Now, market-clearing requires that the sum of labor used by firms equals the total labor supplied by the labor packer, so $\int_0^1 N_t(j)dj = N_t$. Hence:

$$\frac{\Pi_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Plug this into the integrated household budget constraint:

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} N_t(l) dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Now plug in the demand for labor of type l:

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Simplify:

$$C_t = \frac{1}{P_t} W_t^{\epsilon_w} N_t \int_0^1 W_t(l)^{1-\epsilon_w} dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Now, using the aggregate (nominal) wage index, we know: $\int_0^1 W_t(l)^{1-\epsilon_w} dl = W_t^{1-\epsilon_w}$. Making this substitution:

$$C_t = \frac{1}{P_t} W_t^{\epsilon_w} N_t W_t^{1-\epsilon_w} + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Or:

$$C_t = \frac{W_t}{P_t} N_t + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t$$

Since $w_t = \frac{W_t}{P_t}$, we must have:

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj \tag{20}$$

In other words, consumption must equal the sum of real quantities of intermediates. Now, plug in the demand curve for intermediate variety j:

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t dj$$

Take stuff out of the integral where possible:

$$C_t = P_t^{\epsilon_p - 1} Y_t \int_0^1 P_t(j)^{1-\epsilon_p} dj$$

Now, from the definition of the aggregate price level, we have: $\int_0^1 P_t(j)^{1-\epsilon_p} = P_t^{1-\epsilon_p}$. This means the terms involving P_t cancel, so we're left with:

$$C_t = Y_t \tag{21}$$

Now, what is Y_t? From the demand for intermediate variety j, we have:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

Using the production function for each intermediate, this is:

$$A_t N_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t$$

Integrate over j:

$$\int_0^1 A_t N_t(j) dj = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t dj$$

Take stuff out of the integral, with the exception of the price level on the right hand side:

$$A_t \int_0^1 N_t(j) dj = Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj$$

Now define a new variable, v_t^p, as:

$$v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj \tag{22}$$

This is a measure of price dispersion. If there were no pricing frictions, all firms would charge the same price, and $v_t^p = 1$. If prices are different, one can show that this expression is bound from below by unity. Using the definition of aggregate labor input, we can therefore write:

$$Y_t = \frac{A_t N_t}{v_t^p} \tag{23}$$

This is the aggregate production function Since $v_t^p \geq 1$, price dispersion results in an output loss – you produce less output than you would given A_t and aggregate labor input if prices are disperse.

Since I've written the first order conditions for labor in terms of the real wage, let's re-write the aggregate nominal wage index in terms of real wages. Divide both sides by $P_t^{1-\epsilon_w}$:

$$\left(\frac{W_t}{P_t}\right)^{1-\epsilon_w} = \int_0^1 \left(\frac{W_t(l)}{P_t}\right)^{1-\epsilon_w} dl$$

Or:

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(l)^{1-\epsilon_w} dl \tag{24}$$

The full set of equilibrium conditions can then be characterized by:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma}(1+i_t)\frac{P_t}{P_{t+1}} \tag{25}$$

$$w_t^{\#,1+\epsilon_w\eta} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{H_{1,t}}{H_{2,t}} \tag{26}$$

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta\phi_w E_t (1+\pi_{t+1})^{\epsilon_w(1+\eta)} H_{1,t+1} \tag{27}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w E_t (1+\pi_{t+1})^{\epsilon_w - 1} H_{2,t+1} \tag{28}$$

$$mc_t = \frac{w_t}{A_t} \tag{29}$$

$$C_t = Y_t \tag{30}$$

$$Y_t = \frac{A_t N_t}{v_t^p} \tag{31}$$

$$v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj \tag{32}$$

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(l)^{1-\epsilon_w} dl \tag{33}$$

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj \tag{34}$$

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}\frac{X_{1,t}}{X_{2,t}} \tag{35}$$

$$X_{1,t} = C_t^{-\sigma} mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta E_t X_{1,t+1} \tag{36}$$

$$X_{2,t} = C_t^{-\sigma} P_t^{\epsilon_p - 1} Y_t + \phi_p \beta E_t X_{2,t+1} \tag{37}$$

$$i_t = (1-\rho_i)i + \rho_i i_{t-1} + \phi_\pi(\pi_t - \pi) + \varepsilon_{i,t} \tag{38}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \tag{39}$$

$$\pi_t = \frac{P_t}{P_{t-1}} - 1 \tag{40}$$

This is sixteen equations in sixteen aggregate variables:
$\left(C_t, i_t, P_t, w_t^{\#}, H_{1,t}, H_{2,t}, w_t, N_t, \pi_t, mc_t, A_t, Y_t, v_t^p, P_t^{\#}, X_{1,t}, X_{2,t}\right)$.

4.1 Re-Writing Equilibrium Conditions

There are two issues with how I've written these conditions. First, I haven't gotten rid of the heterogeneity – I still have j and l indexes showing up. Second, I have the price level showing up, which, as I mentioned above, may not be stationary. Hence, I want to re-write these conditions (i) only in terms of inflation, eliminating the price level; and (ii) getting rid of the heterogeneity, which the Calvo (1983) assumption allows me to do.

The Euler equation can be trivially re-written in terms of inflation as:

$$C_t^{\sigma} = \beta E_t C_{t+1}^{-\sigma}(1+i_t)(1+\pi_{t+1})^{-1} \tag{41}$$

Let's look at the expressions for the price level and the real wage. The expression for the price level is:

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj$$

Now, a fraction $(1-\phi_p)$ of these firms will update their price to the same reset price, $P_t^{\#}$. The other fraction ϕ_p will charge the price they charged in the previous period. This means we can break up the integral on the right hand side as:

$$P_t^{1-\epsilon_p} = \int_0^{1-\phi_p} P_t^{\#,1-\epsilon_p} dj + \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj$$

This can be written:

$$P_t^{1-\epsilon_p} = (1-\phi_p) P_t^{\#,1-\epsilon_p} + \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj$$

Now, here's the beauty of the Calvo assumption. Because the firms who get to update are randomly chosen, and because there are a large number (continuum) of firms, the integral (sum) of individual prices over some subset of the unit interval will simply be proportional to the integral over the entire unit interval, where the proportion is equal to the subset of the unit interval over which the integral is taken. This means:

$$\int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj = \phi_p \int_0^1 P_{t-1}(j)^{1-\epsilon_p} dj = \phi_p P_{t-1}^{1-\epsilon_p}$$

This means that the aggregate price level (raised to $1-\epsilon_p$) is a convex combination of the reset price and lagged price level (raised to the same power). So:

$$P_t^{1-\epsilon_p} = (1-\phi_p) P_t^{\#,1-\epsilon_p} + \phi_p P_{t-1}^{1-\epsilon_p}$$

In other words, we've gotten rid of the heterogeneity. The Calvo assumption allows us to integrate out the heterogeneity and not worry about keeping track of what each firm is doing from the perspective of looking at the behavior of aggregates. Now, we still have the issue here that we are written in terms of the price level, not inflation. To get it in terms of inflation, divide both sides by $P_{t-1}^{1-\epsilon_p}$, and define $\pi_t^{\#} = \frac{P_t^{\#}}{P_{t-1}} - 1$ as reset price inflation:

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p \tag{42}$$

We can do exactly the analogous thing for wages. The aggregate real wage index is:

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(l)^{1-\epsilon_w} dl$$

Since $1-\phi_w$ of households will update the same reset wage, and ϕ_w will be stuck with last period's *nominal* wage, this is:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#,1-\epsilon_w} + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}}{P_t}\right)^{1-\epsilon_w} dl$$

Note that I have written this in terms of nominal wages in terms of the non-updated wages. We can re-write in terms of real wages as:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#,1-\epsilon_w} + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}}{P_{t-1}}\right)^{1-\epsilon_w} \left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon_w} dl$$

Written in terms of inflation, and taking it out of the integral, this is just:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#,1-\epsilon_w} + (1+\pi_t)^{\epsilon_w-1} \int_{1-\phi_w}^1 w_{t-1}(l)^{1-\epsilon_w} dl$$

Again, the Calvo assumption allows us to get rid of the integral on the right hand side, which will just be proportional to last period's aggregate real wage. So we're left with:

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#,1-\epsilon_w} + \phi_w (1+\pi_t)^{\epsilon_w-1} w_{t-1}^{1-\epsilon_w} \tag{43}$$

We can also use the Calvo assumption to break up the price dispersion term, by again noting that $(1-\phi_p)$ of firms will update to the same price, and ϕ_p firms will be stuck with last period's price. Hence:

$$v_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^{\#}}{P_t}\right)^{-\epsilon_p} dj + \int_{1-\phi_p}^1 \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\epsilon_p} dj$$

This can be written in terms of inflation by multiplying and dividing by powers of P_{t-1} where necessary:

$$v_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^{\#}}{P_{t-1}}\right)^{-\epsilon_p} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon_p} dj + \int_{1-\phi_p}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon_p} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon_p} dj$$

We can take stuff out of the integral:

$$v_t^p = (1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p} \int_{1-\phi_p}^1 \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\epsilon_p} dj$$

By the same Calvo logic, the term inside the integral is just going to be proportional to v_{t-1}^p. This means we can write the price dispersion term as:

$$v_t^p = (1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p}\phi_p v_{t-1}^p \tag{44}$$

In other words, we just have to keep track of v_t^p, not the individual prices.

