Graduate Macro Theory II: Consumption-Saving Models: An Introduction to General Equilibrium

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1 Introduction

Macro is about general equilibrium: all agents must behave optimally and all markets must clear. You have previously learned about *decision problems* in which prices are taken as given (e.g. the permanent income hypothesis for consumption). In general equilibrium, the real interest rate is the intertemporal price of consumption, and adjusts to clear asset markets. We can dress up the problem to have lots of other markets (labor, investment goods, etc.), but the same general principle holds.

This note is going to work through simple, two-period, consumption-saving models. I like to call these "endowment economies," because there is no production and income is exogenous. Though simple, there are some powerful insights. In an endowment economy, the real interest rate adjusts so that agents simply consume their endowment each period – there is no possibility of moving resources across time, so the real interest rate adjusts to undo all the consumption-smoothing you've previously learned about. As a result, the equilibrium real interest rate can tell us something about (i) how plentiful the future is expected to be relative to the present and (ii) how uncertain the future is.

We can also use this simple framework to show how differences at the micro level (i.e. heterogeneity) don't necessarily matter for aggregate dynamics. This justifies the use of representative agent models. To justify the use of representative agent models, you need complete asset markets that allow agents to trade at the micro-level.

2 Deterministic Model

There are a continuum of agents populating the unit interval. These agents are indexed by $i \in [0, 1]$. Agents live for two periods, t and $t + 1$. Agents are endowed with an exogenous and deterministic stream of income, $Y_t(i)$ and $Y_{t+1}(i)$. Think of the units of income being fruit. Fruit is non-storable.

In period t, agents determine how much to consume, $C_t(i)$, and how much to save. Agents have access to a one-period riskless bond, $B_t(i)$, that trades at price q_t and pays out one-for-one in $t+1$. Agents are price-takers.

The budget constraint facing an agent in period t is:

$$C_t(i) + q_t B_t(i) \leq Y_t(i) \tag{1}$$

Note that $B_t(i)$ is a *stock* variable – how much (fruit) one takes from t to $t+1$. $B_t(i) > 0$ means an agent is saving, and $B_t(i) < 0$ means an agetn is borrowing. In principle, agents could carry fruit from $t+1$ into $t+2$, but we disallow this as a terminal condition – agents won't want to die with positive asset holdings, and are not permitted to die in debt. Hence, $B_{t+1}(i) = 0$. The $t+1$ budget constraint is:

$$C_{t+1}(i) \leq Y_{t+1}(i) + B_t(i) \tag{2}$$

Agents have a flow utility function given by $u(\cdot)$, where $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. Agents discount future utility flows by $0 < \beta < 1$. Lifetime utility is:

$$\mathbb{U}(i) = u(C_t(i)) + \beta u(C_{t+1}(i)) \tag{3}$$

The decision problem of an agent is to pick a consumption sequence, $C_t(i)$ and $C_{t+1}(i)$, and bond holdings, $B_t(i)$, to maximize lifetime utility subject to the two budget constraints:

$$\max_{C_t(i), C_{t+1}(i), B_t(i)} \quad u(C_t(i)) + \beta u(C_{t+1}(i))$$
$$\text{s.t.}$$

$$\begin{aligned} C_t(i) + q_t B_t(i) &\leq Y_t(i) \\ C_{t+1}(i) &\leq Y_{t+1}(i) + B_t(i) \end{aligned}$$

A Lagrangian is:

$$\mathbb{L} = u(C_t(i)) + \beta u(C_{t+1}(i)) + \lambda_{1,t}(i)\left(Y_t(i) - C_t(i) - q_t B_t(i)\right) + \lambda_{2,t}(i)\left(Y_{t+1}(i) + B_t(i) - C_{t+1}(i)\right)$$

First-order conditions (FOC):

$$\frac{\partial \mathbb{L}}{\partial C_t(i)} = 0 \iff u'(C_t(i)) = \lambda_{1,t}(i)$$

$$\frac{\partial \mathbb{L}}{\partial C_{t+1}(i)} = 0 \iff \beta u'(C_{t+1}(i)) = \lambda_{2,t}(i)$$

$$\frac{\partial \mathbb{L}}{\partial B_t(i)} = 0 \iff q_t \lambda_{1,t}(i) = \lambda_{2,t}(i)$$

Combining these, we get:

$$q_t = \frac{\beta u'(C_{t+1}(i))}{u'(C_t(i))} \tag{4}$$

(4) is a version of the typical consumption Euler equation. The term on the right-hand side is known as the *stochastic discount factor*, which is just the marginal rate of substitution between current and future consumption. (4) is a special case of a more general asset-pricing condition: the price of an asset (in this case, q_t), equals the expected product of the stochastic discount factor with the payout from the asset. In this simple case, the payout from the asset is known and equal to one, and there is no uncertainty over future consumption either.

Digression: Current vs. Present Value Lagrangians

Above, I wrote a "present value Lagrangian" – the multiplier on the $t+1$ constraint, $\lambda_{2,t}(i)$, equals the present value of marginal utility, $\beta u'(C_{t+1}(i))$. An alternative formulation that I will use more generally is a current value Lagrangian:

$$\mathbb{L} = u(C_t(i)) + \beta u(C_{t+1}(i)) + \lambda_t(i)\left(Y_t(i) - C_t(i) - q_t B_t(i)\right) + \beta \lambda_{t+1}(i)\left(Y_{t+1}(i) + B_t(i) - C_{t+1}(i)\right)$$

Here, instead of $\lambda_{1,t}(i)$ and $\lambda_{2,t}(i)$, I just have one Lagrange multiplier indexed by time, and I discount future constraints by β.[1] The FOC will be:

$$\frac{\partial \mathbb{L}}{\partial C_t(i)} = 0 \iff u'(C_t(i)) = \lambda_t(i)$$

$$\frac{\partial \mathbb{L}}{\partial C_{t+1}(i)} = 0 \iff u'(C_{t+1}(i)) = \lambda_{t+1}(i)$$

$$\frac{\partial \mathbb{L}}{\partial B_t(i)} = 0 \iff q_t \lambda_t(i) = \beta \lambda_{t+1}(i)$$

Which gives rise to exactly the same optimality condition, (4).

