Graduate Macro Theory II: The Real Business Cycle Model

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1 Introduction

This note describes the canonical real business cycle model. A couple of classic references here are Kydland and Prescott (1982), King, Plosser, and Rebelo (1988), and King and Rebelo (2000). The model is essentially just the neoclassical growth model augmented to have variable labor supply.

2 The Decentralized Model

I will set the problem up as a decentralized model, studying first the behavior of households and then the behavior of firms.

There are two primary ways of setting the model up, which both yield identical solutions. In both households own the firms, but management and ownership are distinct, and so households behave as though firm profits are given. In one formulation firms own the capital stock. In another formulation, households own the capital stock and rent it to firms. We will go through both formulations. Absent some other kind of friction, it doesn't matter how we specify ownership.

In both setups I abstract from trend growth, which, as we have seen, does not really make much of a difference anyway.

2.1 Firms Own the Capital Stock

Here we assume that firms own the capital stock. We begin with the household problem.

2.1.1 Household Problem

There is a representative household. It discounts the future by $\beta < 1$. It supplies labor (measured in hours), N_t, and consumes, C_t. It gets utility from consumption and leisure; with the time endowment normalized to unity, leisure is $1 - N_t$. It earns a wage rate, w_t, which it takes as given. It holds bonds, B_t, which pay interest rate r_{t-1}. r_{t-1} is the interest rate known at $t-1$ which pays out in t; r_t is the interest rate known in t which pays out in $t+1$. $B_t > 0$ means that the household has a positive stock of savings; $B_t < 0$ means the household has a stock of debt. Note that "savings" is a stock; "saving" is a flow. The household takes the interest rate as given. Its budget constraint says that each period, total expenditure cannot exceed total income. It earns wage income, $w_t N_t$, profit distributions in the form of dividends, Π_t, and interest income on existing bond holds, $r_{t-1} B_t$ (note this can be negative, so that there is an interest cost of servicing debt). Household expenditure is composed of consumption, C_t and saving, $B_{t+1} - B_t$ (i.e. the accumulation of new savings). Hence we can write the constraint:

$$C_t + (B_{t+1} - B_t) \leq w_t N_t + \Pi_t + r_{t-1} B_t \tag{1}$$

Note a timing convention – r_{t-1} is the interest you have to pay today on existing debt. r_t is what you will have to pay tomorrow, but you choose how much debt to take into tomorrow today. Hence, we assume that the household observes r_t in time t. Hence we can treat r_t as known from the perspective of time t. The household chooses consumption, work effort, and the new stock of savings each period to maximize the present discounted value of flow utility:

$$\max_{C_t, N_t, B_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) \right)$$

s.t.

$$C_t + B_{t+1} \leq w_t N_t + \Pi_t + (1 + r_{t-1}) B_t$$

We can form a current value Lagrangian:

$$\mathbb{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) + \lambda_t (w_t N_t + \Pi_t + (1 + r_{t-1}) B_t - C_t - B_{t+1}) \right)$$

The first order conditions characterizing an interior solution are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \iff u'(C_t) = \lambda_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \iff v'(1 - N_t) = \lambda_t w_t \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \iff \lambda_t = \beta E_t \lambda_{t+1} (1 + r_t) \tag{4}$$

These can be combined together to yield:

$$u'(C_t) = \beta E_t \left(u'(C_{t+1})(1 + r_t) \right) \tag{5}$$

$$v'(1 - N_t) = u'(C_t) w_t \tag{6}$$

(5) and (6) have very intuitive, intermediate micro type interpretations. (5) says that the price you're willing to pay for a bond today (normalize to unity) is the expected value of the product of the stochastic discount factor, $\beta u'(C_{t+1})/u'(C_t)$ (which is just the marginal rate of substitution between future and current consumption), with the return on the bond, $1 + r_t$. (6) says to equate the marginal rate of substitution between leisure and consumption (i.e. $\frac{v'(1-n_t)}{u'(c_t)}$) to the relative price of leisure (i.e. w_t).

In addition, there is the transversality condition:

$$\lim_{t \to \infty} \quad \beta^t B_{t+1} u'(C_t) = 0 \tag{7}$$

This condition just rules out over-saving or dying in debt.

2.1.2 The Firm Problem

There is a representative firm. The firm wants to maximize the present discounted value of (real) net revenues (i.e. cash flows). It discounts future cash flows by the stochastic discount factor of the household. The way I'll write the stochastic discount factor puts cash flows (measured in goods) in terms of *current* consumption (we take the current period to be $t = 0$). Define the stochastic discount factor as:

$$\Lambda_{0,t} = \beta^t \frac{u'(C_t)}{u'(C_0)}$$

The firm discounts by this because this is how consumers value future dividend flows. One unit of dividends returned to the household at time t generates $u'(C_t)$ additional units of utility, which must be discounted back to the present period (which we take to be 0), by β^t. Dividing by $u'(C_0)$ gives the period 0 consumption equivalent value of the future utils. So $\Lambda_{0,t}$ is the relative valuation the household attaches to income received in t from the perspective of period 0; alternatively, $\Lambda_{t,t+j}$ is the valuation in period t of income received in $t + j$. The stochastic discount factor is itself a random variable (hence "stochastic"): from the perspective of period 0, one doesn't know what marginal utility will be in period t.

The firm produces output, Y_t, according to a constant returns to scale production function, $Y_t = A_t F(K_t, N_t)$, with the usual properties. It hires labor, purchases new capital goods, and issues debt. I denote its debt as D_t, and it pays interest on its debt, r_{t-1}. Its revenue each period is equal to output. Its costs each period are the wage bill, investment in new physical capital, and servicing costs on its debt. Investment in new physical capital is assumed to be one-for-one in terms of output. It can raise its cash flow by issuing new debt (i.e. $D_{t+1} - D_t$ raises cash flow). The firm can essentially issue equity to finance capital by reducing its dividend. Its problem can be written as:

$$\max_{N_t, I_t, D_{t+1}, K_{t+1}} \quad V_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \left(A_t F(K_t, N_t) - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t \right)$$

s.t.

$$K_{t+1} = I_t + (1 - \delta) K_t$$

We can re-write the problem by imposing that the constraint hold each period:

$$\max_{N_t, K_{t+1}, D_{t+1}} \quad V_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \left(A_t F(K_t, N_t) - w_t N_t - K_{t+1} + (1 - \delta) K_t + D_{t+1} - (1 + r_{t-1}) D_t \right)$$

The first order conditions are as follows:

$$\frac{\partial V_0}{\partial N_t} = 0 \Leftrightarrow A_t F_N(K_t, N_t) = w_t \tag{8}$$

$$\frac{\partial V_0}{\partial K_{t+1}} = 0 \Leftrightarrow 1 = \mathbb{E}_t \Lambda_{t,t+1} ((A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta)) \tag{9}$$

$$\frac{\partial V_0}{\partial D_{t+1}} = 0 \Leftrightarrow 1 = \mathbb{E}_t \Lambda_{t,t+1} (1 + r_t) \tag{10}$$

Note that (10) is the same as (6). This must therefore must hold in equilibrium as long as the household is optimizing. What this means, in practice, is that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of D_{t+1}. This is essentially the Modigliani-Miller theorem – it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

2.1.3 Closing the Model

To close the model we need to specify a stochastic process for the exogenous variable(s). The only exogenous variable in the model is A_t. We assume that it is well-characterized as following a mean zero AR(1) in the log (we have abstracted from trend growth):

$$\ln A_t = \rho \ln A_{t-1} + s_A \varepsilon_t \tag{11}$$

We assume that $\varepsilon_t \sim N(0, 1)$; scaling by s_A gives us the standard deviation of shocks.