Now, we need to adjust the reset price expression. First, define two new auxiliary variables as follows:

$$x_{1,t} \equiv \frac{X_{1,t}}{P_t^{\epsilon_p}}$$

$$x_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon_p - 1}}$$

Dividing both sides of the reset price expressions by the appropriate power of P_t, we have:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t \frac{X_{1,t+1}}{P_t^{\epsilon_p}}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t \frac{X_{2,t+1}}{P_t^{\epsilon_p - 1}}$$

Multiplying and dividing the $t+1$ terms by the appropriate power of P_{t+1}, we have:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t \frac{X_{1,t+1}}{P_{t+1}^{\epsilon_p}} \left(\frac{P_{t+1}}{P_t}\right)^{\epsilon_p}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t \frac{X_{2,t+1}}{P_{t+1}^{\epsilon_p - 1}} \left(\frac{P_{t+1}}{P_t}\right)^{\epsilon_p - 1}$$

Or, in terms of inflation:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{45}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1} \tag{46}$$

Now, in terms of the reset price expression, since we divided $X_{1,t}$ by $P_t^{\epsilon_p}$ and divided $X_{2,t}$ by $P_t^{\epsilon_p - 1}$, we de-facto multiply the ratio of $\frac{X_{1,t}}{X_{2,t}}$ by P_t^{-1}. Hence, to keep equality, we need to multiply the right hand side by P_t. Hence, the reset price expression can now be written:

$$P_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} P_t \frac{x_{1,t}}{x_{2,t}}$$

Now, simply divide both sides by P_{t-1} to have everything in terms of inflation rates:

$$1 + \pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}(1+\pi_t)\frac{x_{1,t}}{x_{2,t}} \tag{47}$$

The full set of equilibrium conditions can now be expressed:

$$C_t^{-\sigma} = \beta E_t C_{t+1}^{-\sigma}(1+i_t)(1+\pi_{t+1})^{-1} \tag{48}$$

$$w_t^{\#,1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1}\frac{H_{1,t}}{H_{2,t}} \tag{49}$$

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta\phi_w E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} H_{1,t+1} \tag{50}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w E_t(1+\pi_{t+1})^{\epsilon_w - 1} H_{2,t+1} \tag{51}$$

$$mc_t = \frac{w_t}{A_t} \tag{52}$$

$$C_t = Y_t \tag{53}$$

$$Y_t = \frac{A_t N_t}{v_t^p} \tag{54}$$

$$v_t^p = (1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p}\phi_p v_{t-1}^p \tag{55}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w) w_t^{\#,1-\epsilon_w} + \phi_w(1+\pi_t)^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{56}$$

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p \tag{57}$$

$$1+\pi_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}(1+\pi_t)\frac{x_{1,t}}{x_{2,t}} \tag{58}$$

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t(1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \tag{59}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t \left(1+\pi_{t+1}\right)^{\epsilon_p - 1} x_{2,t+1} \tag{60}$$

$$i_t = (1-\rho_i)i + \rho_i i_{t-1} + \phi_\pi(\pi_t - \pi) + \varepsilon_{i,t} \tag{61}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \varepsilon_{a,t} \tag{62}$$

This is now fifteen equations in fifteen variables, where I have eliminated P_t as a variable, replaced $P_t^{\#}$ with $\pi_t^{\#}$, and replaced $X_{1,t}$ and $X_{2,t}$ with $x_{1,t}$ and $x_{2,t}$.

5 Steady State

In the non-stochastic steady state, $A = 1$. Steady state inflation will be equal to target. We can solve for steady state reset price inflation as:

$$1+\pi^{\#} = \left(\frac{(1+\pi)^{1-\epsilon_p} - \phi_p}{1-\phi_p}\right)^{\frac{1}{1-\epsilon_p}} \tag{63}$$

Here we see that if $\pi = 0$, then $\pi^{\#} = 0$ as well. Steady state price dispersion is:

$$v^p = \frac{(1-\phi_p)\left(\frac{1+\pi}{1+\pi^{\#}}\right)^{\epsilon_p}}{1-(1+\pi)^{\epsilon_p}\phi} \tag{64}$$

From this, we can again see that if $\pi = \pi^{\#} = 0$, then $v^p = 1$. The steady state nominal interest rate is:

$$1 + i = \frac{1}{\beta}(1 + \pi) \tag{65}$$

The steady state auxiliary pricing variables are:

$$x_1 = \frac{Y^{1-\sigma} mc}{1 - \phi_p \beta (1+\pi)^{\epsilon_p}} \tag{66}$$

$$x_2 = \frac{Y^{1-\sigma}}{1 - \phi_p \beta (1+\pi)^{\epsilon_p - 1}} \tag{67}$$

This means the ratio is:

$$\frac{x_1}{x_2} = mc \frac{1 - \phi_p \beta (1+\pi)^{\epsilon_p - 1}}{1 - \phi_p \beta (1+\pi)^{\epsilon_p}}$$

Hence, we can solve for steady state marginal cost as:

$$mc = \frac{\epsilon_p - 1}{\epsilon_p} \frac{1 - \phi_p \beta (1+\pi)^{\epsilon_p}}{1 - \phi_p \beta (1+\pi)^{\epsilon_p - 1}} \frac{1 + \pi^{\#}}{1 + \pi} \tag{68}$$

Here again, we see that if $\pi = \pi^{\#} = 0$, then $mc = \frac{\epsilon_p - 1}{\epsilon_p}$, which is the desired flexible price markup.

Let's solve for the optimal reset wage in terms of the steady state real wage:

$$w^{\#} = \left(\frac{\left(1 - \phi_w (1+\pi)^{\epsilon_w - 1}\right)}{1 - \phi_w} \right)^{\frac{1}{1-\epsilon_w}} w \tag{69}$$

This says that the reset wage is proportional to the steady state wage. Note that, if $\phi_w = 0$ (wages were flexible), we would have $w^{\#} = w$. Let's now solve for the steady states of the auxiliary variables related to wage-setting. We have:

$$H_1 = \frac{\psi w^{\epsilon_w (1+\eta)} N^{1+\eta}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}} \tag{70}$$

$$H_2 = \frac{Y^{-\sigma} w^{\epsilon_w} N}{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}} \tag{71}$$

The ratio is just:

$$\frac{H_1}{H_2} = \psi w^{\epsilon_w \eta} Y^{\sigma} N^{\eta} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}$$

Plug this into the FOC for labor:

$$w^{\#, 1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \psi w^{\epsilon_w \eta} Y^{\sigma} N^{\eta} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}$$

Now, as an instructive exercise, suppose that $\phi_w = 0$, so that wages were flexible. This would mean $w^{\#} = w$. Combining these, we'd have:

$$\frac{\epsilon_w - 1}{\epsilon_w} Y^{-\sigma} w = \psi N^{\eta}$$

This is instructive. If $\epsilon_w \to \infty$, this would be the same static FOC that we've had before: the marginal disutility of labor must equal the marginal utility of consumption ($Y^{-\sigma}$ here, since $C = Y$) times the real wage. If you defined the MRS (marginal rate of substitution) between labor and consumption as $\psi N^{\eta} Y^{\sigma}$, then you could re-write this as:

$$w = \frac{\epsilon_w}{\epsilon_w - 1} MRS$$

In other words, households set the wage as a *markup* over the marginal rate of substitution, in an analogous way to how firms set price as a markup over marginal cost.