Digression: Alternative Treatment of Bonds

Above, I model the savings instrument as a one-period discount bond – you pay q_t for it, and it pays out a fixed face value of one in the future. An alternative normalization is to assume that there is a one-for-one transformation between consumptions and savings

[1] Note that we could also have multipliers indexed by time in the present value formulation.

in t, and savings brought from t to $t+1$ payout principle plus interest, $1+r_t$, where r_t is the (net) interest rate ($R_t = 1+r_t$) is gross). The budget constraints would be:

$$C_t(i) + B_t(i) \leq Y_t(i) \tag{5}$$

Note that $B_t(i)$ is a *stock* variable – how much (fruit) one takes from t to $t+1$. $B_t(i) > 0$ means an agent is saving, and $B_t(i) < 0$ means an agetn is borrowing. In principle, agents could carry fruit from $t+1$ into $t+2$, but we disallow this as a terminal condition – agents won't want to die with positive asset holdings, and are not permitted to die in debt. Hence, $B_{t+1}(i) = 0$. The $t+1$ budget constraint is:

$$C_{t+1}(i) \leq Y_{t+1}(i) + (1+r_t)B_t(i) \tag{6}$$

With this formulation, the two budget constraints can be combined into one, intertemporal budget constraint:

$$C_t(i) + \frac{C_{t+1}(i)}{1+r_t} \leq Y_t(i) + \frac{Y_{t+1}(i)}{1+r_t} \tag{7}$$

This says that the present discounted value of the stream of consumption cannot exceed the present discounted value of the stream of income.

The Lagrangian and FOC are:

$$\mathbb{L} = u(C_t(i)) + \beta u(C_{t+1}(i)) + \lambda_{1,t}(i)\left(Y_t(i) - C_t(i) - B_t(i)\right) + \lambda_{2,t}(i)\left(Y_{t+1}(i) + (1+r_t)B_t(i) - C_{t+1}(i)\right)$$

First-order conditions (FOC):

$$\frac{\partial \mathbb{L}}{\partial C_t(i)} = 0 \iff u'(C_t(i)) = \lambda_{1,t}(i)$$

$$\frac{\partial \mathbb{L}}{\partial C_{t+1}(i)} = 0 \iff \beta u'(C_{t+1}(i)) = \lambda_{2,t}(i)$$

$$\frac{\partial \mathbb{L}}{\partial B_t(i)} = 0 \iff \lambda_{1,t}(i) = \lambda_{2,t}(i)(1+r_t)$$

The resulting optimality condition is:

$$1 = \frac{\beta u'(C_{t+1}(i))}{u'(C_t(i))}(1+r_t) \tag{8}$$

This is exactly the same as above, where $q_t = \frac{1}{1+r_t} = \frac{1}{R_t}$. You see bonds treated both ways. These are effectively just different normalizations – are you normalizing the

t or $t+1$ value of the bond to be one?

The key insight from (4), or alternatively (8), is that all agents will choose the same *ratio* of consumption across time because they face a common asset price (q_t or $1/R_t$). To see this concretely, suppose that the utility function is natural log, so the Euler equation becomes:

$$1 = \beta(1+r_t)\frac{C_t(i)}{C_{t+1}(i)} \tag{9}$$

Alternatively, the (gross) growth rate of consumption between t and $t+1$, must be the same for all agents, and equal to the aggregate growth rate:

$$\frac{C_{t+1}(i)}{C_t(i)} = \frac{C_{t+1}}{C_t} = \beta(1+r_t) \tag{10}$$

An important insight here, that will hold for more general specifications of the flow utility function, is that consumption will be expected to grow when the real interest rate is high and vice-versa.

To close the model, we need to define a market-clearing condition. The market-clearing condition is simple: bonds must be in zero net supply. That is:

$$\int_0^1 B_t(i)di = 0$$

This is the market-clearing condition because there is no way, in the aggregate, to transfer resources across time (the good, fruit, is non-storable, and there is nothing like physical capital). Individuals could borrow or save, but in aggregate there can be no-borrowing or saving. Integrating the budget constraints across i, and letting them hold with equality, we get:

$$\int_0^1 C_t(i)di + q_t\int_0^1 B_t(i)di = \int_0^1 Y_t(i)di$$

$$\int_0^1 C_{t+1}(i)di = \int_0^1 Y_{t+1}(i)di + \int_0^1 B_t(i)di$$

Using the market-clearing condition for bonds, and defining aggregate consumption and income as the sums across agents, we have:

$$C_t = Y_t \tag{11}$$

$$C_{t+1} = Y_{t+1} \tag{12}$$

In other words, aggregate consumption just equals the aggregate endowment each period.

The competitive equilibrium of this economy is a price, r_t, and consumption bundles, $C_t(i)$ and $C_{t+1}(i)$, such that all agents behave optimally and all markets clear. But this is particularly straightforward to compute and we need not reference individual heterogeneity because the Euler equation implies that agents choose the same growth rate of consumption. So the competitive equilibrium is characterized by:

$$\frac{1}{1+r_t} = q_t = \frac{\beta u'(Y_{t+1})}{u'(Y_t)} \tag{13}$$

In other words, it is as though there is one agent in the economy – i.e. there exists a representative agent. We can determine the aggregate prices and quantities without reference to household-level heterogeneity. This does not require that all agents are ex-ante identical, nor that they are even ex-post identical.