2.1.4 Equilibrium

A competitive equilibrium is a set of prices (r_t, w_t) and allocations $(C_t, N_t, K_{t+1}, D_{t+1}, B_{t+1})$ taking K_t, D_t, B_t, A_t and the stochastic process for A_t as given; such that the optimality conditions (5)-(6), (8)-(10), and the transversality condition all hold; and all markets clear. The labor market clearing just means that w_t is such that the N_t hired by the is the same as the N_t supplied by the household. The bond market clearing means that $B_t = D_t$ in all periods, so $B_{t+1} = D_{t+1}$ as well). This just means that any debt issued by the firm is held by the household (and vice-versa).

Let's consolidate the household and firm budget constraints. Just combine the definition of flow accounting profit with the capital accumulation equation and the household budget constraint at equality:

$$C_t + (B_{t+1} - B_t) = w_t N_t + r_{t-1} B_t + A_t F(K_t, N_t) - w_t N_t - I_t + D_{t+1} - (1 + r_{t-1}) D_t$$

Imposing the market-clearing conditions, we get:

$$A_t F(K_t, N_t) = C_t + I_t \tag{12}$$

In other words, bond market-clearing plus both budget constraints holding just gives the standard aggregate resource constraint (or accounting identity) that output, $Y_t = A_t F(K_t, N_t)$, must be consumed or invested in new physical capital.

If you combine the household's first order condition for labor supply with the firm's condition, you get:

$$v'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t)$$

If you combine the household's first order condition for bonds with the firm's first order condition for capital, you get:

$$\mathbb{E}_t \Lambda_{t,t+1}(1 + r_t) = \mathbb{E}_t \Lambda_{t,t+1} \left(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \right)$$

Which may be written:

$$1 + r_t = \frac{\mathbb{E}_t \Lambda_{t,t+1} \left(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1 - \delta) \right)}{\mathbb{E}_t \Lambda_{t,t+1}} \tag{13}$$

I can do this because $1 + r_t$ is known from the perspective of period t, so I can move it out side the expectations operator. Now, one would be tempted to "cancel" the expected values of the stochastic discount factors, but one cannot, in general, do this because in the numerator we have the expected value of a product of two random variables (the stochastic discount factor with the future marginal product of capital). In a first-order approximation we would be able to do this, but in general we will not be able to. In effect, there will be a slight "risk premium" to physical capital relative to riskless bonds, but suffice it to say there will be a close connection between the real interest rate on bonds and the marginal product of physical capital.

2.2 Households Own the Capital Stock

Now we consider a version of the decentralized problem in which the household owns the capital stock and rents it to firms. Otherwise the structure of the problem is the same.

2.2.1 Household Problem

As before, the household consumes and supplies labor. Now it also owns the capital stock. It earns a rental rate for renting out the capital stock to firms each period, R_t^k. The household budget constraint is:

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t = w_t N_t + R_t^k K_t + r_{t-1}B_t + \Pi_t \tag{14}$$

The household has income comprised of labor income, capital income, interest income, and profits (again it takes profits as given). It can consume this, accumulate more capital (this is the $K_{t+1} - (1-\delta)K_t$ term), or accumulate more saving. Its problem is:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1-N_t) \right)$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + B_{t+1} - B_t = w_t N_t + R_t^k K_t + r_{t-1}B_t + \Pi_t$$

Form a current value Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big(u(C_t) + v(1-N_t) + \lambda_t (w_t N_t + R_t^k K_t + (1+r_{t-1})B_t + \Pi_t - C_t - K_{t+1} + (1-\delta)K_t - B_{t+1}) \Big)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \iff u'(C_t) = \lambda_t \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = 0 \iff v'(1-N_t) = \lambda_t w_t \tag{16}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \iff \lambda_t = \beta \mathbb{E}_t \lambda_{t+1} (R_{t+1}^k + (1-\delta)) \tag{17}$$

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = 0 \iff \lambda_t = \beta \mathbb{E}_t \lambda_{t+1} (1 + r_t) \tag{18}$$

These first order conditions can be combined to yield:

$$v'(1-N_t) = u'(C_t)w_t \tag{19}$$

$$u'(C_t) = \beta \mathbb{E}_t u'(C_{t+1})(R^k_{t+1} + (1-\delta)) \tag{20}$$

$$u'(C_t) = \beta \mathbb{E}_t u'(C_{t+1})(1+r_t) \tag{21}$$

Note that the labor supply condition and the Euler equation for bonds are the same as in the earlier setup.

2.2.2 The Firm Problem

The firm problem is similar to before, but now it doesn't choose investment. Rather, it chooses capital *today* given the rental rate, R^k_t. Note that the firm can vary capital today even though the household cannot given that capital is predetermined. The labor choice and debt choice are similar. In fact, because the amount of the debt is going to end up being indeterminate, it is common to just assume that firms don't issue/hold debt and just solve a static problem. Again, the firm wants to maximize the present discounted value of cash flows.

$$\max_{N_t, K_t, D_{t+1}} \quad V_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \left(A_t F(K_t, N_t) - w_t N_t - R^k_t K_t + D_{t+1} - (1+r_{t-1})D_t \right)$$

The first order conditions are:

$$\frac{\partial V_0}{\partial N_t} = 0 \iff A_t F_K(K_t, N_t) = w_t \tag{22}$$

$$\frac{\partial V_0}{\partial K_t} = 0 \iff A_t F_K(K_t, N_t) = R_t \tag{23}$$

$$\frac{\partial V_0}{\partial D_{t+1}} = 0 \iff 1 = \mathbb{E}_t \Lambda_{t,t+1}(1+r_t) \tag{24}$$

The first order condition for debt is again identical to the one for the household; so again the amount of debt is indeterminate.

2.2.3 Equivalence to the Other Setup

It is pretty straightforward to see that either setup gives you identical conditions. Take the FOC for capital demand for the firm and plug it into the household's Euler equation for capital:

$$u'(C_t) = \beta E_t(u'(C_{t+1})(A_{t+1}F_K(K_{t+1}, N_{t+1}) + (1-\delta)) \tag{25}$$

This is identical to the firm FOC for capital when the firm owns the capital stock. Hence, all the first order conditions are the same. The definition of equilibrium is the same. Both the firm and household budget constraints holding again give rise to the accounting identity (14). Hence, these setups give rise to identical solutions. It simply does not matter whether households own the capital stock and lease it to firms or whether firms own the capital stock. Since households own firms, these are equivalent ownership structures.

3 A Planner's Version of the Problem

A central planner just picks allocations to maximize the objective of the household in the model. Prices do not appear in the planner's problem. It is:

$$\max_{C_t, N_t, K_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1 - N_t) \right)$$

$$\text{s.t.}$$

$$C_t + K_{t+1} - (1-\delta)K_t \leq A_t F(K_t, N_t)$$

A Lagrangian is:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[u(C_t) + v(1 - N_t) + \lambda_t \left(A_t F(K_t, N_t) - C_t - K_{t+1} + (1-\delta)K_t \right) \right]$$

The FOC are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \iff u'(C_t) = \lambda_t$$
$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \iff v'(1 - N_t) = \lambda_t A_t F_N(K_t, N_t)$$
$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \iff \lambda_t = \beta \mathbb{E}_t \lambda_{t+1} (A_{t+1} F_K(K_{t+1}, N_{t+1} + (1-\delta))$$

Eliminating the multipliers yields:

$$u'(C_t) = \beta \mathbb{E}_t u'(C_{t+1}) \left(A_{t+1} F_K(K_{t+1}, N_{t+1}) + (1-\delta) \right) \tag{26}$$

$$v'(1 - N_t) = u'(C_t) A_t F_N(K_t, N_t) \tag{27}$$

These, plus the law of motion for capital (and a transversality condition) constitute the solution to the planner's problem. Note that if we eliminate the prices in the decentralized problem, we arrive at exactly these same conditions. In other word's, the planner's solution coincides with the decentralized equilibrium. The competitive equilibrium is efficient.