Now, go back to our earlier expression from the FOC for labor. Eliminating the reset wage, we have:

$$\left(\frac{\left(1 - \phi_w (1+\pi)^{\epsilon_w - 1}\right)}{1 - \phi_w} \right)^{\frac{1+\epsilon_w \eta}{1-\epsilon_w}} w^{1+\epsilon_w \eta} = \frac{\epsilon_w}{\epsilon_w - 1} \psi w^{\epsilon_w \eta} Y^{\sigma} N^{\eta} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}$$

Simplify:

$$N^{\eta} = \frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} Y^{-\sigma} w \left(\frac{\left(1 - \phi_w (1+\pi)^{\epsilon_w - 1}\right)}{1 - \phi_w} \right)^{\frac{1+\epsilon_w \eta}{1-\epsilon_w}} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}}$$

Now, what does this do for us? Well, we know that $w = mc$, and we know that $N = Y v^p$. Plugging these in:

$$N^{\eta} = \frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} N^{-\sigma} \left(v^p\right)^{-\sigma} mc \left(\frac{\left(1 - \phi_w (1+\pi)^{\epsilon_w - 1}\right)}{1 - \phi_w} \right)^{\frac{1+\epsilon_w \eta}{1-\epsilon_w}} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}}$$

Now we can solve for N:

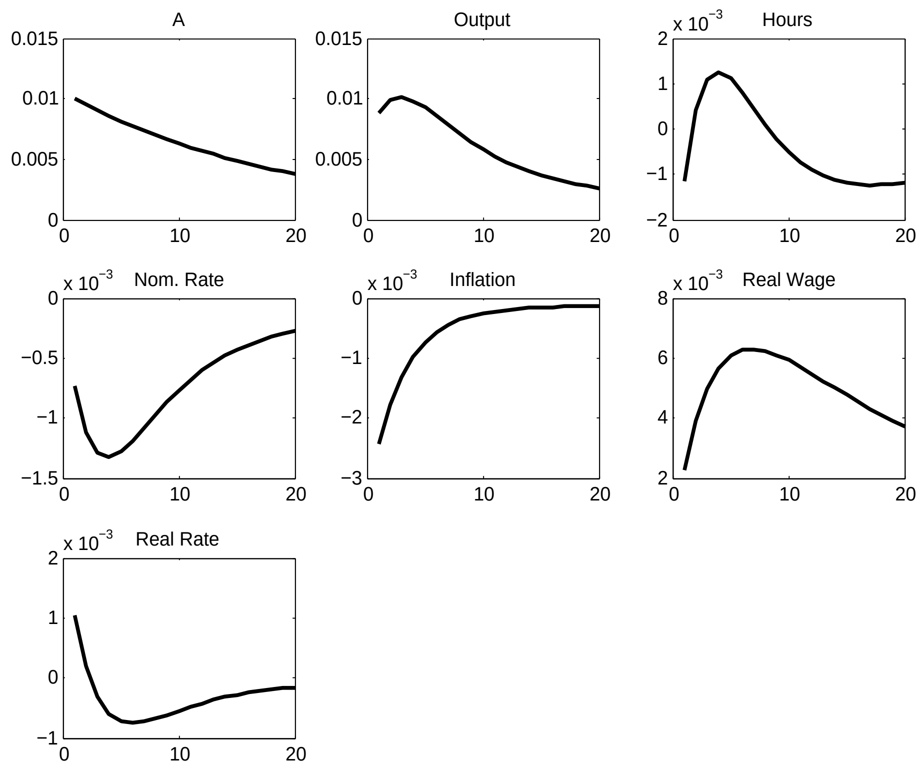
$$N = \left(\frac{\epsilon_w - 1}{\epsilon_w} \frac{1}{\psi} \left(v^p\right)^{-\sigma} mc \left(\frac{\left(1 - \phi_w (1+\pi)^{\epsilon_w - 1}\right)}{1 - \phi_w} \right)^{\frac{1+\epsilon_w \eta}{1-\epsilon_w}} \frac{1 - \phi_w \beta (1+\pi)^{\epsilon_w (1+\eta)}}{1 - \phi_w \beta (1+\pi)^{\epsilon_w - 1}} \right)^{\frac{1}{\sigma+\eta}} \tag{72}$$

Once we know N, we know Y as well.

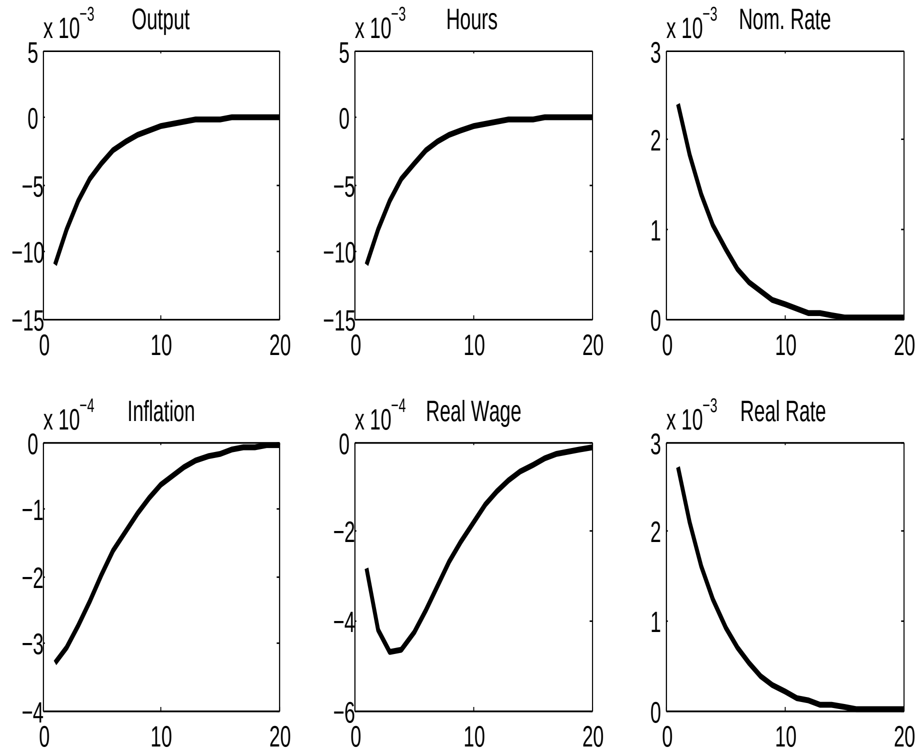
6 Quantitative Analysis

I solve the model quantitatively using a first order approximation in Dynare. I use the following parameter values: $\epsilon_p = \epsilon_w = 10$, $\phi_p = \phi_w = 0.75$, $\beta = 0.99$, $\sigma = 1$, $\psi = 1$, $\eta = 1$, $\rho_a = 0.95$, $\rho_i = 0.8$, $\phi_\pi = 1.5$, and I set the standard deviations of the productivity and monetary policy shocks to 0.01 and 0.0025, respectively.

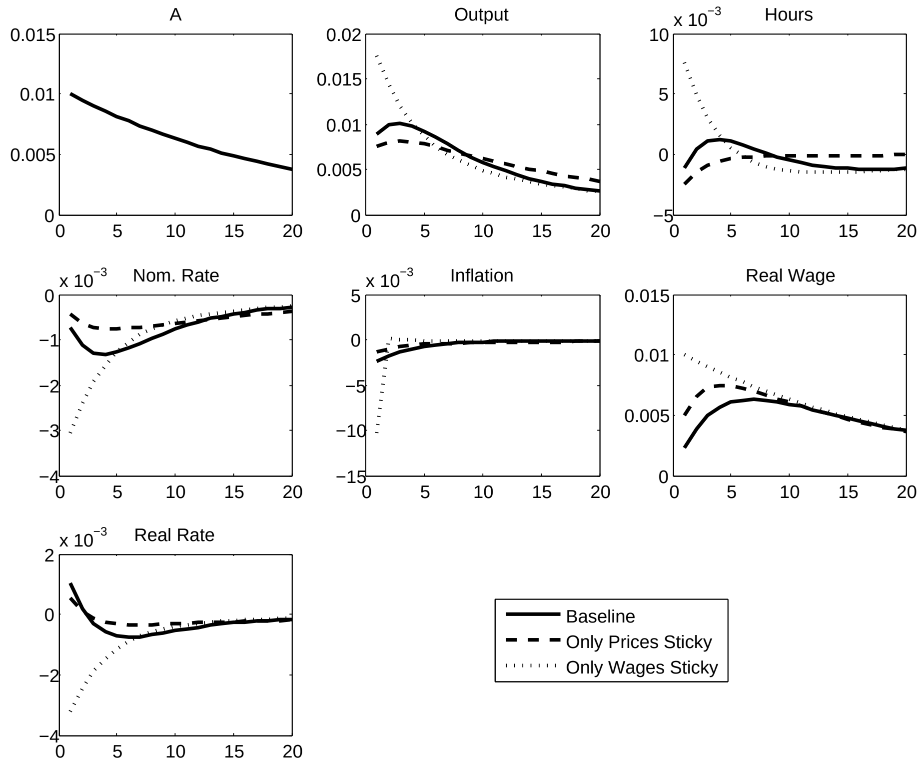
Below are the impulse responses to a productivity shock. Output rises, by an amount fairly close to its "flexible price" level, hours decline, the nominal interest rate declines, the real interest rate rises, and the real wage rises.