To see this concretely, suppose that there are two kinds of agents. Type 1 agents have an endowment stream of $(Y_t(1), Y_{t+1}(1)) = (1, 0)$, and type 2 agents have an endowment stream of $(Y_t(2), Y_{t+1}(2)) = (0, 1)$. Suppose that there is a mass, $\alpha \in [0, 1]$ of type 1 agents, $1 - \alpha$ of type 2 agents. Suppose that both types of agents have log preferences. This means that the intertemporal budget constraints for both types of agents can be written:

$$(1+\beta)C_t(i) = Y_t(i) + \frac{Y_{t+1}(i)}{1+r_t} \tag{14}$$

This mean that the consumption and savings function for type 1 agents are:

$$C_t(1) = \frac{1}{1+\beta} \tag{15}$$

$$B_t(1) = \frac{\beta}{1+\beta} \tag{16}$$

Note that both consumption and saving for the type 1 agents are independent of the value of the real interest rate, r_t. This is because the income and substitution effects of r_t exactly cancel out with log preferences. For type 2 agents, we have:

$$C_t(2) = \frac{1}{(1+\beta)(1+r_t)} \tag{17}$$

$$B_t(2) = -\frac{1}{(1+\beta)(1+r_t)} \tag{18}$$

For type 2 agents, consumption demand is decreasing in r_t, and saving supply is increasing in r_t. The aggregate market-clearing condition is the same:

$$\int_0^1 B_t(i)di = \alpha B_t(1) + (1-\alpha)B_t(2)$$

Which is:

$$\alpha \frac{\beta}{1+\beta} - \frac{1-\alpha}{(1+\beta)(1+r_t)} = 0$$

Which may be written:

$$\alpha\beta(1 + r_t) - (1 - \alpha) = 0$$

Or:

$$1 + r_t = \frac{1}{\beta}\left(\frac{1-\alpha}{\alpha}\right) \tag{19}$$

This is the equilibrium (gross) interest rate as a function of exogenous parameters. We can then solve for the consumption bundles of both types of agents as:

$$C_t(1) = \frac{1}{1+\beta} \tag{20}$$

$$C_t(2) = \frac{1}{1+\beta}\frac{\beta\alpha}{1-\alpha} \tag{21}$$

Aggregate consumption is:

$$C_t = \int_0^1 C_t(i)di = \alpha C_t(1) + (1-\alpha)C_t(2) = \frac{\alpha}{1+\beta} + \frac{\alpha\beta}{1+\beta} = \alpha$$

Note that the aggregate endowment is:

$$Y_t = \int_0^1 Y_t(i)di = \alpha$$

In other words, the aggregate market-clearing condition is just $C_t = Y_t$ (which will also be the case for $t+1$). The aggregate enddowment in $t+1$ will be $1-\alpha$. This means that equilibrium real interest rate, from (19), satisfies:

$$1 + r_t = \frac{1}{\beta}\frac{Y_{t+1}}{Y_t} \tag{22}$$

Hopefully you're catching onto something here. The equilibrium of this endowment economy with two types of agents is *identical* to the equilibrium economy where there is one, representative agent. The representative agent problem with log utility would be:

$$\max_{C_t, C_{t+1}, B_t} \quad \ln C_t + \beta \ln C_{t+1}$$

$$\text{s.t.}$$

$$\begin{aligned} C_t + B_t &\leq Y_t \\ C_t &\leq Y_{t+1} + (1+r_t)B_t \end{aligned}$$

The optimality condition for the representative agent is the standard Euler equation:

$$1 = \beta(1+r_t)\frac{C_t}{C_{t+1}} \tag{23}$$

Imposing market clearing for the representative agent, we would have $B_t = 0$, implying $C_t = Y_t$ and hence $C_{t+1} = Y_{t+1}$, giving:

$$1 + r_t = \frac{1}{\beta}\frac{Y_{t+1}}{Y_t} \tag{24}$$

Which is *identical* to (22). Put differently: even though individual households are neither ex-ante identical (nor ex-post, because they will, in general, have different consumption bundles), the aggregate economy behaves as though there is just *one* household. Basically, what we need for this result is that agents can freely trade with one another.

2.1 The Equilibrium Real Interest Rate

Unless otherwise noted, let us just assume we are starting off with a representative agent economy. In this kind of two-period endowment economy, the key condition is the Euler equation combined with the market-clearing condition, giving:

$$1 = \beta(1+r_t)\frac{u'(Y_{t+1})}{u'(Y_t)} \tag{25}$$

For simplicity, assume log utility, so we have:

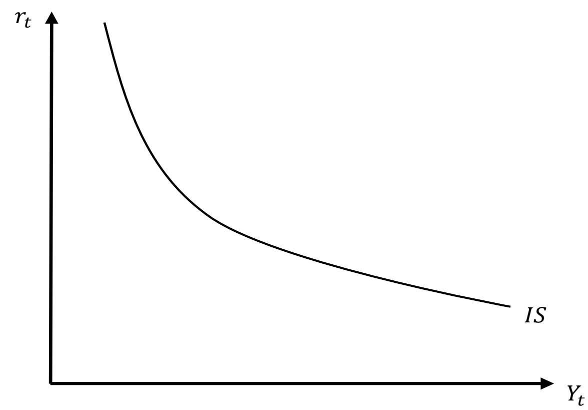
$$1 = \beta(1+r_t)\frac{Y_t}{Y_{t+1}}$$

Or:

$$Y_t = \frac{Y_{t+1}}{\beta(1+r_t)} \tag{26}$$

(26) is essentially an "IS curve" in undergraduate parlance (IS: investment equals saving). It shows how much fruit is *demanded* in the present as a function of the future *future* endowment, Y_{t+1}, the real interest rate, r_t, and a preference parameter governing impatience, β.