4 Analysis of the Decentralized Model

We can combined first order conditions from the firm and household problems (in either setup) to yield the equilibrium conditions:

$$u'(C_t) = \beta \mathbb{E}_t \left(u'(C_{t+1})(R^k_{t+1} + (1-\delta) \right) \tag{28}$$

$$v'(1-N_t) = u'(C_t)w_t \tag{29}$$

$$K_{t+1} = A_t F(K_t, N_t) - C_t + (1-\delta)K_t \tag{30}$$

$$\ln A_t = \rho \ln A_{t-1} + s_A \varepsilon_t \tag{31}$$

$$Y_t = A_t F(K_t, N_t) \tag{32}$$

$$Y_t = C_t + I_t \tag{33}$$

$$u'(C_t) = \beta \mathbb{E}_t u'(C_{t+1})(1 + r_t) \tag{34}$$

$$w_t = A_t F_N(K_t, N_t) \tag{35}$$

$$R^K_t = A_t F_K(K_t, N_t) \tag{36}$$

(28) is the Euler equation for capital – I'm going to write this in terms of the rental rate on capital (in the setup where household's own the capital stock), but it doesn't matter. This is functionally just an asset pricing condition, as discussed above. (29) is a labor supply condition. (30) is the law of motion for capital, while (31) is an exogenous process for productivity. (32) is the aggregate production function. (33) is the aggregate resource constraint. (34) is the Euler equation for bonds. (35) is a labor demand condition, and (36) is a capital demand condition.

We have nine variables, C_t, N_t, K_t, Y_t, I_t, r_t, R^k_t, w_t, and A_t. We always need the same number of variables as we have equations. Three of these are prices – r_t, w_t, and R^K_t. One is exogenous (A_t). The others are quantities or flow units of time (N_t). Note that we do not need to keep track of bonds. The level of debt is indeterminate absent some other kind of friction. So it is common to assume that no one issues debt or holds bonds in these models.

We need to specify functional forms. For simplicity, assume that $u(C_t) = \ln C_t$ and $v(1-N_t) = \theta \ln(1-N_t)$. Assume that the production function is Cobb-Douglas: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$.

Given these parameter values we can analyze the steady state. I will use a lack of a time subscript, rather than a $*$ superscript, to denote steady state values. The steady state is a situation in which $A = 1$ (its non-stochastic unconditional mean), $K_{t+1} = K_t = K$, and $C_{t+1} = C_t = C$. Given the steady state values of these variables, the steady state values of the static variables can be backed out.

Start with the Euler equation for capital.

$$1 = \beta(R^k + (1-\delta))$$

This means:

$$R^k = \frac{1}{\beta} - (1-\delta) \tag{37}$$

Similar, the steady state real interest rate is:

$$r = \frac{1}{\beta} - 1 \tag{38}$$

This means: $R^k = r + \delta$. The key is to solve ro the capital-labor ratio and go from there. From the capital demand condition, we have:

$$R^K = \alpha \left(\frac{K}{N}\right)^{\alpha-1}$$

But this means:

$$\frac{K}{N} = \left(\frac{\alpha}{\frac{1}{\beta} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{39}$$

Now that we have the capital-labor ratio, we can get the steady state wage in terms of that:

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha} \tag{40}$$

Now, combine the capital accumulation equation with the resource constraint, but divide everything by N:

$$\left(\frac{K}{N}\right)^{\alpha} = \frac{C}{N} + \delta\frac{K}{N}$$

This means we can write the consumption-labor ratio in terms of the capital-labor ratio as:

$$\frac{C}{N} = \left(\frac{K}{N}\right)^{\alpha} - \delta\frac{K}{N}$$

Now go to the labor supply condition, written in terms of K/N:

$$\theta\frac{1}{1-N} = \frac{1}{C}(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$

Multiply and divide both sides by N:

$$\theta\frac{N}{1-N} = \frac{N}{C}(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$

Using our expression for the consumption-labor ratio, we can write this as:

$$\theta\frac{N}{1-N} = \frac{(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}}{\left(\frac{K}{N}\right)^{\alpha} - \delta\frac{K}{N}}$$

This can be written:

$$\frac{1-N}{N} = \theta \frac{\left(\frac{K}{N}\right)^{\alpha} - \delta \frac{K}{N}}{(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}}$$

Which implies:

$$\frac{1}{N} = 1 + \theta \frac{\left(\frac{K}{N}\right)^{\alpha} - \delta \frac{K}{N}}{(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}}$$

Or:

$$N = \left(1 + \theta \frac{\left(\frac{K}{N}\right)^{\alpha} - \delta \frac{K}{N}}{(1-\alpha)\left(\frac{K}{N}\right)^{\alpha}}\right)^{-1} \tag{41}$$

Here, we naturally see that N is decreasing in θ – the bigger is θ, the more the household likes leisure, and the less labor it supplies. Instead of doing things this way, we will often specify a target value of labor, such as $N = 1/3$, and pick θ to be consistent with that target. As we shall see, with this specification of preferences, the target value of N (or equivalently the value of θ) is not innocuous from the perspective of model dynamics. There is an alternative preference specification in which the steady state level of N is just a normalization that does not matter for dynamics.

Once we have N and K/N, the rest of the steady state is straightforward to compute.

4.1 Log-Linearization

To build intuition, it is helpful to log-linearize the model by hand. It's much easier to build intuition with linear equations.

Let's analyze the decentralized model, characterized by (28)-(36). We will log-linearize about the non-stochastic steady state using the functional forms given above.

Start with the capital Euler equation, (28). I will (i) take logs, (ii) totally differentiate, and then (iii) simplify to put in "tilde" notation. Note that I will ignore expectations operators until the end when I am done with the linearization, as discussed earlier.

$$\begin{aligned} -\ln C_t &= \ln \beta - \ln C_{t+1} + \ln\left(R^k_{t+1} + (1-\delta)\right) \\ -\frac{dC_t}{C} &= -\frac{dC_{t+1}}{C} + \beta dR^k_{t+1} \\ \mathbb{E}_t \widetilde{C}_{t+1} &= \widetilde{C}_t + \beta R^k \mathbb{E}_t \widetilde{R}_{t+1} \end{aligned} \tag{42}$$

(42) says that expected consumption growth is positively related to the expected return on capital (which is equal to the marginal product of capital). Now go to the labor supply condition.

$$\ln \theta - \ln(1-N_t) = -\ln C_t + \ln w_t$$

$$\frac{dN_t}{1-N} = -\frac{dC_t}{C} + \frac{dw_t}{w}$$

$$\frac{N}{1-N}\widetilde{N}_t = -\widetilde{C}_t + \widetilde{w}_t$$

Define $\gamma = \frac{N}{1-N}$. We can then write this as:

$$\gamma\widetilde{N}_t = -\widetilde{C}_t + \widetilde{w}_t \tag{43}$$

$\frac{1}{\gamma}$ is what is called the *Frisch labor supply elasticity*. It measures the partial of $\widetilde{N}_t$ with respect to $\widetilde{w}_t$, holding $\widetilde{C}_t$ fixed. In words, the Frisch elasticity is defined as the elasticity of labor supply holding the marginal utility of wealth fixed. In this model with separable preferences, the marginal utility of wealth is just the marginal utility of consumption. This parameter, γ, ends up mattering for the dynamics in the model. This means that the steady state value of N matters. We will return to this issue later. There is a way to write preferences in which steady state N is just a normalization.