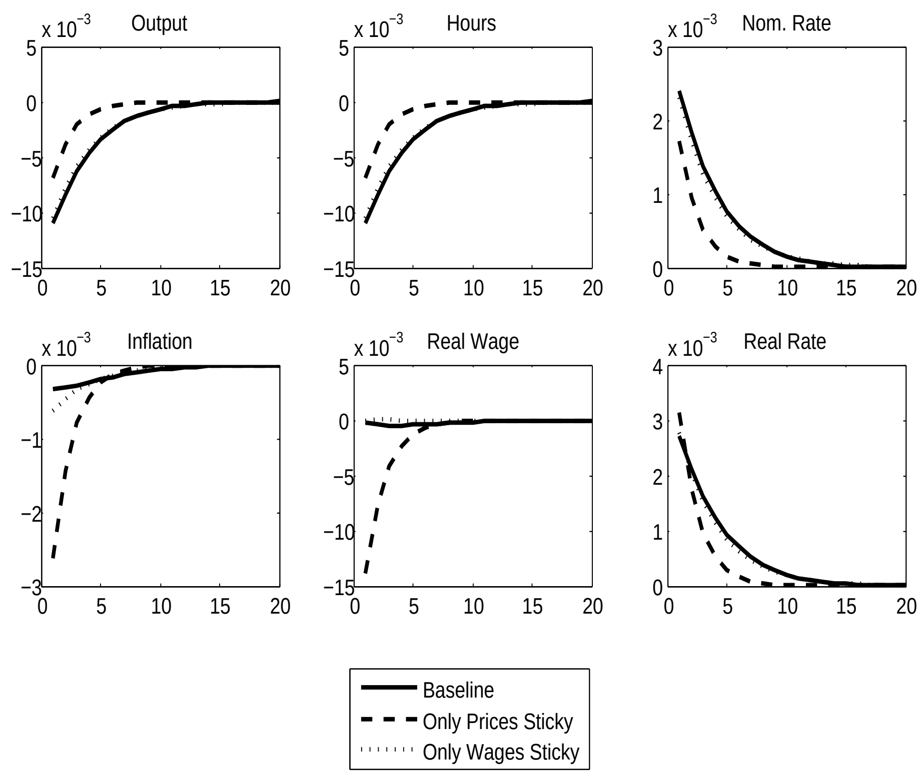


The responses to a monetary policy shock are shown below. Both the real and nominal interest rates rise. Output, hours, inflation, and the real wage all decline.



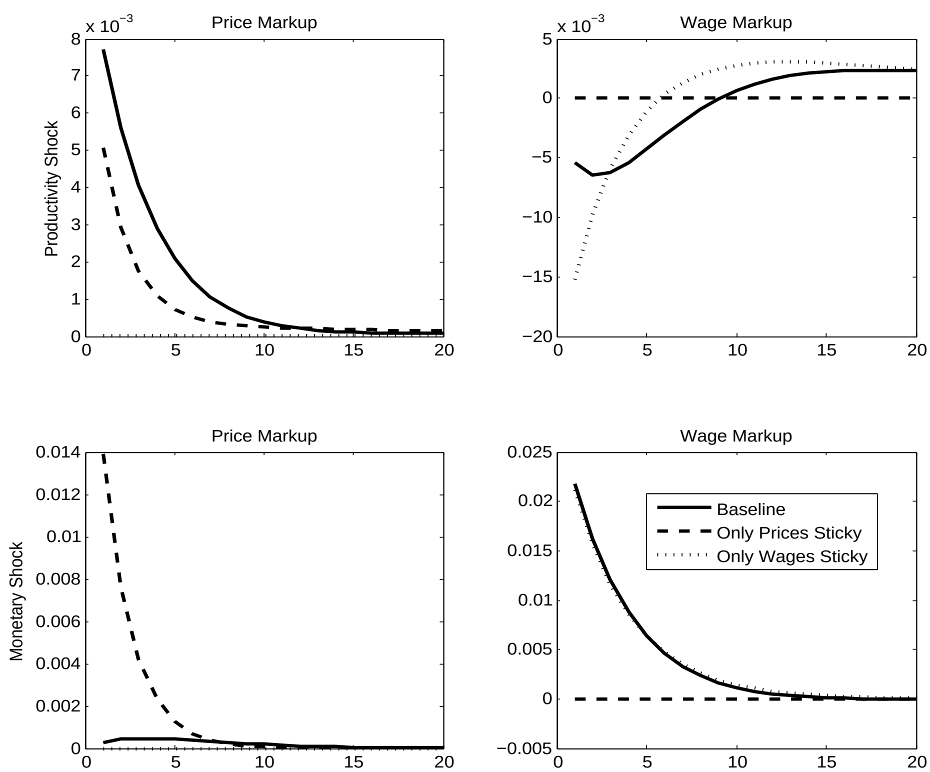
What is the relative importance of price and wage stickiness in accounting for this pattern of impulse responses? We observe that wage rigidity (in isolation) actually amplifies the responses of real variables to a productivity shock, whereas price rigidity (in isolation) dampens those responses. The responses with both price and wage rigidity are somewhere in between. In contrast, the reactions to the monetary policy shock are pretty similar with either wage or price rigidity.


A
Output
Hours
Nom. Rate
Inflation
Real Wage
Real Rate
Baseline
Only Prices Sticky
Only Wages Sticky


Output
Hours
Nom. Rate
Inflation
Real Wage
Real Rate
Baseline
Only Prices Sticky
Only Wages Sticky

We can understand the pattern of impulse responses by noting that there are two monopoly distortions in the model. One relates to price-setting (price would be a fixed markup over marginal cost if prices were flexible) and one relates to wage-setting (the real wage would be a fixed markup over the marginal rate of substitution). We can think of price and wage rigidity causing these markups to vary endogenously in the short run in response to shocks. The price markup is just the negative of real marginal cost, and the wage markup is the ratio of the real wage to the marginal rate of substitution. Output will respond by less than it would if prices were flexible if either of these markups rise in response to a shock; if either markup falls, this is relatively expansionary.

In the figure below, I plot the price and wage markups to both a productivity and monetary policy shock under three regimes: one where both prices and wages are sticky, one where only prices are sticky, and one where only wages are sticky. In response to either shock, when only wages are sticky, the price markup is fixed. In contrast, when only prices are sticky, the wage markup is fixed.



Above, we see that both the price and wage markups go up in response to the monetary policy shock, but they go in opposite directions after a productivity shock. A productivity shock puts upward pressure on real wages and downward pressure on prices. Downward pressure on prices means that some firms will end up with prices that are higher than they would like, hence, if prices are sticky, the price markup rises, which effectively means the economy is more distorted. This is why output rises by less than it would if prices were flexible in a sticky price model in response to a productivity shock. The opposite pattern occurs with wages. Real wages need to rise after a positive productivity shock; because some households can't adjust their nominal wages, they end up with wage markups that are too low. Hence, the aggregate wage markup falls, which means that the economy is relatively undistorted along that dimension, which is relatively expansionary. This is why, when only wages are sticky, output rises by more than it would under flexible prices and wages, because the wage markup gets "squeezed." The differential behavior of the price and wage markups in response to the productivity shock accounts for why the output responses to the shock look so different when one of the stickiness parameters is "turned off."

In response to the monetary policy shock, both the wage and price markups move in the same direction. The contractionary monetary policy shock puts downward pressure on prices, so some firms end up with prices that are too high relative to what they would optimally like – the price markup rises. If prices and wages were both flexible, there would be no effect on real wages of a monetary policy shock. The downward pressure on prices therefore means that there is downward pressure on wages. Since some households can't adjust their wages downward, they end up with wages that are too high, and the economy-wide wage markup rises. The increases in both the price and wage markups are contractionary in the case of a monetary policy shock. Since the markups behave in the same way, we observe that there is a much smaller difference in the responses to a monetary policy shock when either prices or wages are sticky, relative to the case of a productivity shock.

7 Log-Linearization

Now, let's log-linearize the equilibrium conditions. We are going to do this about a zero inflation steady state, which makes life much easier. Start with the Euler equation, going ahead and imposing the accounting identity that $C_t = Y_t$. We have:

$$\begin{aligned} -\sigma \ln Y_t &= \ln \beta - \sigma E_t \ln Y_{t+1} + i_t - E_t \pi_{t+1} \\ -\sigma \widetilde{Y}_t &= -\sigma E_t \widetilde{Y}_{t+1} + \widetilde{i}_t - E_t \widetilde{\pi}_{t+1} \end{aligned}$$

Where $\widetilde{Y}_t = \frac{Y_t - Y}{Y}$, $\widetilde{i}_t = i_t - i$, and $\widetilde{\pi}_t = \pi_t - \pi$. In other words, the variables already in rate form (interest rate and inflation) are expressed as absolute deviations, and variables not already in rate form as percent (log) deviations. We can re-write this as:

$$\widetilde{Y}_t = E_t \widetilde{Y}_{t+1} - \frac{1}{\sigma}\left(\widetilde{i}_t - E_t \widetilde{\pi}_{t+1}\right) \tag{73}$$

This is sometimes called the "New Keynesian IS Curve."