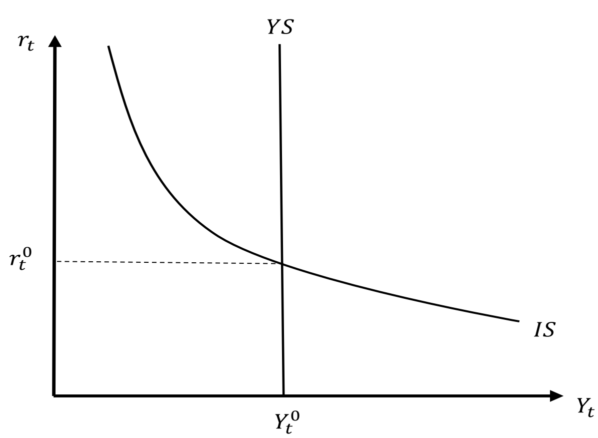
Figure 1: IS Graph



The IS curve shifts right if Y_{t+1} is higher (or left if lower). It shifts down (equivalently left) if β goes up (i.e. if there is an increase in patience).

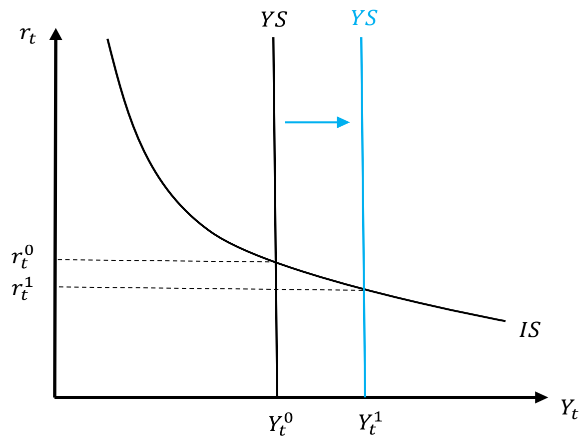
You graphically compute the equilibrium interest rate by combining this demand relationship with a supply relationship. Supply in this economy is very straightforward – it's exogenous (inelastic). Call this the YS curve, given at some exogenous level of the current endowment. This is shown below:

Figure 2: Graphical Equilibrium



Suppose that there is an increase in the current aggregate endowment, from Y_t^0 to Y_t^1. Let's call this a supply shock. We can see how the equilibrium real interest rate will react:

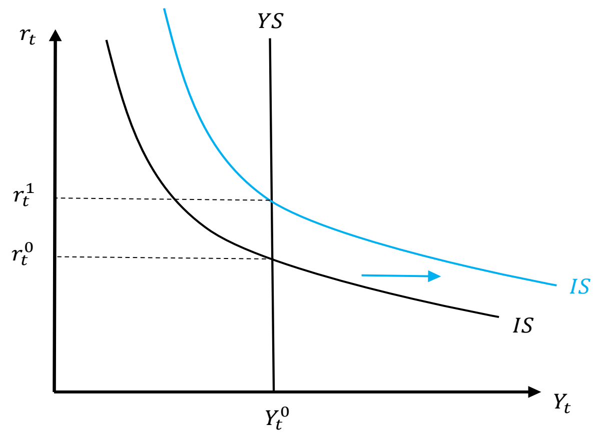
Figure 3: Supply Shock



We see that the equilibrium interest rate must call whenever the current endowment increases. Let's stop a second and think about why. *Holding the interest rate fixed*, we know from our earlier analysis of consumption (i.e. the PIH), agents would like to increase their saving so as to smooth out their consumption relative to their income. But, in equilibrium in an endowment economy like this, it is not possible to increase saving. The real interest rate must fall in an amount sufficient to make saving more unattractive. In equilibrium, the representative household will just eat the extra endowment today, even though, for a given interest rate, it would like to smooth.

Now let's consider a "demand shock." In particular, suppose the representative household knows that its future endowment, Y_{t+1}, is going to be higher. Holding the interest rate fixed, this makes the household want to consume more in the present (equivalently, save less, or borrow more). In other words, aggregate demand goes up (the IS curve shifts out). But, in equilibrium the household cannot save less – it has to just consume its current endowment, which is unchanged. To make the household satisfied with this, the real interest rate must rise. We can see this in the graph below.

Figure 4: Demand Shock



As an aside, a change in preferences, such as a reduction in β (a decrease in patience), would qualitatively produce the same effects as shown above. The household would want to consume more (save less) in the present, but given that this is not possible, the real interest rate would have to rise to make the household content just eating its (unchanged) current endowment.

There is a key point here that is quite transparent in a two-period consumption-saving model. That is this: *general equilibrium "undoes" the consumption smoothing that is desirable in the decision problem of the household.* This means that the equilibrium real interest rate is going to tell us something about how plentiful the future is relative to the present. We know from studying the decision problem that agents would like to save when the present is plentiful, and borrow when the future is expected to be relatively plentiful. The real interest rate works to undo this. When the present is relatively plentiful, the real interest rate falls. When the future is relatively plentiful, the real interest rate rises. This basic mechanism will be at play even in much larger equilibrium models.