I'm going to go out of order, but let's now log-linearize the factor demand conditions, which are already log-linear:

$$\widetilde{w}_t = \widetilde{A}_t + \alpha\widetilde{K}_t - \alpha\widetilde{N}_t \tag{44}$$

$$\widetilde{R}^k_t = \widetilde{A}_t - (1-\alpha)\widetilde{K}_t + (1-\alpha)\widetilde{N}_t \tag{45}$$

(44) says that labor demand is downward-sloping; it shifts right with an increase in productivity or when there is more capital. (45) is similar for capital demand. Note that labor and capital are complements – more of one increases the marginal product of the other.

The production function is also already log-linear:

$$\widetilde{Y}_t = \widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{N}_t \tag{46}$$

The resource constraint can be written in log-linear terms as:

$$\widetilde{Y}_t = \frac{C}{Y}\widetilde{C}_t + \frac{I}{Y}\widetilde{I}_t \tag{47}$$

The linearized accumulation equation for capital is:

$$\widetilde{K}_{t+1} = \delta\widetilde{I}_t + (1-\delta)\widetilde{K}_t \tag{48}$$

The Euler equation for bonds is:

$$\mathbb{E}_t\widetilde{C}_{t+1} = \widetilde{C}_t + \widetilde{r}_t \tag{49}$$

(49) says that expected consumption growth is positively related to the current real interest rate. If you combine this with the Euler equation for capital, we get:

$$\widetilde{r}_t = \beta R \mathbb{E}_t \widetilde{R}^k_{t+1} \tag{50}$$

In other words, the real interest rate is closely tied to the marginal product of capital. Intuitively, this makes sense. Bonds and capital are alternative means to transfer resources intertemporally. Bonds are risk-free, capital is risky (in the sense that one doesn't know what future productivity will look like). In a linearization, agents behave as though they are risk-neutral, so we basically have a one-to-one arbitrage condition between capital and bonds.

The law of motion for productivity is already log-linear.

$$\widetilde{A}_t = \rho \widetilde{A}_{t-1} + s_A \varepsilon_t \tag{51}$$

For completeness, the full set of linearized equilibrium conditions put together is:

$$\mathbb{E}_t \widetilde{C}_{t+1} = \widetilde{C}_t + \beta R^k \mathbb{E}_t \widetilde{R}_{t+1} \tag{52}$$

$$\gamma \widetilde{N}_t = -\widetilde{C}_t + \widetilde{w}_t \tag{53}$$

$$\widetilde{w}_t = \widetilde{A}_t + \alpha \widetilde{K}_t - \alpha \widetilde{N}_t \tag{54}$$

$$\widetilde{R}^k_t = \widetilde{A}_t - (1-\alpha)\widetilde{K}_t + (1-\alpha)\widetilde{N}_t \tag{55}$$

$$\widetilde{Y}_t = \widetilde{A}_t + \alpha \widetilde{K}_t + (1-\alpha)\widetilde{N}_t \tag{56}$$

$$\widetilde{Y}_t = \frac{C}{Y}\widetilde{C}_t + \frac{I}{Y}\widetilde{I}_t \tag{57}$$

$$\widetilde{K}_{t+1} = \delta \widetilde{I}_t + (1-\delta)\widetilde{K}_t \tag{58}$$

$$\mathbb{E}_t \widetilde{C}_{t+1} = \widetilde{C}_t + \widetilde{r}_t \tag{59}$$

$$\widetilde{A}_t = \rho \widetilde{A}_{t-1} + s_A \varepsilon_t \tag{60}$$

5 Graphical Analysis

It is not terribly common to do so, but I think it is extremely helpful for building intuition to try to graphically analyze this model.

To do this, as a first step we want to build a phase diagram in the forward-looking jump variable ($\widetilde{C}_t$) and the endogenous state variable ($\widetilde{K}_t$). We have to do some laborious algebra to substitute out all other variables. Atart with the Euler equation for capital. Eliminate the rental rate to write it as:

$$\mathbb{E}_t \widetilde{C}_{t+1} = \widetilde{C}_t + \beta R^k \mathbb{E}_t \left(\widetilde{A}_{t+1} - (1-\alpha)\widetilde{K}_{t+1} + (1-\alpha)\widetilde{N}_{t+1} \right)$$

Now eliminate $\widetilde{N}_{t+1}$ using the labor supply/demand conditions. As a first step, substitute out the real wage by combining labor demand and supply:

$$\widetilde{A}_t + \alpha \widetilde{K}_t - \alpha \widetilde{N}_t = \gamma \widetilde{N}_t + \widetilde{C}_t$$

So:

$$\widetilde{N}_t = \frac{1}{\alpha+\gamma}\widetilde{A}_t + \frac{\alpha}{\alpha+\gamma}\widetilde{K}_t - \frac{1}{\alpha+\gamma}\widetilde{C}_t$$

Now, plug this in for $\widetilde{N}_{t+1}$

$$\mathbb{E}_t\widetilde{C}_{t+1} = \widetilde{C}_t + \beta R^k \mathbb{E}_t \left[\widetilde{A}_{t+1} - (1-\alpha)\widetilde{K}_{t+1} + (1-\alpha)\left(\frac{1}{\alpha+\gamma}\widetilde{A}_{t+1} + \frac{\alpha}{\alpha+\gamma}\widetilde{K}_{t+1} - \frac{1}{\alpha+\gamma}\widetilde{C}_{t+1}\right)\right]$$

Now group terms:

$$\left(1 + \frac{\beta(1-\alpha)R^k}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{C}_{t+1} = \widetilde{C}_t + \beta R^k \left(1 + \frac{1-\alpha}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{A}_{t+1} + \beta(1-\alpha)R^k\left(\frac{\alpha}{\alpha+\gamma} - 1\right)\widetilde{K}_{t+1}$$

This can be simplified a bit to yield:

$$\left(1 + \frac{\beta(1-\alpha)R^k}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{C}_{t+1} = \widetilde{C}_t + \beta R^k \left(\frac{1+\gamma}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{A}_{t+1} - \beta(1-\alpha)R^k\left(\frac{\gamma}{\alpha+\gamma}\right)\widetilde{K}_{t+1} \tag{61}$$

Now proceed similarly with the accumulation equation for capital. We have:

$$\widetilde{K}_{t+1} = \delta\left(\frac{Y}{I}\widetilde{Y}_t - \frac{C}{I}\widetilde{C}_t\right) + (1-\delta)\widetilde{K}_t$$

Note that $Y = \left(\frac{K}{N}\right)^\alpha N$ and $I = \delta\left(\frac{K}{N}\right)N$, so $Y/I = \left(\frac{K}{N}\right)^{\alpha-1}\delta^{-1}$. We can write $C/I = C/(\delta K)$, so we can write the above as:

$$\widetilde{K}_{t+1} = \left(\frac{K}{N}\right)^{\alpha-1}\widetilde{Y}_t - \frac{C}{K}\widetilde{C}_t + (1-\delta)\widetilde{K}_t$$