Real marginal cost is already log-linear:

$$\widetilde{mc}_t = \widetilde{w}_t - \widetilde{A}_t \tag{74}$$

Log-linearize the production function:

$$\widetilde{Y}_t = \widetilde{A}_t + \widetilde{N}_t + \widetilde{v}_t^p$$

Now, what is $\widetilde{v}_t^p$? Let's take logs and go from there:

$$\ln v_t^p = \ln\left((1-\phi)(1+\pi_t^{\#})^{\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p}\phi v_{t-1}^p\right)$$

Now, from our discussion above, we know that $v^p = 1$ when $\pi = 0$. Totally differentiating:

$$\widetilde{v}_t^p = \frac{1}{1}\Big(-\epsilon_p(1-\phi)(1+\pi^{\#})^{-\epsilon_p-1}(1+\pi)^{\epsilon_p}(\pi_t^{\#}-\pi^{\#}) + \epsilon_p(1-\phi)(1+\pi^{\#})^{-\epsilon_p}(1+\pi)^{\epsilon_p-1}(\pi_t-\pi)\ldots$$
$$\ldots\epsilon_p(1+\pi)^{\epsilon_p-1}\phi v^p(\pi_t-\pi) + (1+\pi)^{\epsilon_p}\phi(v_{t-1}^p - v^p)\Big)$$

Using now known facts about the steady state, this reduces to:

$$\widetilde{v}_t^p = -\epsilon_p(1-\phi)\widetilde{\pi}_t^{\#} + \epsilon_p(1-\phi)\widetilde{\pi}_t + \epsilon_p\phi\widetilde{\pi}_t + \phi\widetilde{v}_{t-1}^p$$

This can be written:

$$\widetilde{v}_t^p = -\epsilon_p(1-\phi)\widetilde{\pi}_t^{\#} + \epsilon_p\widetilde{\pi}_t + \phi\widetilde{v}_{t-1}^p$$

Now, log-linearize the equation for the evolution of inflation:

$$(1-\epsilon_p)\pi_t = \ln\left((1-\phi)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi\right)$$
$$(1-\epsilon_p)(\pi_t - \pi) = (1+\pi)^{\epsilon_p-1}\left((1-\epsilon_p)(1-\phi)(1+\pi^{\#})^{-\epsilon_p}(\pi_t^{\#}-\pi^{\#})\right)$$

In the last line above, the $(1+\pi)^{\epsilon_p-1}$ shows up because the term inside parentheses is equal to $(1+\pi)^{1-\epsilon_p}$ evaluated in the steady state, and when taking the derivative of the log this term gets inverted evaluated at that point. Using facts about the zero inflation steady state, we have:

$$(1-\epsilon_p)\widetilde{\pi}_t = (1-\epsilon_p)(1-\phi)\widetilde{\pi}_t^{\#}$$

Or:

$$\widetilde{\pi}_t = (1-\phi)\widetilde{\pi}_t^{\#} \tag{75}$$

In other words, actual inflation is just proportional to reset price inflation, where the constant is equal to the fraction of firms that are updating their prices. This is pretty intuitive. Now, use this in the expression for price dispersion:

$$\widetilde{v}_t^p = \epsilon_p \left(\widetilde{\pi}_t - (1-\phi)\widetilde{\pi}_t^{\#} \right) + \phi \widetilde{v}_{t-1}^p$$

But from above, the first term drops out, so we are left with:

$$\widetilde{v}_t^p = \phi \widetilde{v}_{t-1}^p \tag{76}$$

If we are approximating about the zero inflation steady state in which $v^p = 1$, then we're starting from a position in which $\widetilde{v}_{t-1}^p = 0$, so this means that $\widetilde{v}_t^p = 0$ at all times. In other words, about a zero inflation steady state, price dispersion is a second order phenomenon, and we can just ignore it in a first order approximation about a zero inflation steady state.

Given this, the log-linearized production function is just:

$$\widetilde{Y}_t = \widetilde{A}_t + \widetilde{N}_t \tag{77}$$

Now, let's log-linearize the reset price expression. This is multiplicative, and so is already in log-linear form. We have:

$$\widetilde{\pi}_t^{\#} = \widetilde{\pi}_t + \widetilde{x}_{1,t} - \widetilde{x}_{2,t} \tag{78}$$

Now we need to log-linearize the auxiliary variables. Imposing the identity that $Y_t = C_t$, we have:

$$\ln x_{1,t} = \ln \left(Y_t^{1-\sigma} mc_t + \phi \beta E_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \right)$$

Totally differentiating:

$$\frac{x_{1,t} - x_1}{x_1} = \frac{1}{x_1} \left((1-\sigma) Y^{-\sigma} mc (Y_t - Y) + Y^{1-\sigma}(mc_t - mc) + \epsilon_p \phi \beta (1+\pi)^{\epsilon_p - 1} x_1 (\pi_{t+1} - \pi) + \phi \beta (1+\pi)^{\epsilon_p} (x_{1,t+1} - x_1) \right)$$

Distributing the $\frac{1}{x_1}$ and multiplying, dividing where necessary to get in to percent deviation terms, and making use of the continued assumption of linearization about a zero inflation steady state, we have:

$$\widetilde{x}_{1,t} = \frac{(1-\sigma) Y^{1-\sigma} mc}{x_1} \widetilde{Y}_t + \frac{Y^{1-\sigma} mc}{x_1} \widetilde{mc}_t + \epsilon_p \phi \beta E_t \widetilde{\pi}_{t+1} + \phi \beta E_t \widetilde{x}_{1,t+1}$$

Now, with zero steady state inflation, we know that $x_1 = \frac{Y^{1-\sigma} mc}{1-\phi\beta}$. This simplifies the first two terms:

$$\widetilde{x}_{1,t} = (1-\sigma)(1-\phi\beta)\widetilde{Y}_t + (1-\phi\beta)\widetilde{mc}_t + \epsilon_p \phi \beta E_t \widetilde{\pi}_{t+1} + \phi \beta E_t \widetilde{x}_{1,t+1} \tag{79}$$

Now, log-linearize $x_{2,t}$:

$$\ln x_{2,t} = \ln\left(Y_t^{1-\sigma} + \phi\beta E_t(1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1}\right)$$

Totally differentiating:

$$\frac{x_{2,t}-x_2}{x_2} = \frac{1}{x_2}\left((1-\sigma)Y^{-\sigma}(Y_t - Y) + (\epsilon_p - 1)\phi\beta(1+\pi)^{\epsilon_p-2}x_2(\pi_{t+1}-\pi) + \phi\beta(1+\pi)^{\epsilon_p-1}(x_{2,t+1}-x_2)\right)$$

Distributing the x_2, multiplying and dividing by appropriate terms, and making use of the fact that $\pi =$, we have:

$$\widetilde{x}_{2,t} = \frac{(1-\sigma)Y^{1-\sigma}}{x_2}\widetilde{Y}_t + (\epsilon_p - 1)\phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t \widetilde{x}_{2,t+1}$$

Since $x_2 = \frac{Y^{1-\sigma}}{1-\phi\beta}$, this can be written:

$$\widetilde{x}_{2,t} = (1-\sigma)(1-\phi\beta)\widetilde{Y}_t + (\epsilon_p - 1)\phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\widetilde{x}_{2,t+1} \tag{80}$$

Now, subtracting $\widetilde{x}_{2,t}$ from $\widetilde{x}_{1,t}$, we have:

$$\widetilde{x}_{1,t} - \widetilde{x}_{2,t} = (1-\phi\beta)\widetilde{mc}_t + \phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\left(\widetilde{x}_{1,t+1} - \widetilde{x}_{2,t+1}\right)$$

From above, we also know that:

$$\widetilde{x}_{1,t} - \widetilde{x}_{2,t} = \widetilde{\pi}_t^{\#} - \widetilde{\pi}_t$$

But $\widetilde{\pi}_t^{\#} = \frac{1}{1-\phi}\widetilde{\pi}_t$, so we must also have:

$$\widetilde{x}_{1,t} - \widetilde{x}_{2,t} = \frac{\phi}{1-\phi}\widetilde{\pi}_t$$

Make this substitution above:

$$\frac{\phi}{1-\phi}\widetilde{\pi}_t = (1-\phi\beta)\widetilde{mc}_t + \phi\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\left(\frac{\phi}{1-\phi}E_t\widetilde{\pi}_{t+1}\right)$$

Multiplying through:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widetilde{mc}_t + (1-\phi)\beta E_t\widetilde{\pi}_{t+1} + \phi\beta E_t\widetilde{\pi}_{t+1}$$

Or:

$$\widetilde{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widetilde{mc}_t + \beta E_t\widetilde{\pi}_{t+1} \tag{81}$$

This is the standard New Keynesian Phillips Curve. Its basic structure is unaltered by the presence of wage rigidity.