3 Stochastic Model

Having established some work above, for now let us assume that there is just a representative agent. The basic consumption-saving problem is the same as before, but the future endowment is *uncertain.*

To make life as simple as possible, suppose there are two states of the world in the future: state 1 and state 2 (e.g. high endowment, low endowment). State 1 materializes with probability p, and state 2 with probability $1 - p$. The household want to maximize *expected* lifetime utility:

$$\mathbb{U} = u(C_t) + p\beta u(C_{t+1}(1)) + (1-p)\beta u(C_{t+1}(2)) = u(C_t) + \beta \mathbb{E}_t u(C_{t+1})$$

Note that the expectations operator is taken over expected utility, $\mathbb{E}_t u(C_{t+1})$, not utility of expected consumption. The period t budget constraint is the same as it ever was:

$$C_t + B_t \leq Y_t \tag{27}$$

There are two states of the world, and the budget constraint in period $t+1$ must hold in both states of the world (in other words, the budget constraint in $t+1$ must not only hold in expectation, it must hold ex-post as well). This means there are two budget constraints in $t+1$:

$$C_{t+1}(1) \leq Y_{t+1}(1) + (1+r_t)B_t \tag{28}$$
$$C_{t+1}(2) \leq Y_{t+2}(2) + (1+r_t)B_t \tag{29}$$

The household's problem is to pick C_t, B_t, and hence $C_{t+1}(1)$ and $C_{t+1}(2)$, to maximize expected utility subject to these budget constraints. Form a current value Lagrangian:

$$\begin{aligned}\mathbb{L} = u(C_t) + p\beta u(C_{t+1}(1)) + (1-p)\beta u(C_{t+1}(2)) + \lambda_t(Y_t - C_t - B_t) + \\ \beta\lambda_{t+1}(1)(Y_{t+1}(1) + (1+r_t)B_t - C_{t+1}(1)) + \beta\lambda_{t+1}(2)(Y_{t+1}(2) + (1+r_t)B_t - C_{t+1}(2))\end{aligned}$$

The FOC are:

$$\begin{aligned}\frac{\partial \mathbb{L}}{\partial C_t} &= 0 \iff u'(C_t) = \lambda_t \\ \frac{\partial \mathbb{L}}{\partial B_t} &= 0 \iff \lambda_t = \beta(1+r_t)(\lambda_{t+1}(1) + \lambda_{t+1}(2)) \\ \frac{\partial \mathbb{L}}{\partial C_{t+1}(1)} &= 0 \iff pu'(C_{t+1}(1)) = \lambda_{t+1}(1) \\ \frac{\partial \mathbb{L}}{\partial C_{t+1}(2)} &= 0 \iff (1-p)u'(C_{t+1}(2)) = \lambda_{t+1}(2)\end{aligned}$$

These can be combined to get:

$$u'(C_t) = \beta(1+r_t)\left(pu'(C_{t+1}(1)) + (1-p)u'(C_{t+1}(2))\right) = \beta(1+r_t)\mathbb{E}_t u'(C_{t+1}) \tag{30}$$

In other words, we can write this as:

$$q_t = \frac{1}{1+r_t} = \mathbb{E}_t\left(\frac{\beta u'(C_{t+1})}{u'(C_t)}\right) \tag{31}$$

I can write it this way because I can move $1/u'(C_t)$ and β inside or outside the expectations operator at will. This is the same Euler equation we had before, just with an expectations operator. It is a special case of the general point that the price of the bond equals the expected value of the product of the stochastic discount factor, $\frac{\beta u'(C_{t+1})}{u'(C_t)}$, with the payout on the asset (in this case, 1).

Note that, rather than assuming just two future states, we can consider more general probability distributions over the future endowment. We can write the Lagrangian more succinctly as:

$$\mathbb{L} = u(C_t) + \beta \mathbb{E}_t u(C_{t+1}) + \lambda_t \left(Y_t - C_t - B_t\right) + \beta \lambda_{t+1} \left(Y_{t+1} + (1+r_t)B_t - C_{t+1}\right)$$

Note that in writing the Lagrangian expectations are over future utility; the future budget constraint must hold ex-post in any state of the world. The FOC are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \iff u'(C_t) = \lambda_t$$
$$\frac{\partial \mathbb{L}}{\partial B_t} = 0 \iff \lambda_t = \beta(1+r_t)\lambda_{t+1}$$
$$\frac{\partial \mathbb{L}}{\partial C_{t+1}} = 0 \iff \mathbb{E}_t u'(C_{t+1}) = \lambda_{t+1}$$

Eliminating the multipliers, we get the same Euler equation:

$$u'(C_t) = \beta(1+r_t)\mathbb{E}_t u'(C_{t+1}) \tag{32}$$

The basic logic of how r_t (equivalently, q_t) adjusts to demand and supply is the same as before. However, with the future endowment uncertain, there is the potential for a different mechanism, and that is uncertainty. To see this concretely, suppose that we have log utility, that $Y_t = 1$, and that $\mathbb{E}_t Y_{t+1} = 1$. Suppose now that there is no uncertainty over the future endowment. The equilibrium real interest rate is easy to compute:

$$1 + r_t = \frac{1}{\beta}\frac{Y_{t+1}}{Y_t} = \frac{1}{\beta} \tag{33}$$

So, for example, if $\beta = 0.98$, $r_t = 0.0204$. Now suppose that the future endowment is uncertain. In particular, suppose there are two states, $Y_{t+1}(1) = 1.5$ and $Y_{t+1}(2) = 0.5$, each occurring with probability 1/2. With log utility, the equilibrium real interest rate will be equal to:

$$1 + r_t = \frac{1}{\beta}\frac{1}{Y_t}\frac{1}{\mathbb{E}_t u'(Y_{t+1})}$$

The expected value of future marginal utility is not the marginal utility of expected future consumption/endowment (an application of Jensen's inequality). Working this out using the numbers above, we have:

$$1 + r_t = \frac{1}{\beta}\left(\frac{1}{2}\times\frac{1}{1.5} + \frac{1}{2}\times\frac{1}{0.5}\right)^{-1} = \frac{1}{\beta}2\left(\frac{2}{3}+2\right)^{-1} = \frac{1}{\beta}2\times\frac{3}{8} = \frac{3}{4}\frac{1}{\beta}$$

In other words, the equilibrium real interest rate in this particular example where the future endowment is uncertainty is 3/4 its value when there is no uncertainty. Why is the equilibrium real rate lower? Because of *precautionary saving*. When the future is uncertain, the household would like to save more than when the future is deterministic. But, in equilibrium, the household cannot save more. Hence, the real interest rate has to be lower to make the household content to just eat its current endowment.