Now plug in for $\widetilde{Y}_t$ from the production function:

$$\widetilde{K}_{t+1} = \left(\frac{K}{N}\right)^{\alpha-1}\left(\widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\widetilde{N}_t\right) - \frac{C}{K}\widetilde{C}_t + (1-\delta)\widetilde{K}_t$$

Now eliminate $\widetilde{N}_t$ from what we found earlier:

$$\widetilde{K}_{t+1} = \left(\frac{K}{N}\right)^{\alpha-1}\left(\widetilde{A}_t + \alpha\widetilde{K}_t + (1-\alpha)\left(\frac{1}{\alpha+\gamma}\widetilde{A}_t + \frac{\alpha}{\alpha+\gamma}\widetilde{K}_t - \frac{1}{\alpha+\gamma}\widetilde{C}_t\right)\right) - \frac{C}{K}\widetilde{C}_t + (1-\delta)\widetilde{K}_t$$

Grouping terms, we get:

$$\widetilde{K}_{t+1} = \left(\frac{K}{N}\right)^{\alpha-1}\left(1+\frac{1-\alpha}{\alpha+\gamma}\right)\widetilde{A}_t - \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)+\frac{C}{K}\right)\widetilde{C}_t + \left(\alpha\left(\frac{K}{N}\right)^{\alpha-1}\left(1+\frac{1-\alpha}{\alpha+\gamma}\right)+(1-\delta)\right)\widetilde{K}_t$$

We can simplify some terms:

$$\widetilde{K}_{t+1} = \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1+\gamma}{\alpha+\gamma}\right)\right)\widetilde{A}_t - \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)+\frac{C}{K}\right)\widetilde{C}_t + \left(\alpha\left(\frac{K}{N}\right)^{\alpha-1}+\left(\frac{1-\alpha}{\alpha+\gamma}\right)\alpha\left(\frac{K}{N}\right)^{\alpha-1}+(1-\delta)\right)\widetilde{K}_t$$

Now, note that $\alpha\left(\frac{K}{N}\right)^{\alpha-1}+(1-\delta)=\frac{1}{\beta}$. So we can write this as:

$$\widetilde{K}_{t+1} = \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1+\gamma}{\alpha+\gamma}\right)\right)\widetilde{A}_t - \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)+\frac{C}{K}\right)\widetilde{C}_t + \left(\frac{1}{\beta}+\alpha\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)\right)\widetilde{K}_t \tag{62}$$

(61) and (62) are the key dynamic (linearized) equations, subbing out all the redundant/static variables. Okay, that was laborious. What is this buying us? Capital being constant, $\widetilde{K}_{t+1}=\widetilde{K}_t$, requires:

$$0 = \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1+\gamma}{\alpha+\gamma}\right)\right)\widetilde{A}_t - \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)+\frac{C}{K}\right)\widetilde{C}_t + \left(\frac{1}{\beta}-1+\alpha\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)\right)\widetilde{K}_t$$

Re-arranging:

$$\widetilde{C}_t = \left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)+\frac{C}{K}\right)^{-1}\left[\left(\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1+\gamma}{\alpha+\gamma}\right)\right)\widetilde{A}_t + \left(\frac{1}{\beta}-1+\alpha\left(\frac{K}{N}\right)^{\alpha-1}\left(\frac{1-\alpha}{\alpha+\gamma}\right)\right)\right]\widetilde{K}_t \tag{63}$$

(63) is the $\widetilde{K}_{t+1}=\widetilde{K}_t$ isocline: it shows combinations of $\widetilde{C}_t$ and $\widetilde{K}_t$, taking $\widetilde{A}_t$ as given, where $\widetilde{K}_{t+1}=\widetilde{K}_t$. In a graph with $\widetilde{C}_t$ on the vertical axis and $\widetilde{K}_t$ on the horizontal axis, it must be upward-sloping (since $\frac{1}{\beta}-1>0$). It shifts up whenever $\widetilde{A}_t$ increases. This is exactly what we would have in the neoclassical model with fixed labor, which has an upward-sloping isocline (at least in the region of the non-stochastic steady state, before it reaches a hump and starts sloping down).

Now, let's fiddle with (61). For $\mathbb{E}_t\widetilde{C}_{t+1}=\widetilde{C}_t$, we must have:

$$\left(\frac{\beta(1-\alpha)R^k}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{C}_{t+1} = \beta R^k\left(\frac{1+\gamma}{\alpha+\gamma}\right)\mathbb{E}_t\widetilde{A}_{t+1} - \beta(1-\alpha)R^k\left(\frac{\gamma}{\alpha+\gamma}\right)\widetilde{K}_{t+1}$$

Now, we have the annoying issue that this in terms of $t+1$ variables. Let's "cheat" and assume these are all dated t (which would be fine in continuous time):

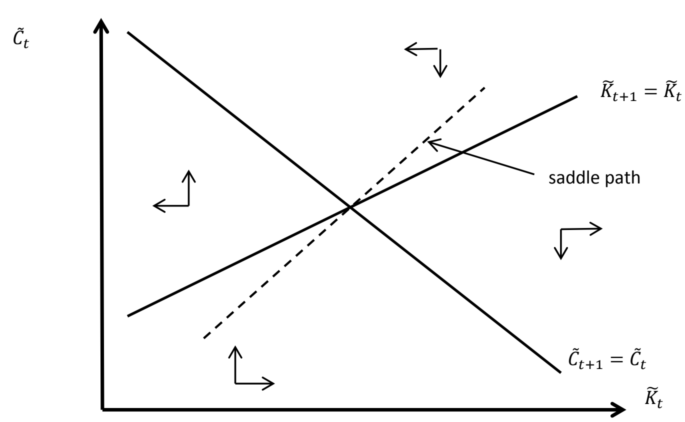
$$\left(\frac{\beta(1-\alpha)R^k}{\alpha+\gamma}\right)\widetilde{C}_t = \beta R^k\left(\frac{1+\gamma}{\alpha+\gamma}\right)\widetilde{A}_t - \beta(1-\alpha)R^k\left(\frac{\gamma}{\alpha+\gamma}\right)\widetilde{K}_t$$

This simplifies greatly, yielding:

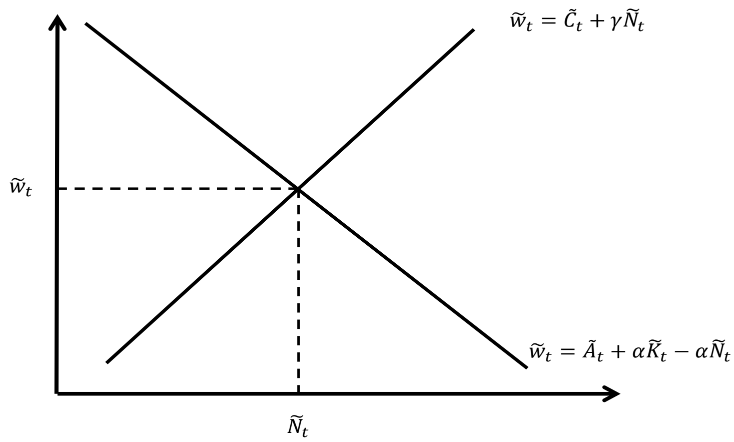
$$\widetilde{C}_t = \left(\frac{1+\gamma}{1-\alpha}\right)\widetilde{A}_t - \gamma\widetilde{K}_t \tag{64}$$

(64) is the $\mathbb{E}_t\widetilde{C}_{t+1} = \widetilde{C}_t$ isocline. It is downward-sloping in $\widetilde{K}_t$ (which is different than the basic neoclassical model). This is the case *unless* $\gamma \to \infty$ (which would correspond to labor being inelastically supplied, because the labor supply elasticity is $1/gamma$), in this case, which is isomorphic to the neoclassical model with fixed labor, it would be vertical at $\widetilde{K}_t = 0$ (i.e. at the steady state). This isocline shifts up whenever $\widetilde{A}_t$ increases.