Now, let's go log-linearize the wage-setting equations. Begin by taking logs of the aggregate real wage series:

$$(1-\epsilon_w)\ln w_t = \ln\Big((1-\phi_w)w_t^{\#,1-\epsilon_w} + \phi_w(1+\pi_t)^{\epsilon_w-1}w_{t-1}^{1-\epsilon_w}\Big)$$

Totally differentiate:

$$(1-\epsilon_w)\frac{w_t-w}{w} = \frac{1}{w^{1-\epsilon_w}}\Big((1-\epsilon_w)(1-\phi_w)w^{\#,-\epsilon_w}(w_t^{\#}-w^{\#}) + (\epsilon_w-1)\phi_w w^{1-\epsilon_w}(\pi_t-\pi) + (1-\epsilon_w)\phi_w w^{-\epsilon_w}(w_{t-1}-w)\Big)$$

Above, I have made use of the fact that we are linearizing about the point $\pi = 0$, which simplifies things a bit. Also, since we are linearizing about a zero inflation steady state, we know from above that $w^{\#} = w$. Making use of this, we have:

$$(1-\epsilon_w)\widetilde{w}_t = (1-\epsilon_w)(1-\phi_w)\widetilde{w}_t^{\#} - (1-\epsilon_w)\phi_w\widetilde{\pi}_t + (1-\epsilon_w)\phi_w\widetilde{w}_{t-1}$$

Simplifying:

$$\widetilde{w}_t = (1-\phi_w)\widetilde{w}_t^{\#} + \phi_w\widetilde{w}_{t-1} - \phi_w\widetilde{\pi}_t \tag{82}$$

This is pretty intuitive. It says that the current real wage is a convex combination of the reset real wage and last period's real wage, minus an adjustment for inflation. The reason the adjustment for inflation is because nominal wages are fixed.

Now, let's log-linearize the reset wage equation. Since it is multiplicative, it is already log-linear:

$$(1+\epsilon_w\eta)\widetilde{w}_t^{\#} = \widetilde{H}_{1,t} - \widetilde{H}_{2,t} \tag{83}$$

Now we need to log-linearize the auxiliary wage-setting variables. Let's log:

$$\ln H_{1,t} = \ln\Big(\psi w_t^{\epsilon_w(1+\eta)}N_t^{1+\eta} + \beta\phi_w E_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)}H_{1,t+1}\Big)$$

$$\widetilde{H}_{1,t} = \frac{1}{H_1}\Big(\epsilon_w(1+\eta)\psi w^{\epsilon_w(1+\eta)-1}N^{1+\eta}(w_t-w) + (1+\eta)\psi w^{\epsilon_w(1+\eta)}N^{\eta}(N_t-N) + \epsilon_w(1+\eta)\beta\phi_w H_1(\pi_{t+1}-\pi) + \dots$$
$$\dots + \phi_w\beta(H_{1,t+1}-H_1)\big)$$

In simplifying this, note that $H_1 = \frac{\psi w^{\epsilon_w(1+\eta)}N^{1+\eta}}{1-\phi_w\beta}$. Distributing things:

$$\widetilde{H}_{1,t} = (1-\phi_w\beta)\epsilon_w(1+\eta)\widetilde{w}_t + (1-\phi_w\beta)(1+\eta)\widetilde{N}_t + \epsilon_w(1+\eta)\phi_w\beta E_t\widetilde{\pi}_{t+1} + \phi_w\beta E_t\widetilde{H}_{1,t+1}$$

Now, let's do $H_{2,t}$:

$$\ln H_{2,t} = \ln\left(Y_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w E_t (1+\pi_{t+1})^{\epsilon_w - 1} H_{2,t+1}\right)$$

Totally differentiate:

$$\widetilde{H}_{2,t} = \frac{1}{H_2}\left(-\sigma Y^{-\sigma-1} w^{\epsilon_w} N(Y_t - Y) + \epsilon_w Y^{-\sigma} w^{\epsilon_w - 1} N(w_t - w) + Y^{-\sigma} w^{\epsilon_w}(N_t - N) + (\epsilon_w - 1)\phi_w\beta H_2(\pi_{t+1} - \pi) + \dots\right.$$
$$\left.\dots + \phi_w\beta(H_{2,t+1} - H_2)\right)$$

In simplifying this, note that $H_2 = \frac{Y^{-\sigma} w^{\epsilon_w} N}{1-\phi_w\beta}$. We can now write this:

$$\widetilde{H}_{2,t} = -(1-\phi_w\beta)\sigma\widetilde{Y}_t + (1-\phi_w\beta)\epsilon_w\widetilde{w}_t + (1-\phi_w\beta)\widetilde{N}_t + (\epsilon_w - 1)\phi_w\beta E_t\widetilde{\pi}_{t+1} + \phi_w\beta E_t\widetilde{H}_{2,t+1}$$

Hence, we have:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\widetilde{w}_t + (1-\phi_w\beta)\eta\widetilde{N}_t + (1-\phi_w\beta)\sigma\widetilde{Y}_t + \phi_w\beta(1+\epsilon_w\eta)E_t\widetilde{\pi}_{t+1} + \phi_w\beta(E_t\widetilde{H}_{1,t+1} - \widetilde{H}_{2,t+1})$$

The MRS between labor and consumption, as introduced above, is $MRS_t = \psi N_t^{\eta} Y_t^{\sigma}$. In log-linear terms, this is:

$$\widetilde{mrs}_t = \eta\widetilde{N}_t + \sigma\widetilde{Y}_t$$

This means we can write:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\widetilde{w}_t + (1-\phi_w\beta)\widetilde{mrs}_t + \phi_w\beta(1+\epsilon_w\eta)E_t\widetilde{\pi}_{t+1} + \phi_w\beta(E_t\widetilde{H}_{1,t+1} - \widetilde{H}_{2,t+1})$$

Let's define $\widetilde{\mu}_t = \widetilde{mrs}_t - \widetilde{w}_t$ as the gap between the MRS and the real wage. Playing around:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\widetilde{w}_t + (1-\phi_w\beta)\widetilde{\mu}_t + (1-\phi_w\beta)\widetilde{w}_t + \phi_w\beta(1+\epsilon_w\eta)E_t\widetilde{\pi}_{t+1} + \phi_w\beta(E_t\widetilde{H}_{1,t+1} - \widetilde{H}_{2,t+1})$$

Or:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = (1-\phi_w\beta)(1+\epsilon_w\eta)\widetilde{w}_t + (1-\phi_w\beta)\widetilde{\mu}_t + \phi_w\beta(1+\epsilon_w\eta)E_t\widetilde{\pi}_{t+1} + \phi_w\beta(E_t\widetilde{H}_{1,t+1} - \widetilde{H}_{2,t+1})$$

Now, from above we know that we can write the reset wage as:

$$\widetilde{w}_t^{\#} = \frac{1}{1-\phi_w}\widetilde{w}_t - \frac{\phi_w}{1-\phi_w}\widetilde{w}_{t-1} + \frac{\phi_w}{1-\phi_w}\widetilde{\pi}_t$$

And we also know that:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = (1+\epsilon_w \eta)\widetilde{w}_t^{\#}$$

Combining these expressions, we have:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\widetilde{w}_t - \frac{(1+\epsilon_w\eta)\phi_w}{1-\phi_w}\widetilde{w}_{t-1} + \frac{(1+\epsilon_w\eta)\phi_w}{1-\phi_w}\widetilde{\pi}_t$$

It is helpful to re-write this in terms of the nominal wage: $\widetilde{W}_t = \widetilde{w}_t + \widetilde{P}_t$. Doing so, we have:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\left(\widetilde{W}_t - \widetilde{P}_t\right) - \frac{(1+\epsilon_w\eta)\phi_w}{1-\phi_w}\left(\widetilde{W}_{t-1} - \widetilde{P}_{t-1}\right) + \frac{(1+\epsilon_w\eta)\phi_w}{1-\phi_w}\widetilde{\pi}_t$$

Now, define $\widetilde{\pi}_t^w = \widetilde{W}_t - \widetilde{W}_{t-1}$ as nominal wage inflation. We can further simplify as:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\left(\widetilde{W}_t - \widetilde{P}_t - \phi_w\widetilde{W}_{t-1} + \phi_w\widetilde{P}_{t-1} + \phi_w\widetilde{\pi}_t\right)$$

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\left(\widetilde{W}_t - \widetilde{W}_{t-1} + (1-\phi_w)\widetilde{W}_{t-1} - \phi_w\left(\widetilde{P}_t - \widetilde{P}_{t-1}\right) - (1-\phi_w)\widetilde{P}_t + \phi_w\widetilde{\pi}_t\right)$$

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\widetilde{\pi}_t^w + (1+\epsilon_w\eta)\widetilde{W}_{t-1} - (1+\epsilon_w\eta)\widetilde{P}_t$$