3.1 Idiosyncratic Uncertainty

Now, let us allow for the individual-level endowments to be uncertain. Suppose that there are two types of agents, 1 and 2. For simplicity, assume that the aggregate future endowment is deterministic. The $t+1$ endowments of the two types of agents are:

$$Y_{t+1}(1) = Y_{t+1} + e_t$$
$$Y_{t+1}(2) = Y_{t+1} - e_t$$

e_t is a mean-zero random variable. Suppose it takes on two values: $1/2$ and $-1/2$. In other words, sometimes the type 1 agent gets more of the endowment and sometimes the type 2 agent gets more. This means that there are effectively two states of nature in the future facing each agent, even though there is no aggregate uncertainty. Assume that agents can buy state-contingent bonds in period t. There are two bonds for both states. $B_{1,t}$ trades at $q_{1,t}$ and pays out when $e_t = 1/2$. $B_{2,t}$ trades at $q_{2,t}$ and pays out one-for-one when $e_t = -1/2$.

The period t budget constraints are:

$$C_t(1) + B_t(1) + q_{1,t}B_{1,t}(1) + q_{2,t}B_{2,t}(1) = Y_t$$
$$C_t(2) + B_t(2) + q_{1,t}B_{1,t}(1) + q_{2,t}B_{2,t}(1) = Y_t$$

The $t+1$ budget constraints facing each type of agent in both states of nature are therefore:

$$C_{1,t+1}(1) = Y_{t+1} + 1/2 + (1+r_t)B_t(1) + B_{1,t}(1)$$
$$C_{2,t+1}(1) = Y_{t+1} - 1/2 + (1+r_t)B_t(1) + B_{2,t}(1)$$
$$C_{1,t+1}(2) = Y_{t+1} - 1/2 + (1+r_t)B_t(2) + B_{1,t}(2)$$
$$C_{2,t+1}(2) = Y_{t+1} + 1/2 + (1+r_t)B_t(2) + B_{2,t}(2)$$

A Lagrangian for the type 1 household is:

$$
\begin{aligned}
\mathbb{L} = u(C_t(1)) + \frac{1}{2}\beta u(C_{1,t+1}(1)) + \frac{1}{2}\beta u(C_{2,t+1}(1)) + \lambda_t(1)\left(Y_t - C_t(1) - B_t(1) - q_{1,t}B_{1,t}(1) - q_{2,t}B_{2,t}(1)\right) \\
+ \beta\lambda_{1,t+1}(1)\left(Y_{t+1} + 1/2 + (1+r_t)B_t(1) + B_{1,t}(1) - C_{1,t+1}(1)\right) + \\
\beta\lambda_{2,t+1}\left(Y_{t+1} - 1/2 + (1+r_t)B_t(1) + B_{2,t}(1) - C_{2,t+1}(1)\right)
\end{aligned}
$$

The notation here is admittedly poor. I'm using 1 and 2 to denote the two states of the world (as well as the two agents). The FOC are:

$$
\begin{aligned}
\frac{\partial \mathbb{L}}{\partial C_t(1)} = 0 &\iff u'(C_t(1)) = \lambda_t(1) \\
\frac{\partial \mathbb{L}}{\partial B_t(1)} = 0 &\iff \lambda_t(1) = (1+r_t)\beta\left[\lambda_{1,t+1}(1) + \lambda_{2,t+1}(1)\right] \\
\frac{\partial \mathbb{L}}{\partial B_{1,t}(1)} = 0 &\iff \lambda_t(1)q_{1,t} = \beta\lambda_{1,t+1}(1) \\
\frac{\partial \mathbb{L}}{\partial B_{2,t}(1)} = 0 &\iff \lambda_t(1)q_{2,t} = \beta\lambda_{2,t+1}(1) \\
\frac{\partial \mathbb{L}}{\partial C_{1,t+1}(1)} = 0 &\iff \frac{1}{2}u'(C_{1,t+1}(1)) = \lambda_{1,t+1}(1) \\
\frac{\partial \mathbb{L}}{\partial C_{1,t+1}(1)} = 0 &\iff \frac{1}{2}u'(C_{2,t+1}(1)) = \lambda_{2,t+1}(1)
\end{aligned}
$$

Simplifying these, and making use of expectation operators, we get:

$$
\begin{aligned}
u'(C_t(1)) =& \beta(1+r_t)\mathbb{E}_t u'(C_{t+1}(1)) && (34) \\
u'(C_t(1))q_{1,t} =& \frac{\beta}{2}u'(C_{1,t+1}(1)) && (35) \\
u'(C_t(1))q_{2,t} =& \frac{\beta}{2}u'(C_{2,t+1}(1)) && (36)
\end{aligned}
$$

These can be re-written as asset pricing conditions:

$$
\begin{aligned}
q_t = \frac{1}{1+r_t} =& \mathbb{E}_t\left(\frac{\beta u'(C_{t+1}(1))}{u'(C_t(1))}\right) && (37) \\
q_{1,t} =& \mathbb{E}_t\left(\frac{\beta u'(C_{t+1}(1))}{u'(C_t(1))}D_{1,t+1}\right) && (38) \\
q_{2,t} =& \mathbb{E}_t\left(\frac{\beta u'(C_{t+1}(1))}{u'(C_t(1))}D_{2,t+1}\right) && (39)
\end{aligned}
$$

The first is the standard pricing contingent for the non-state-contingent bond. In the last two, I have introduced the notation that $D_{j,t+1}$ is the payout of each bond, $j = 1, 2$. $D_{1,t+1} = 1$ in state

1 and 0 otherwise, and the oppostie for $D_{2,t+1}$. The last two conditions look a little funny, but they are just specialized cases of the general stochastic discount factor pricing condition. The price you are willing to pay for an asset is just how you value that asset's payout in relative to how you value consumption today (i.e. the marginal rate of substitution).