We can graph both (64) and (63). This is shown below. The two isoclines must intersect at the steady state (since variables are linearized about the steady, the point of intersection is actually "$0-0$" on the graph). In terms of dynamics, if we are to the right of the consumption-constant isocline ($\widetilde{K}_t$ is "too big"), consumption will be declining (because the rental rate on capital, and hence the real interest rate, will be small / below steady state). Hence, arrows point down to the right of this isocline, and up to the left. If we have too little consumption relative to the capital-constant isocline, capital will be increasing (arrows point to the right below), and the reverse (arrows point to the left above). From this, we can infer that there will exist a saddle path that goes from southwest to northeast. This is shown as the dotted line. Note this is exactly the same idea as the phase diagram in the basic neoclassical growth model – the only difference that consumption constant isocline is (locally) downward-sloping (rather than vertical, due to endogenous labor supply).

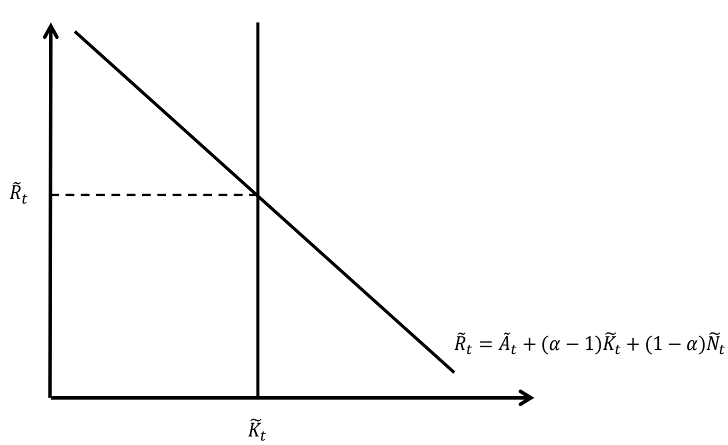


What the phase diagram does is tell us what $\widetilde{C}_t$ needs to be, given the endogenous ($\widetilde{K}_t$) and exogenous ($\widetilde{A}_t$) state variables. Once we have $\widetilde{C}_t$, we can then determine labor market variables. We can graph labor demand and supply as follows:



$\widetilde{C}_t$ essentially tells us where the labor supply curve should sit. $\widetilde{A}_t$ and $\widetilde{K}_t$ tell us where the labor demand curve should sit. The intersection determines $\widetilde{N}_t$ and $\widetilde{w}_t$. Once we have $\widetilde{N}_t$ pinned down, we can get $\widetilde{Y}_t$ (and hence also $\widetilde{I}_t$.

Now, the real interest rate, which in this model is closely related to the rental rate on capital, can be inferred by the expected consumption dynamics in the phase diagram. But we can also think about there existing a demand-supply graph for physical capital that determines $\widetilde{R}_t^k$. Capital demand is downward-sloping – the firm rents capital up until the point at which the rental rate equals the marginal product. The supply of capital is a bit trickier. In the *short run*, the supply of capital is completely inelastic – capital is pre-determined within period, after all. In the long run, the supply of capital is, in contrast, effectively perfectly elastic, since the steady-state rental rate on capital is determined by preferences. This is shown below.



Once we have these three graphs, we can *qualitatively* think about how all the endogenous variables will react to shocks in the model. Although everything happens simultaneously in general equilibrium, it is helpful (to me, at least) to think about progressing in stages. We will consider exercises in which we initially sit in a steady state, and then there is an exogenous shock (in this simple model, the only exogenous shock is to productivity). To intuit what happens, follow these steps:

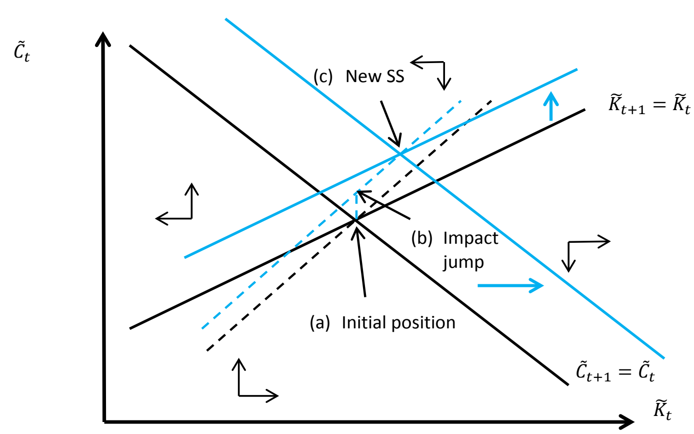
1. Given the exogenous state variable, $\widetilde{A}_t$, determine what $\widetilde{C}_t$ should be in the phase diagram
2. Given $\widetilde{C}_t$ from the phase diagram, determine $\widetilde{N}_t$ and $\widetilde{w}_t$ in the labor market diagram. The real interest rate, $\widetilde{r}_t$, is governed by expected consumption dynamics.
3. Determine $\widetilde{R}_t^k$ in the capital market diagram
4. Determine $\widetilde{Y}_t$ and $\widetilde{I}_t$ given $\widetilde{N}_t$ and $\widetilde{C}_t$
5. Determine dynamics of $\widetilde{C}_t$ and $\widetilde{K}_t$ from the phase diagram, and then repeat the steps above to infer the dynamics of other variables.

6 Dynamic Responses to Productivity Shocks

Let us use this machinery to get some intuition for the *qualitative* responses to shocks to $\widetilde{A}_t$. I'm going to consider two types of shocks to productivity: a *permanent* increase in $\widetilde{A}_t$ and a *purely temporary* increase in $\widetilde{A}_t$ (by purely temporary, I mean lasting only one period, i.e. iid).

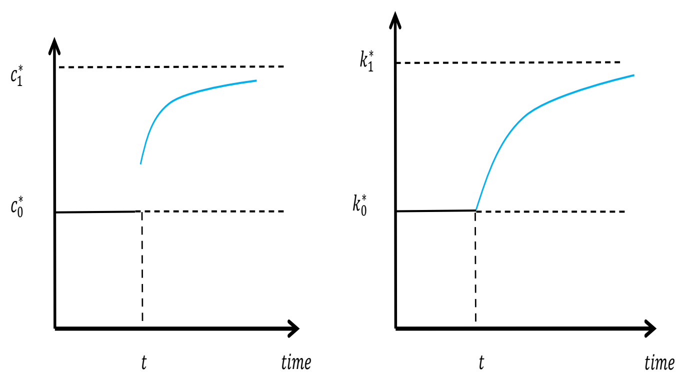
6.1 Permanent Productivity Shock

Assume that we initially sit in a steady state (so $\widetilde{A}_t = 0$, along with all other variables). Then, in period t, $\widetilde{A}_t$ jumps up, and is expected to stay forever at this higher level.