In terms of the real wage again, this can be re-written:

$$\widetilde{H}_{1,t} - \widetilde{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\widetilde{\pi}_t^w + (1+\epsilon_w\eta)\widetilde{w}_{t-1} - (1+\epsilon_w\eta)\widetilde{\pi}_t$$

Now, combine this with our earlier expression for the difference between the auxiliary variables. We have:

$$\frac{1}{1-\varphi_w}\widetilde{\pi}_t^w + \widetilde{w}_{t-1} - \widetilde{\pi}_t = (1-\phi_w\beta)\widetilde{w}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \phi_w\beta E_t\widetilde{\pi}_{t+1} + \frac{\phi_w\beta}{1+\epsilon_w\eta}(E_t\widetilde{H}_{1,t+1} - E_t\widetilde{H}_{2,t+1})$$

Plugging in the same expression for the expected future difference in the auxiliary variables, we have:

$$\frac{1}{1-\varphi_w}\widetilde{\pi}_t^w + \widetilde{w}_{t-1} - \widetilde{\pi}_t = (1-\phi_w\beta)\widetilde{w}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \phi_w\beta E_t\widetilde{\pi}_{t+1} + \dots$$
$$\dots + \frac{\phi_w\beta}{1+\epsilon_w\eta}\left(\frac{1+\epsilon_w\eta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w + (1+\epsilon_w\eta)\widetilde{w}_t - (1+\epsilon_w\eta)E\widetilde{\pi}_{t+1}\right)$$

Simplifying:

$$\frac{1}{1-\varphi_w}\widetilde{\pi}_t^w + \widetilde{w}_{t-1} - \widetilde{\pi}_t = (1-\phi_w\beta)\widetilde{w}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \phi_w\beta E_t\widetilde{\pi}_{t+1} + \dots$$
$$\frac{\phi_w\beta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w + \phi_w\beta\widetilde{w}_t - \phi_w\beta E_t\widetilde{\pi}_{t+1}$$

Simplifying:

$$\frac{1}{1-\varphi_w}\widetilde{\pi}_t^w + \widetilde{w}_{t-1} - \widetilde{\pi}_t = \widetilde{w}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w$$

Re-arranging:

$$\frac{1}{1-\phi_w}\widetilde{\pi}_t^w = \widetilde{w}_t - \widetilde{w}_{t-1} + \widetilde{\pi}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w$$

Now, note that $\widetilde{w}_t - \widetilde{w}_{t-1} + \widetilde{\pi}_t = \widetilde{\pi}_t^w$. Hence, we have:

$$\left(\frac{1}{1-\phi_w} - 1\right)\widetilde{\pi}_t^w = \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w$$

Or:

$$\frac{\phi_w}{1-\phi_w}\widetilde{\pi}_t^w = \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\widetilde{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}E_t\widetilde{\pi}_{t+1}^w$$

And finally:

$$\widetilde{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}\widetilde{\mu}_t + \beta\widetilde{E}_t\pi_{t+1}^w \tag{84}$$

This is the wage Phillips Curve. It looks almost the same as the price Phillips Curve, but there is an extra terms $(1+\epsilon_w\eta)$ in the denominator. Since $\epsilon_w\eta > 0$, this means that the wage Phillips Curve is always "flatter" than the price Phillips Curve for equal values of the Calvo parameters, ϕ_p and ϕ_w. Also, differently than for prices, the elasticity parameter ϵ_w shows up in the log-linearized Phillips Curve for wages, whereas it does not for prices.

The full set of log-linearized first order conditions can be written:

$$\widetilde{Y}_t = E_t\widetilde{Y}_{t+1} - \frac{1}{\sigma}\left(\widetilde{i}_t - E_t\widetilde{\pi}_{t+1}\right) \tag{85}$$

$$\widetilde{mc}_t = \widetilde{w}_t - \widetilde{A}_t \tag{86}$$

$$\widetilde{Y}_t = \widetilde{A}_t + \widetilde{N}_t \tag{87}$$

$$\widetilde{\pi}_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{mc}_t + \beta E_t\widetilde{\pi}_{t+1} \tag{88}$$

$$\widetilde{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}\widetilde{\mu}_t + \beta\widetilde{E}_t\pi_{t+1}^w \tag{89}$$

$$\widetilde{\mu}_t = \widetilde{mrs}_t - \widetilde{w}_t \tag{90}$$

$$\widetilde{mrs}_t = \eta \widetilde{N}_t + \sigma \widetilde{Y}_t \tag{91}$$

$$\widetilde{\pi}_t^w = \widetilde{w}_t - \widetilde{w}_{t-1} + \widetilde{\pi}_t \tag{92}$$

$$\widetilde{i}_t = \rho_i \widetilde{i}_{t-1} + (1-\rho_i)\phi_\pi \widetilde{\pi}_t + \varepsilon_{i,t} \tag{93}$$

$$\widetilde{A}_t = \rho_a \widetilde{A}_{t-1} + \varepsilon_{a,t} \tag{94}$$

This is ten equations in ten unknowns ($\widetilde{Y}_t, \widetilde{N}_t, \widetilde{mc}_t, \widetilde{i}_t, \widetilde{mrs}_t, \widetilde{\mu}_t, \widetilde{w}_t, \widetilde{A}_t, \widetilde{\pi}_t, \widetilde{\pi}_t^w$).

7.1 Gap Formulation

As in the simpler NK model, there are some redundant variables here that could be eliminated, and we might like to write the Phillips Curves for prices and inflation in terms of "gaps."

As we did earlier, let's define variables with a superscript f as the flexible price/wage variables: the values of endogenous variables which would obtain in the absence of *both* price and wage stickiness. If both prices and wages were flexible, we'd have $\widetilde{\mu}_t = \widetilde{mc}_t = 0$. This would imply:

$$\begin{aligned}
\widetilde{w}_t^f &= \widetilde{A}_t \\
\widetilde{w}_t^f &= \eta \widetilde{N}_t^f + \sigma \widetilde{Y}_t^f \\
\widetilde{Y}_t^f &= \widetilde{A}_t + \widetilde{N}_t^f
\end{aligned}$$

Plugging the first and third of the above expressions into the middle, we get:

$$\widetilde{A}_t = \eta \left(\widetilde{Y}_t^f - \widetilde{A}_t \right) + \sigma \widetilde{Y}_t^f$$

Simplifying:

$$\widetilde{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \widetilde{A}_t \tag{95}$$

Unsurprisingly, this is the same log-linearized expression for the natural rate of output as we had before. Let's play around with the definition of μ_t a little bit:

$$\begin{aligned}
\widetilde{\mu}_t &= \eta \widetilde{N}_t + \sigma \widetilde{Y}_t - \widetilde{w}_t \\
\widetilde{\mu}_t &= \eta \left(\widetilde{Y}_t - \widetilde{A}_t \right) + \sigma \widetilde{Y}_t - \widetilde{w}_t \\
\widetilde{\mu}_t &= (\sigma+\eta)\widetilde{Y}_t - \eta \widetilde{A}_t - \widetilde{w}_t
\end{aligned}$$

Now, add and subtract $\widetilde{A}_t$ from the right hand side:

$$\widetilde{\mu}_t = (\sigma+\eta)\widetilde{Y}_t - (1+\eta)\widetilde{A}_t + \widetilde{A}_t - \widetilde{w}_t$$

Simplifying:

$$\widetilde{\mu}_t = (\sigma + \eta) \left(\widetilde{Y}_t - \frac{1+\eta}{\sigma+\eta} \widetilde{A}_t \right) - (w_t - \widetilde{A}_t)$$

Now, define the *real wage gap* as $\widetilde{X}_t^w = \widetilde{w}_t - \widetilde{A}_t$, since we know that the flexible price real wage would just be $\widetilde{w}_t^f = \widetilde{A}_t$. The output gap, $\widetilde{X}_t = \widetilde{Y}_t - \widetilde{Y}_t^f$, is the same as before. This means we can write this expression as:

$$\widetilde{\mu}_t = (\sigma + \eta)\widetilde{X}_t - \widetilde{X}_t^w \tag{96}$$

We can then plug this into the wage Phillips Curve to get:

$$\widetilde{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}(\sigma+\eta)\widetilde{X}_t - \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}\widetilde{X}_t^w + \beta E_t \widetilde{\pi}_{t+1}^w \tag{97}$$

The price Phillips Curve can be written in terms of the real wage gap, since real marginal cost is the same thing as the real wage gap:

$$\widetilde{\pi}_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{X}_t^w + \beta E_t \widetilde{\pi}_{t+1} \tag{98}$$

The Euler/IS equation can be written in terms of the output gap:

$$\widetilde{X}_t = E_t \widetilde{X}_{t+1} - \frac{1}{\sigma}\left(\widetilde{i}_t - E_t\widetilde{\pi}_{t+1} - \widetilde{r}_t^f\right) \tag{99}$$