One important thing to note. The FOC for both the type 1 and type 2 agents will take the identical form – the additive uncertainty in the budget constraints is irrelevant for the FOC. To make life a little easier, suppose that we have log utility. Since the FOC are the same, and the agents face the same prices, we have:

$$\frac{C_t(1)}{C_{1,t+1}(1)} = \frac{C_t(2)}{C_{1,t+1}(2)}$$
$$\frac{C_t(1)}{C_{2,t+1}(1)} = \frac{C_t(2)}{C_{2,t+1}(2)}$$

This means that:

$$\frac{C_t(1)}{C_t(2)} = \frac{C_{1,t+1}(1)}{C_{1,t+1}(2)} = \frac{C_{2,t+1}(1)}{C_{2,t+1}(2)} \tag{40}$$

In other words, the *relative* consumption of the two types of agents is the same, across both *time* and across *states*. Suppose that there is a mass 1/2 of each type of agent. The bond market-clearing conditions are:

$$\frac{1}{2}B_t(1) + \frac{1}{2}B_t(2) = 0$$
$$\frac{1}{2}B_{1,t}(1) + \frac{1}{2}B_{1,t}(2) = 0$$
$$\frac{1}{2}B_{2,t}(1) + \frac{1}{2}B_{2,t}(2) = 0$$

For the state-contingent bonds, this means that, if one type of agent is saving (has a positive position), the other must be borrowing (has a negative position), i.e. $B_{1,t}(1) = -B_{1,t}(2)$. Let us now integrate the period $t+1$ budget constraints together across agents for each state:

$$\frac{1}{2}C_{1,t+1}(1) + \frac{1}{2}C_{1,t+1}(2) = Y_{t+1} + (1+r_t)\frac{1}{2}(B_t(1) + B_t(2)) + \frac{1}{2}(B_{1,t}(1) + B_{1,t}(2))$$
$$\frac{1}{2}C_{2,t+1}(1) + \frac{1}{2}C_{2,t+1}(2) = Y_{t+1} + (1+r_t)\frac{1}{2}(B_t(1) + B_t(2)) + \frac{1}{2}(B_{2,t}(1) + B_{2,t}(2))$$

But with the market-clearing conditions, these just become:

$$\frac{1}{2}C_{1,t+1}(1) + \frac{1}{2}C_{1,t+1}(2)) = Y_{t+1} \tag{41}$$

$$\frac{1}{2}C_{2,t+1}(1) + \frac{1}{2}C_{2,t+1}(2)) = Y_{t+1} \tag{42}$$

I will skip a few steps, but we get exactly the same thing for the period t budget constraints, in particular:

$$\frac{1}{2}C_t(1) + \frac{1}{2}C_t(2) = Y_t \tag{43}$$

(41)-(43) simply say that aggregate consumption equals the aggregate endowment in each period and in each state of nature in the future.

Now, combine the two aggregate resource constraints in the two states of nature for $t+1$ together to get:

$$C_{1,t+1}(1) + C_{1,t+1}(2) = C_{2,t+1}(1) + C_{2,t+1}(2) \tag{44}$$

We can write this as:

$$C_{1,t+1}(2)\left(\frac{C_{1,t+1}(1)}{C_{1,t+1}(2)} + 1\right) = C_{2,t+1}(2)\left(\frac{C_{2,t+1}(1)}{C_{2,t+1}(2)} + 1\right) \tag{45}$$

But from the FOC above, (40), the terms in parentheses are equal, which means:

$$C_{1,t+1}(2) = C_{2,t+1}(2) \tag{46}$$

In other words, consumption for type 2 agents in $t+1$ will be *identical* regardless of which idiosyncratic state materializes. But if this is true for type 2 agents, then it's also true for type 1 agents in $t+1$. Which then also means that each type of agent has the same consumption in period t. Which means that the whole problem collapses to a representative agent problem. The idiosyncratic risk does not matter for aggregate dynamics, because it is functionally insured away with the state-contingent bonds, $B_{1,t}$ and $B_{2,t}$. As long as we have a set of complete markets that cover all idiosyncratic risk, we can sweep that under the rug and justify a representative agent representation of the problem.

4 Infinite Horizon Problem

The infinite horizon version of these problems is not much different. In a representative agent version, we would have the problem:

$$\max_{C_{t+j}, B_{t+j}} \quad \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j u(C_{t+j})$$

s.t.

$$C_{t+j} + B_{t+j} \leq Y_{t+j} + (1 + r_{t+j-1})B_{t+j-1}$$

Here, B_{t-1} (the initial stock of debt) is given (assume zero). The idea is to pick a sequence of consumption and saving to maximize expected lifetime utility. We can form a current value Lagrangian:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(C_t) + \lambda_t \left(Y_t + (1 + r_{t-1})B_{t-1} - C_t - B_t \right) \right]$$

Note that I am switching up timing notation here a bit in the formulation of the Lagrangian relative to the specification of the problem. When time "starts" is arbitrary – I wrote the problem as starting in t and going forward. I wrote the Lagrangian as starting in period 0 and going forward. The starting point is completely arbitrary – what matters is correctly keeping track of the past, the future, and the present. It saves space to start the Lagrangian in period 0 – because then you can just index time by t instead of $t + j$.

