

Graduate Macro Theory II: Notes on Value Function Iteration

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Spring 2024

1 Introduction

These notes discuss how to solve dynamic economic models using value function iteration.

2 A Deterministic Growth Model

The solution to the deterministic growth model can be written as a Bellman equation as follows:

$$V(K) = \max_C \left\{ \frac{C^{1-\sigma} - 1}{1-\sigma} + \beta V(K') \right\}$$

s.t.

$$K' = K^\alpha - C + (1 - \delta)K$$

Variables with a $'$ denote future values. K is a predetermined state variable. C is a jump variable. You can impose that the constraint holds and re-write the problem as choosing the future state:

$$V(K) = \max_{K'} \left\{ \frac{(K^\alpha + (1 - \delta)K - K')^{1-\sigma} - 1}{1-\sigma} + \beta V(K') \right\}$$

The basic idea of value function iteration is as follows. Create a grid of possible values of the state, K , with N elements. Then make a guess of the value function, $V^0(k)$. This guess will be a $N \times 1$ vector – one value for each possible state. For that guess of the value function, compute $V^1(k)$ as follows:

$$V^1(K) = \max_{K'} \left\{ \frac{(K^\alpha + (1 - \delta)K - K')^{1-\sigma} - 1}{1-\sigma} + \beta V^0(K') \right\}$$

In doing this process, it is important to remember to take care of the “max” operator inside the Bellman equation: conditional on $V^0(K)$, you have to choose K' to maximize the right hand side. For each grid value of K , this gives you a new value for the value function, call this $V^1(K)$. Then compare $V^1(K)$ to your initial guess, $V^0(K)$. If it is not close, then take your new value function, $V^1(K)$, as the “guess” and repeat the process. On the n^{th} iteration, you will have $V^n(K)$ and $V^{n-1}(K)$. For n large, if your problem is well-defined, these should be very close together and you can stop.

2.1 A Numerical Example

Suppose that the parameter values are as follows: $\sigma = 2$, $\beta = 0.95$, $\delta = 0.1$, $\alpha = 0.33$. Basically, this can be taken to be a reasonable parameterization at an annual frequency. The Euler equation for this problem is:

$$C^{-\sigma} = \beta C'^{\sigma} (\alpha K'^{\alpha-1} + (1 - \delta))$$

In steady state, $C' = C = C^*$. This occurs where $K' = K = K^*$ where:

$$K^* = \left(\frac{\alpha}{1/\beta - (1 - \delta)} \right)^{\frac{1}{1-\alpha}} \quad (1)$$

For the parameters I have chose, I get a value of $K^* = 3.169$. We then need to construct a grid of possible values of K . I will construct a grid with $N = 100$ values, ranging between 25 percent of the steady state value of K and 175 percent of its steady state value. The MATLAB code is:

```

1 kmin = 0.25*kstar;
2 kmax = 1.75*kstar;
3 grid = (kmax-kmin)/kgrid;
4
5 kmat = kmin:grid:kmax;
6 kmat = kmat';
7 [N,n] = size(kmat);

```

The resulting vector is called “kmat” and is 100×1 . Next I guess a value function. For simplicity I will just guess a vector of zeros:

```

1 v0 = zeros(N,1);

```

I need to specify a tolerance for when to stop searching over value functions. In particular, my objective is going to be the norm of the of the absolute value of the difference between $V^n(K)$ and $V^{n-1}(K)$. The norm I’m going to use is the square root of the sum of squared deviations. One could do other measures of closeness of the value functions. I need to set a tolerance for this for when MATLAB to stop. I set this tolerance at 0.01. I also can specify the maximum number of

iterations that MATLAB can take. I set this at 300. The MATLAB code for these preliminaries is:

```
1 tol = 0.01;
2 maxits = 1000;
```

I'm going to write a loop over all possible values of the state. For each value in K , find the K' that maximizes the Bellman equation given my guess of the value function. I have a function file to do the maximizing that we will return to in a minute. Outside of the loop I have a “while” statement, which tells MATLAB to keep repeating the text as long as the difference between value functions is greater than the tolerance. The MATLAB code for this part is:

```
1 while dif>tol & its < maxits
2     for i = 1:N
3         k0 = kmat(i,1);
4         k1 = fminbnd(@valfun,kmin,kmax);
5         v1(i,1) = -valfun(k1);
6         k11(i,1) = k1;
7     end
8 dif = norm(v1-v0);
9 v0 = v1;
10 its = its+1
11 end
```