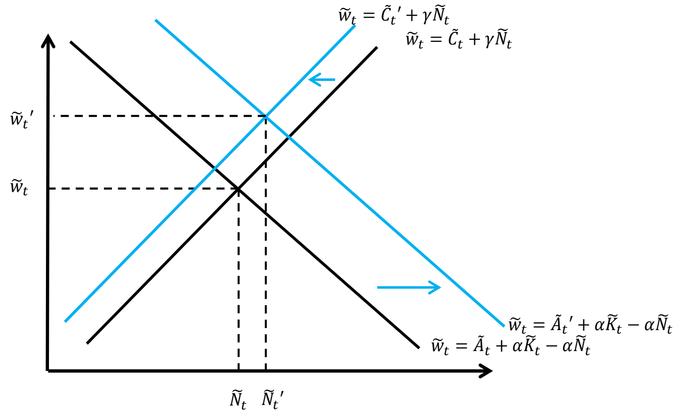
Let's start with the phase diagram. The increase in $\widetilde{A}_t$ shifts both isoclines up. Although from the shifts it appears ambiguous, we know that the isoclines must cross at a point with higher capital and consumption – that is, the steady state of both of these variables will be higher. There is a new saddle path associated with the new isoclines. Because we are dealing with a permanent change, consumption must jump immediately to the new saddle path.

Now, we know that the saddle path must be upward-sloping (from southwest to northeast). I have drawn the figure where consumption jumps up on impact. It is conceivable that consumption could jump down. There are two competing economic effects governing the response of consumption. On the one hand is the income effect – the household is wealthier, and so wants to increase consumption. But there is a competing substitution effect – we need to accumulate more capital, and the marginal product of capital is higher (and hence the real interest rate will be higher). This force encourage the household to defer its consumption to take advantage of the high return on capital. It is conceivable that consumption could jump down (i.e. the saddle path is very steep, as opposed to comparatively flat). This is likely to happen when the intertemporal elasticity of substitution, IES, is quite high (i.e., with iso-elastic utility, σ is quite low, for example less than log utility). But for most values of the IES, consumption is going to jump up, so that is how I have drawn it.



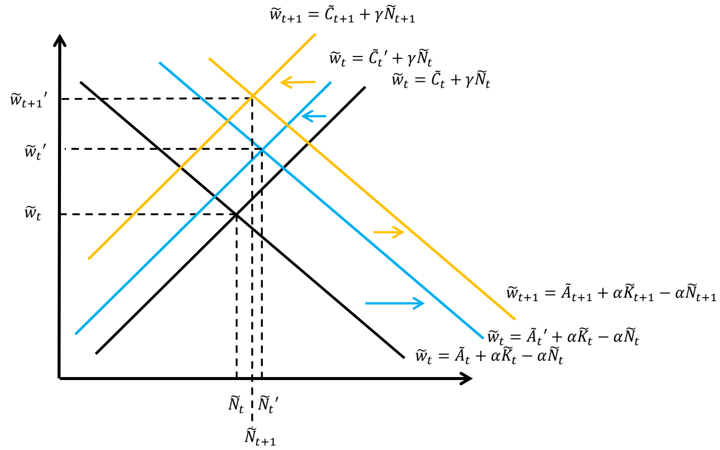
Once we know the initial jump in consumption, the dynamics of consumption and capital are easy to infer from the saddle path. Both will increase, and will approach the new higher steady state. These impulse response diagrams are shown above.

Now, let us turn to the labor market. The increase in $\widetilde{A}_t$ shifts the labor demand curve to the right. If $\widetilde{C}_t$ jumps up, as shown above (and is it will under typical parameterizations), then the labor supply curve will shift up (or into the left). This is shown in the figure below.



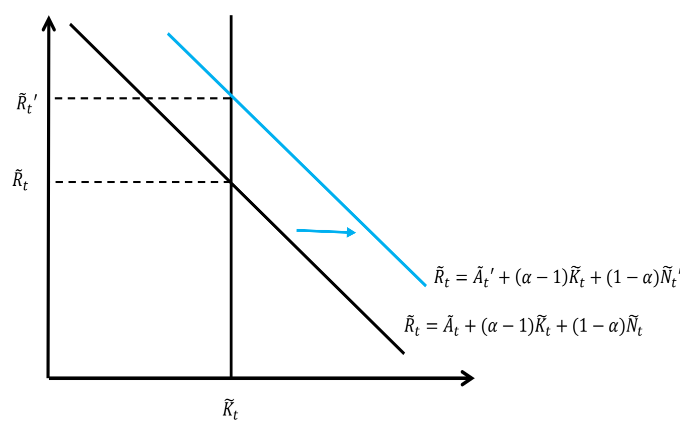
There are, again, competing income and substitution effects. The way the labor supply curve is drawn, movements along the labor supply curve measure the substitution effect. Higher productivity leads to a higher wage, which causes labor input to rise (and leisure to fall). But the income effect is to take more leisure (assuming consumption goes up), meaning to work less. This causes the labor supply curve to shift in. The size of the income effect depends on the persistence of the shock, because this governs how much consumption reacts. As drawn, I have shown it where the labor supply curve shifts in, but not by so much that $\widetilde{N}_t$ falls. We can unambiguously conclude that the real wage rises. The effect on labor is ambiguous, but likely to go up for standard parameterizations.

Now, what is going to happen dynamically? Consumption is going to continue to increase, which is going to cause the labor supply curve to continue to shift in. While $\widetilde{A}_t$ stays at a higher level, we do start to accumulate more capital, which causes the labor demand curve to continue to shift out. Since capital is slow-moving, the effects on labor demand are likely to be small relative to labor supply. So, for most parameterizations we would expect labor input to start falling (after its initial upward-jump) while the wage continues to rise. This is how things are shown in the graph below.

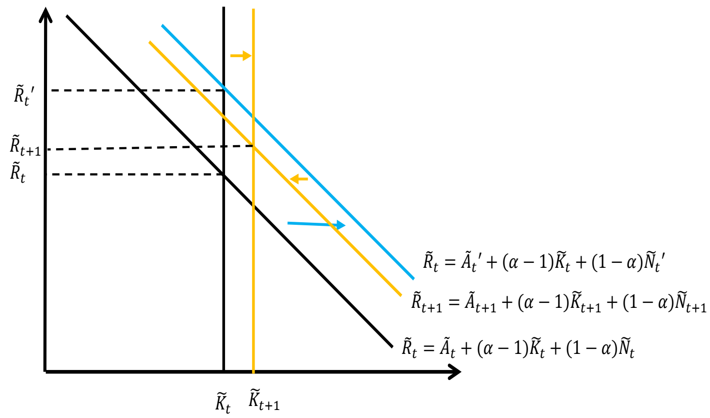


Whether labor ends up higher or lower in the new steady state is not obvious qualitatively. In much of macro, we work with preferences where long-run income and substitution effects exactly offset (as in King-Plosser-Rebelo), so labor eventually returns to where it started.