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \iff u'(C_t) = \lambda_t$$
$$\frac{\partial \mathbb{L}}{\partial B_t} = 0 \iff \lambda_t = \beta \mathbb{E}_t \lambda_{t+1}(1 + r_t)$$

Combining, we get:

$$1 = \mathbb{E}_t \left(\frac{\beta u'(C_{t+1})}{u'(C_t)} (1 + r_t) \right) \tag{47}$$

This is the same Euler equation as before. It is necessary for an optimal consumption plan. But it is not sufficient. There will in general be many consumption plans consistent with the Euler equation. What we need to close things out is a terminal condition. In the two-period model, the terminal condition was simple – we imposed that the household died with no savings (nor any debt). Here that is a little more complicated. The transversality condition here is:

$$\lim_{T \to \infty} \quad \mathbb{E}_t \beta^T \lambda_{t+T} B_{t+T} = 0 \tag{48}$$

The transversality condition says that the expected discounted utility value of any leftover savings "at the end of time" is zero. λ_{t+T} is the current value multiplier (marginal utility of consumption at $t + T$, measured in $t + T$ utils). Multiplication by β^T discounts it back to the present. The transversality condition says you place zero value on any leftover saving. This rules out over-saving (or over-borrowing).

4.1 A Value Function Representation

In this model, we have two state variables (Y_t and B_{t-1}), and one choice variable, C_t. A value function represents a multi-horizon problem as a two-period problem. Drop the t subscripts, and denote the present with no superscript and the future with a $'$ superscript. In particular:

$$V(Y, B) = \max_{C, B'} \quad u(C) + \beta \mathbb{E} V(Y', B')$$

s.t.

$$C + B' = Y + (1+r)B$$

The value of being in a particular state, $V(\cdot)$, is the maximized value of the current flow objective function, $u(C)$, plus the discounted value of being in the next state, subject to the budget constraint. In writing the budget constraint, I have made a slight change in notation for the timing on bond holdings: B is what you enter the period with, and B' is what you choose today that is available to you tomorrow. $V(\cdot)$ is an unknown function, but assume that it is differentiable.

Re-write the problem by eliminating C and write it in terms of choosing the future state (if there were no endogenous state variable, the problem would effectively be static):

$$V(Y,B) = \max_{B'} \quad u\left(Y + (1+r)B - B'\right) + \beta \mathbb{E} V(Y', B')$$

We need a first a first order condition to take care of the max operator. Differentiate with respect to B' and set equal to zero.

$$u'\left(Y + (1+r)B - B'\right) = \beta \mathbb{E} \frac{\partial V(Y', B')}{\partial B'}$$

Now, since we don't know the function $V(\cdot)$, this doesn't look so useful. But we can make use of an envelope condition (Benveniste-Scheinkman) to make progress. Assume that B' has been chosen optimally, so the max operator is taken care of:

$$V(Y,B) = u\left(Y + (1+r)B - B'\right) + \beta \mathbb{E} V(Y', B')$$

Differentiate this with respect to B (the existing stock of savings):

$$\frac{\partial V(Y,B)}{\partial B} = (1+r)u'(Y + (1+r)B - B') - u'(Y + (1+r)B - B')\frac{dB'}{dB} + \beta \mathbb{E} \frac{\partial V(Y', B')}{\partial B'}\frac{dB'}{dB}$$

But this can be re-arranged:

$$\frac{\partial V(Y,B)}{\partial B} = (1+r)u'(Y + (1+r)B - B') - \left[u'(Y + (1+r)B - B') - \beta \mathbb{E} \frac{\partial V(Y', B')}{\partial B'}\right]\frac{dB'}{dB}$$

But the term in brackets is zero from the first-order condition! So we are left with:

$$\frac{\partial V(Y,B)}{\partial B} = (1+r)u'(Y + (1+r)B - B') = (1+r)u'(C)$$

Hence:

$$\frac{\partial V(Y', B')}{\partial B'} = (1+r)u'(C')$$

So, plugging back into the FOC:

$$u'(C) = \beta(1+r)\mathbb{E}u'(C') \tag{49}$$

This is exactly the same FOC we get with the Lagrangian formulation.

4.2 Permanent vs. Transitory Changes in the Endowment

We know from our study of the Permanent Income Hypotheses (PIH) that, holding the real interest rate fixed, households want to *save* most of a transitory increase in income but *consume* most of a permanent increase in income. This logic is based on the decision problem, taking the intertemporal price of consumption as given. What happens in equilibrium where r_t adjusts to make asset markets clear (which, in an endowment economy, is that $B_t = 0$?

The market-clearing condition in an endowment economy requires that $C_t = Y_t$ each period. Imposing this in the Euler equation, we have:

$$1 = \mathbb{E}_t \left(\frac{\beta u'(Y_{t+1})}{u'(Y_t)} (1+r_t) \right)$$

Since $1+r_t$ is known relative to the expectation operator, we can write the equilibrium condition as:

$$1 + r_t = \left(\mathbb{E}_t \left(\frac{\beta u'(Y_{t+1})}{u'(Y_t)} \right) \right)^{-1} \tag{50}$$

From this, we can intuit how r_t will react to more or less persistent changes in income. Suppose that Y_t increases but there is no change in expected future endowments (i.e. a transitory change in income). $u'(Y_t)$ will fall a lot, which will push up the stochastic discount factor, which means that r_t must fall. In contrast, suppose that there is a permanent change in income. Y_{t+1} will go up as much as Y_t, so the stochastic discount factor won't change, and hence r_t won't change much (or at all).

What is going on here? r_t is adjusting to *undo* the saving behavior that the household wants to do. When faced with a transitory increase in income, the household wants to increase its saving. But in equilibrium it can't. So the real interest rate must fall to make the household content to not increase its saving. In contrast, when there is a permanent change in income, the household doesn't want to adjust its saving/borrowing behavior – it is fine just eating its extra endowment. But then the real interest rate doesn't need to change much (or at all).