This code can be interpreted as follows. For each value i in the state space, I get the initial value of K . Then the command on line 4 finds the argmax of the Bellman equation, which is found in the function file “valfun2.m”. Line 5 collects the optimized value into the new value function (called $v1$), and line 6 finds the policy function associated with this choice (ie. the optimal choice of K' , which is called $k1$ in this code). After the loop over the possible values of the state I calculate the difference and write out the iteration number. I do not include a semi-colon after these statements so that I can see the converge taking place in the command window.

A complication arises in function file because “fminbnd” assumes a continuous choice set – basically it will search over *all* values of $k1$ between “kmin” and “kmax”, which will include points not in the original capital grid. For this reason, we need to interpolate values of the value function off of the grid. I will use linear interpolation. Suppose that Matlab chooses a point $k1$ that is off the grid. I can approximate the value function at that point assuming that the value function is linear between the two points on the grid immediately around that point. In particular, I want to find $K(i) < K < K(i + 1)$. Then I can approximate the value function at this point as:

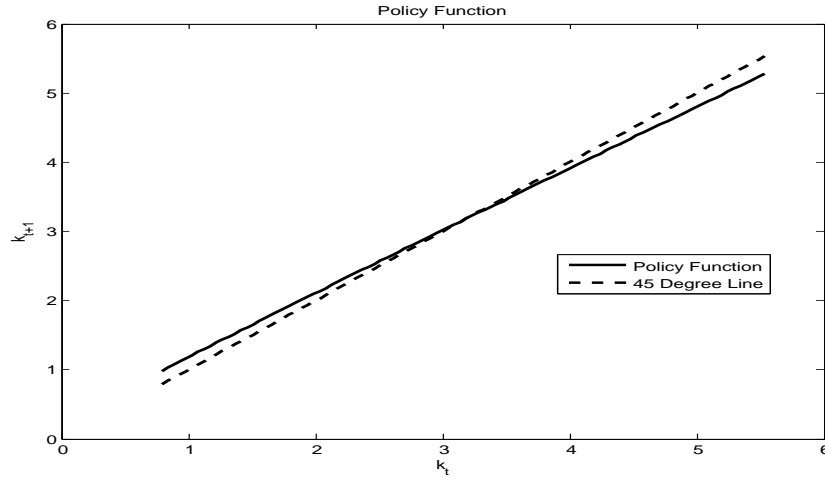
$$V(k) \approx V(K(i)) + \frac{v(K(i+1)) - v(K(i))}{K(i+1) - K(i)}(K - K(i))$$

One can do other forms of interpolation. If the problem is not too non-linear, and your grid points are close enough to one another, linear interpolation ought to be pretty good. Matlab has a built-in interpolation code called “interp” that will do this for you.