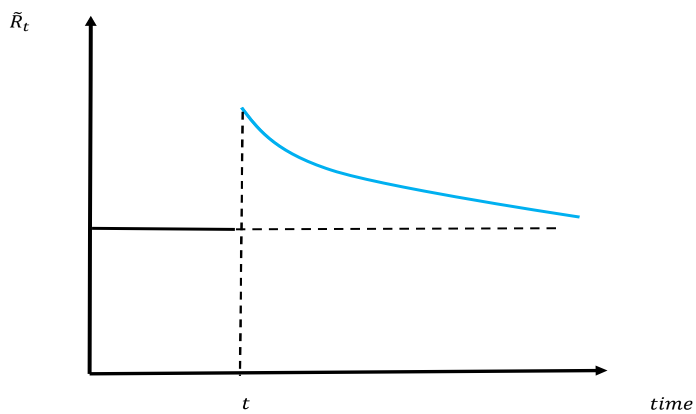
Now, let's turn to thinking about the rental rate on capital. In the short run, the supply of capital is fixed. The demand for capital rises due to higher $\widetilde{A}_t$ and higher $\widetilde{N}_t$. Even if labor falls on impact, which is possible depending on parameters, the demand for capital must still shift out because the rental rate needs to rise as the economy accumulates more capital. So, we can conclude that $\widetilde{R}_t^k$ rises on impact.



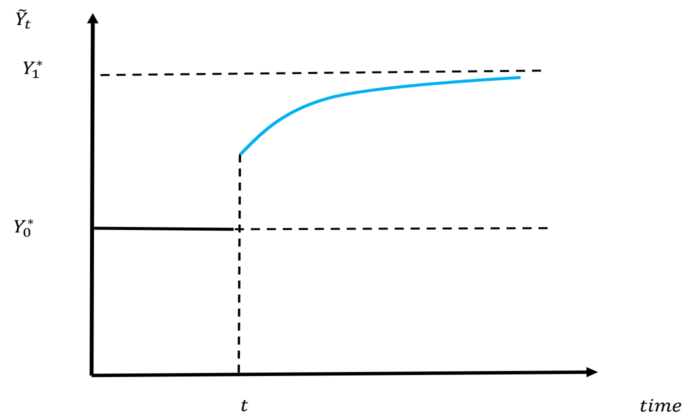
What about dynamically? As we move forward in time, the supply of capital shifts to the right as the economy accumulates capital. The demand for capital, in contrast, starts to shift as labor input declines (this assumes that labor input rises on impact). Both of these effects put downward pressure on the rental rate, as shown below.



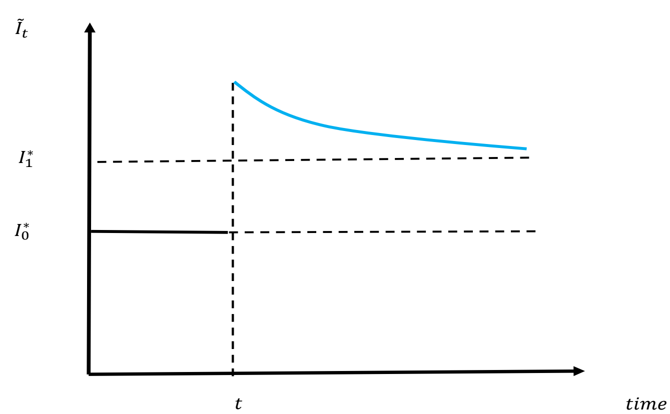
In the long run, we know that the rental rate on capital is unrelated to the level of productivity; the steady-state rental rate is $R^k = \frac{1}{\beta} - (1 - \delta)$. Effectively, the long-run supply of capital is perfectly elastic (horizontal). So we know that the rental rate must eventually return to its original starting point. This is shown below.



What about output and investment? We can infer these from our analysis above. On impact, we know that output must go up – because productivity is higher and labor input is (likely) higher. Output must increase even if $\widetilde{N}_t$ declines. We know this because we know that investment must increase (to get to a higher steady-state capital stock). The only way for labor input to decline on impact is for there to be a big impact-increase in consumption. If both consumption and investment rise, then output must also rise. After impact, output dynamics will be governed by the dynamics of capital – it will continue to rise until it approaches the new, higher steady state, as shown below.



Given the dynamics of the capital stock from the phase diagram, we know that investment must increase on impact. Investment must also be higher in the new steady state to support a higher steady state capital stock. We would expect that investment will "overshoot" on impact, rising by more than its new steady state value, because capital is going to accumulate fastest early on.



The real interest rate is pinned down by the expected rental rate on capital (as well as the consumption dynamics it). Therefore, the real interest rate should rise on impact, and then decline back to its original steady state.

To summarize, following a *permanent* increase in productivity in this model, we should have:

1. Permanently higher capital, output, consumption, investment, and real wage. In the long run, output rises by more than the direct effect of higher productivity because of higher capital.

2. On impact, investment and output must go up. So, too, must the real wage. It is, in theory, ambiguous what happens to consumption and labor hours. Consumption is likely

to go up (unless the IES is really large). Labor input is also likely to rise, but how much it rises is inversely related to how much consumption rises. Labor rising provides short-run *amplification* of the productivity shock – output rises by more than the direct effect of higher productivity. Capital accumulation provides long-run amplification.

6.2 Temporary Productivity Shock

Now, let's think about the polar opposite of a permanent productivity shock – a one-time, iid shock to $\widetilde{A}_t$. I will called this "purely temporary." For this exercise, I am not going to show graphs, but rather just reason through things with words.

First, think about the phase diagram. The isoclines shift in the same way as earlier, but consumption cannot jump to the new saddle path. Rather, it needs to jump, "ride" explosive dynamics for one period, and then be back on the original saddle path. If the shock is only one period, approximately consumption will do *nothing at all.* In actuality, it will jump up a very little bit, but this change will be negligible.

Now turn to the labor market. The effect on labor demand is the same whether the shock to productivity is permanent or just one time. This means labor demand shifts right, exactly as in the permanent shock case. But because consumption does very little, the labor supply curve essentially does not move. Therefore, we have labor hours rising and the real wage rising. But note: *relative to the permanent shock case, labor rises more, and the real wage less.* After the period of the shock, the labor market will quickly back to its original steady state.

With labor rising, output will rise on impact. In fact, it will rise *more* relative to the permanent shock case because of the bigger impact on labor. With output rising (more) and consumption essentially not reacting, investment must also rise, also by more than in the permanent shock case.

In terms of capital demand, the capital demand curve shifts right immediately. This means $\widetilde{R}_t^k$ rises. Dynamically, however, the capital demand curve is going to shift right back in the next period, but the capital supply curve will shift out slightly because of higher investment on impact. Therefore, we would expect $\mathbb{E}_t \widetilde{R}_{t+1}^k$ to slightly fall. That means that the real interest rate will fall. Which means that consumption will be expected to decline (from its very small initial upward jump). This makes sense, because consumption has to go back to the original steady state. Everything will go back to steady state quickly.

From reasoning through all this, we can conclude a couple of things in response to productivity shocks in the RBC model. We have considered two knife-edge cases: a permanent shock and a purely temporary shock. Most of the time, we will be considering shocks that are transitory but somewhere in between permanent and purely temporary. The equilibrium effects on endogenous variables will lie somewhere in between the two polar cases. The more persistent a shock, the more the responses will look like the permanent case. The more transitory the shock, the more the responses will look like the iid case.

1. When productivity improves, output, investment, and the real wage are going to go up. Output and investment will go up *more* the *less* persistent is the productivity shock; the real

wage will go up *less* the less persistent is the shock.

2. Consumption is likely to go up, and will go up *more* the more persistent is the shock. In contrast, labor hours will go up *less* the more persistent is the shock.

3. The rental rate on capital will go up when productivity improves. It will always return to steady state, even if the shock is permanent. The real interest rate tracks the one-period ahead change in the rental rate.