Where:

$$\widetilde{r}_t^f = \sigma \frac{1+\eta}{\sigma+\eta}(\rho_a - 1)\widetilde{A}_t \tag{100}$$

We can re-write the wage inflation evolution equation in terms of the real wage gap as well:

$$\begin{aligned}
\widetilde{\pi}_t^w &= \widetilde{w}_t - \widetilde{A}_t + \widetilde{A}_t - \widetilde{w}_{t-1} + \widetilde{A}_{t-1} - \widetilde{A}_{t-1} + \widetilde{\pi}_t \\
\widetilde{\pi}_t^w &= \widetilde{X}_t^w - \widetilde{X}_{t-1}^w + \widetilde{A}_t - \widetilde{A}_{t-1} + \widetilde{\pi}_t
\end{aligned}$$

We can write the full system of equilibrium conditions as:

$$\widetilde{X}_t = E_t \widetilde{X}_{t+1} - \frac{1}{\sigma}\left(\widetilde{i}_t - E_t\widetilde{\pi}_{t+1} - \widetilde{r}_t^f\right) \tag{101}$$

$$\widetilde{\pi}_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{X}_t^w + \beta E_t \widetilde{\pi}_{t+1} \tag{102}$$

$$\widetilde{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}(\sigma+\eta)\widetilde{X}_t - \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}\widetilde{X}_t^w + \beta E_t \widetilde{\pi}_{t+1}^w \tag{103}$$

$$\widetilde{r}_t^f = \sigma \frac{1+\eta}{\sigma+\eta}(\rho_a - 1)\widetilde{A}_t \tag{104}$$

$$\widetilde{\pi}_t^w = \widetilde{X}_t^w - \widetilde{X}_{t-1}^w + \widetilde{A}_t - \widetilde{A}_{t-1} + \widetilde{\pi}_t \tag{105}$$

$$\widetilde{i}_t = \rho_i \widetilde{i}_{t-1} + (1-\rho_i)\phi_\pi \widetilde{\pi}_t + \varepsilon_{i,t} \tag{106}$$

$$\widetilde{A}_t = \rho_a \widetilde{A}_{t-1} + \varepsilon_{a,t} \tag{107}$$

There is no way to write the Price Phillips Curve solely in terms of the output gap when wages are sticky. In the model with just price stickiness, we were able to write marginal cost in terms of the output gap by eliminating the real wage using the static first order condition for labor supply, so we could write marginal cost just in terms of $\widetilde{Y}_t$ and $\widetilde{A}_t$. Here that isn't straightforward since the first order condition for labor supply is substantially more complicated.

7.2 Optimal Monetary Policy

As in the model with just price stickiness, it is possible to derive a welfare loss function from taking a second order approximation to the household's value function while using a first order approximation to the equilibrium conditions. The loss function now depends on the squared values of the output gap, price inflation, and wage inflation:

$$L = \frac{\lambda_p}{\epsilon_p}\widetilde{X}_t^2 + \widetilde{\pi}_t^2 + \frac{\lambda_p}{\lambda_w}\frac{\epsilon_w}{\epsilon_p}(\widetilde{\pi}_t^w)^2$$

These coefficients are given by:

$$\lambda_p = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}(\sigma+\eta)$$

$$\lambda_w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}$$

Hence, the relative weight on the output gap is the same as in the simpler model. λ_w is just the coefficient on the real wage gap in the wage Phillips Curve. The relative weight on wage inflation depends on (i) the relative coefficients λ_p and λ_w, and (ii) the relative elasticities of goods and labor demand, ϵ_p and ϵ_w.

Why is wage inflation an argument in the loss function? It shows up for an analogous reason to why price inflation shows up.

To think about aggregate welfare, we need to come up with a social welfare function since there is not a representative agent in this model. The easiest aggregate welfare function is the utilitarian one in which we sum up welfare of individual households. Individual welfare is:

$$V_t(l) = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \psi\frac{N_t(l)^{1+\eta}}{1+\eta} + \beta E_t V_{t+1}(l)$$

Define aggregate welfare as:

$$\mathcal{W}_t = \int_0^1 V_t(l)dl$$

$$\mathcal{W}_t = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{\psi}{1+\eta}\int_0^1 N_t(l)^{1+\eta}dl + \beta E_t \mathcal{W}_{t+1}$$

Now, note that the demand for labor of variety l:

$$N_t(l) = \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w} N_t$$

Plug this in to the expression for welfare above:

$$\mathcal{W}_t = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{\psi}{1+\eta}\int_0^1 \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w(1+\eta)} N_t^{1+\eta}dl + \beta E_t \mathcal{W}_{t+1}$$

Define:

$$v_t^w = \int_0^1 \left(\frac{W_t(l)}{W_t}\right)^{-\epsilon_w(1+\eta)}$$

This is a measure of wage dispersion, and it is bound from below by 1. This can be written recursively if we want as we previously did for prices, using the assumptions of the Calvo mechanism. Aggregate welfare then becomes:

$$\mathcal{W}_t = \frac{C_t^{1-\sigma}-1}{1-\sigma} - \frac{\psi}{1+\eta} v_t^w N_t^{1+\eta} + \beta E_t \mathcal{W}_{t+1} \tag{108}$$

Hence, the reason wage inflation matters is that wage dispersion effectively drives a wedge between labor supplied and labor used in production: if $v_t^w > 1$, there is some labor "lost" in the process, in a way analogous to how price dispersion results in some "lost" output.

Going back to the approximated loss function, the intuition for the relative weight on wage inflation is fairly intuitive. Price or wage inflation are costly to the extent to which prices or wages are sticky: if aggregate prices or wages move around, and prices are sticky, this induces price or wage dispersion. The bigger is ε_p, the more costly is price dispersion (the lower is the weight on wage inflation); the bigger is ε_w, the more costly is wage dispersion (the bigger is the weight on wage inflation). The stickier are prices, the smaller is λ_p, and hence the smaller is the relative weight on wage inflation (the bigger is the relative weight on price inflation). Conversely, the stickier are wages, the smaller λ_w is, and the bigger the relative weight on inflation.

It is instructive to think about what the relative weight on wage inflation ought to look like by considering some numeric values. Suppose that $\phi_p = \phi_w = 0.75$, and $\epsilon_p = \epsilon_w = 10$, with $\sigma = \eta = 1$, and $\beta = 0.99$. We get $\lambda_p = 0.1717$, but $\lambda_w = 0.0078$. This means that the relative weight on wage inflation is 22 – wage inflation is 22 times more important than price inflation. What really drives this is that the wage Phillips Curve is much "flatter" than the price Phillips Curve because of the presence of ϵ_w in the denominator of the slope coefficient.

Before doing anything quantitatively, it is useful to stop and think for a minute. In the basic NK model, it was possible to completely stabilize both inflation and the output gap, and therefore achieve a zero welfare loss. This was because stabilizing prices led to a stable output gap, and vice-versa. Here, it is in general not possible to simultaneously stabilize price inflation, the output gap, and wage inflation. This is easy to see. For the output gap to be zero, the real wage must equal its natural rate (which is in turn equal to $\widetilde{A}_t$). But for the real wage to equal its natural rate, either wages or prices must adjust to the extent to which $\widetilde{A}_t$ moves around. Hence, you can't simultaneously get the real wage to fluctuate (which it must if there are real shocks) without either prices or wages moving around. In other words, the presence of wage stickiness makes a central bank face a non-trivial tradeoff without having to resort to including a cost-push shock in the model.

Below I present welfare losses from a quantitative version of the model (using the parameters described above), along with $\rho_a = 0.95$ and $s_a = 0.01$, for different types of monetary policy:

Policy	L
Taylor Rule	-0.0020
Price Inflation Targeting	-1.0021
Wage Inflation Targeting	-0.0010
Gap Targeting	-0.0010

For the Taylor rule specification I use $\widetilde{i}_t = \rho_i \widetilde{i}_{t-1} + (1 - \rho_i)\left(\phi_\pi \widetilde{\pi}_t + \phi_x \widetilde{X}_t\right)$. This is pretty interesting in that we see that price inflation targeting does very poorly. The reason why is fairly transparent. If you target zero price inflation, then the real wage gap must be equal to zero from the price Phillips Curve. But a zero real wage gap means that real wages must move around one-for-one with $\widetilde{A}_t$. Given that prices can't move, this means we have to have a lot of wage inflation, and the relative weight on wage inflation is very high. Hence, price inflation targeting does poorly. Wage inflation targeting does very well, which makes sense given the high weight on wage inflation. Interestingly, output gap targeting does well too. Stabilizing the gap results in little wage inflation (and comparatively much price inflation), but given the relative weights this ends up doing well from a welfare perspective.

As in the simpler model with just sticky prices, we can derive formal first order conditions to characterize the optimal policy, but I leave that as an exercise for the interested reader.