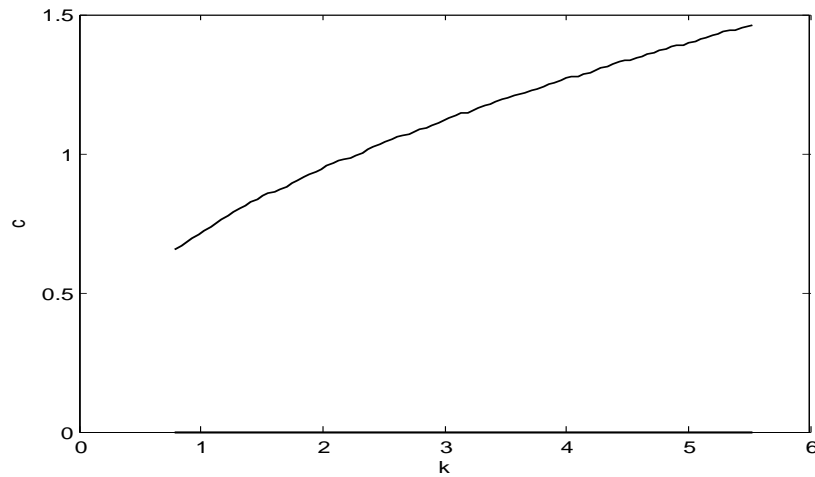

```

27 maxits = 1000;
28 dif = tol+1000;
29 its = 0;
30
31 kgrid = 99; % grid points + 1
32 kstar = (alpha/(1/beta - (1-Δ)))^(1/(1-alpha)); % steady state k
33 cstar = kstar^alpha - Δ*kstar;
34 istar = Δ*kstar;
35 ystar = kstar^alpha;
36
37 kmin = 0.25*kstar;
38 kmax = 1.75*kstar;
39 grid = (kmax-kmin)/kgrid;
40
41 kmat = kmin:grid:kmax;
42 kmat = kmat';
43 [N,n] = size(kmat);
44
45 polfun = zeros(kgrid+1,3);
46
47 v0 = zeros(N,1);
48 dif = 10;
49 its = 0;
50
51 while dif>tol & its < maxits
52     for i = 1:N
53         k0 = kmat(i,1);
54         k1 = fminbnd(@valfun,kmin,kmax);
55         v1(i,1) = -valfun(k1);
56         k11(i,1) = k1;
57     end
58 dif = norm(v1-v0);
59 v0 = v1;
60 its = its+1
61 end
62
63 for i = 1:N
64     con(i,1) = kmat(i,1)^alpha - k11(i,1) + (1-Δ)*kmat(i,1);
65     polfun(i,1) = kmat(i,1)^alpha - k11(i,1) + (1-Δ)*kmat(i,1);
66 end
67
68
69 figure
70 plot(kmat,v1,'-k','Linewidth',1)
71 xlabel('k')
72 ylabel('V(k)')
73
74 figure
75 plot(kmat,polfun,'-k','Linewidth',1)

```

Once I know K' , I can get C . Here is a plot of C against K :



3 The Stochastic Growth Model

Making the problem stochastic presents no special challenges, other than an expansion of the state space and having to deal with expectations and probabilities. The problem can be written:

$$V(K, A) = \max_C \left\{ \frac{C^{1-\sigma} - 1}{1-\sigma} + \beta EV(K', A') \right\}$$

s.t.

$$K' = AK^\alpha - C + (1 - \delta)K$$

You can impose that the constraint holds and re-write the problem as choosing the future state:

$$V(K, A) = \max_{K'} \left\{ \frac{(AK^\alpha + (1 - \delta)K - K')^{1-\sigma} - 1}{1 - \sigma} + \beta EV(K', A') \right\}$$

The stochastic part comes in because A is stochastic. I assume that A follows a Markov process, which generically means that the current state is a sufficient statistic for forecasting future values of the state. Let \tilde{A} denote a $q \times 1$ vector denoting possible realizations of A . Then let P be a $q \times q$ matrix whose (i, j) element is taken to be the probability of transitioning to state j from state i ; hence the rows must sum to one. I assume that A can take on the following three values: low, medium, and high:

$$\tilde{A} = \begin{pmatrix} 0.9 \\ 1.0 \\ 1.1 \end{pmatrix}$$

I define the probability matrix so that all of its elements are $\frac{1}{3}$:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

I can specify the A process as follows:

```
1 amat = [0.9 1 1.1]';
2 prob = (1/3)*ones(3,3);
```

This just means that the expected value of A next period is always 1, regardless of what current productivity is. It also means that the non-stochastic steady state value A is also 1.

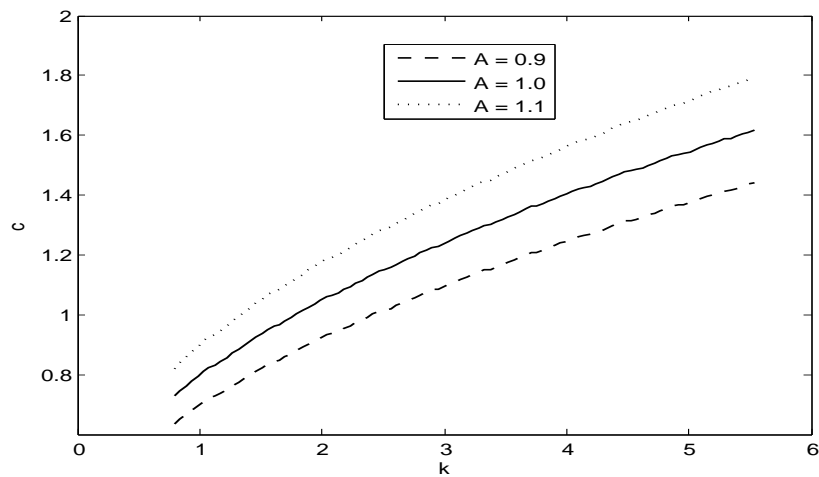
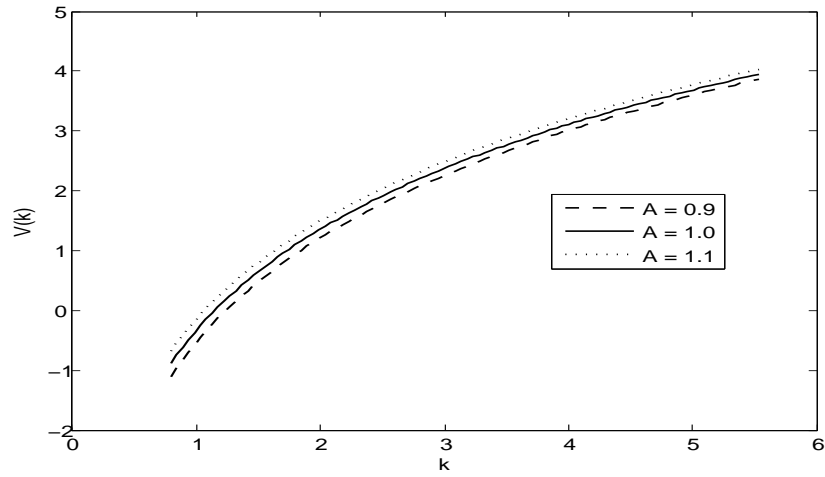
Another complication now is that my value function is going to have $N \times q$ elements. In other words, there is a different value to being in every different capital/productivity combination. I construct this as a $N \times q$ matrix, though I could conceivably do Kronecker products and make the state space a one dimensional object.

The value function iteration proceeds similarly to above, but I have to take account of the fact that there are expectations over future states now. I start with an initial guess of the value function, $V^0(K, A)$. Then I construct the Bellman equation:

$$V^1(K, A) = \max_{k'} \left\{ \frac{(AK^\alpha + (1 - \delta)K - K')^{1-\sigma} - 1}{1 - \sigma} + \beta E (V^0(K', A')) \right\}$$

Here the expectation is taken over the future value function. The uncertainty comes in over future values of A' – you are choosing K' based on an expectation of A' . Given a choice of K' ,

because my specification of P is such that there is no real persistence in A .



4 Dealing with “Static” Variables

As written above, all of the variables are dynamic – choices today directly affect the states tomorrow. In many models we will also want to work with “static” variables, variables whose choice today does not directly influence the value of future states. An example here is variable employment. Suppose that we take the neoclassical growth model with variable employment.

Let the time endowment of the agents be normalized to 1, and let their work effort be given by n . They get utility from leisure, which is $1 - n$. The dynamic program can be written as:

$$V(K) = \max_{C,n} \left\{ \frac{C^{1-\sigma} - 1}{1-\sigma} + \theta \ln(1-n) + \beta V(K') \right\}$$

s.t.

$$K' = K^\alpha n^{1-\alpha} - C + (1 - \delta)K$$

The first order condition for the choice of employment is:

$$\frac{\theta}{1-n} = C^{-\sigma} (1-\alpha) K^\alpha n^{-\alpha} \quad (2)$$

This first order condition must hold for any choice of C (equivalently, K') along an optimal path. Hence, you can effectively treat this first order condition as a “constraint” that must hold. It essentially determines n implicitly as a function of C and K . Then you can write the modified Bellman equation as:

$$V(K) = \max_{K'} \left\{ \frac{(K^\alpha n^{1-\alpha} + (1-\delta)K - K')^{1-\sigma} - 1}{1-\sigma} + \theta \ln(1-n) + \beta V(K') \right\}$$

s.t.

$$\frac{\theta}{1-n} = (K^\alpha n^{1-\alpha} + (1-\delta)K - K')^{-\sigma} (1-\alpha) K^\alpha n^{-\alpha}$$

The “constraint” (the FOC for labor) implicitly defines n as a function of K and K' . So you can effectively plug that in everywhere you see n showing up, leaving the problem as one of just choosing K' given K .