Problem Set 2

Graduate Macro II, Spring 2024
The University of Notre Dame
Professor Sims

Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. When asked to solve a model in Dynare, unless otherwise instructed you (i) should solve via a first-order approximation, (ii) may give Dynare the equilibrium conditions and do not need to do the linearization by hand, (iii) should show impulse responses up to a horizon of 20 periods, and (iv) should plot impulse responses (or analyze moments) of logged variables. This problem set is due on Canvas by 5:00 pm on February 9.

1. In this problem you will compare policy functions from a deterministic neoclassical growth model using (a) value function iteration and (b) log-linearization.

$$\max_{C_t, K_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$K_{t+1} = K_t^{\alpha} - C_t + (1-\delta)K_t$$

$$K_0 \quad \text{given}$$

   (a) Use L'Hopital's rule to prove that, as $\sigma \to 1$, the within period utility function goes to $\ln C_t$.

   (b) Set this problem up as a dynamic programming problem. What is the state variable? What is the control variable? Write down the Bellman equation. Find the first order condition necessary for an optimal solution.

   (c) Find an expression for the steady state capital stock and the steady state value of consumption.

   (d) Suppose that the parameter values are as follows: $\beta = 0.95$, $\alpha = 0.36$, $\sigma = 2$, and $\delta = 0.1$. What are the numerical values of the steady state capital stock and consumption for these parameters? *Write your own code* to numerically solve for the value and policy functions. To do so, create a grid of the capital stock, with the minimum value 0.25 of the steady state capital stock and the maximum value 1.75 times the steady state capital stock, with 300 grid points between. Use linear interpolation to evaluate points off the grid. Show a graph of both the final value function and the policy function.

   (e) Now set up the problem using a Lagrangian. Write out the first order conditions, including the transversality condition. Provide a verbal explanation for the intuition behind the transversality condition.

(f) Log-linearize the first order conditions about the steady state. Form a VAR(1) of the form:

$$\mathbf{X}_{t+1} = \mathbf{M}\mathbf{X}_t$$

Where $\mathbf{X}_t$ contains the variables expressed as percentage deviations about the steady state. Write out an expression for $\mathbf{M}$.

(g) Solve for the linear policy function mapping the state variable into the jump variable. Write out the numerical policy function here.

(h) Show a plot of the linearized policy function and the policy function from the value function iteration procedure obtained above together. Be sure to transform the linearized policy function, which is expressed as a percentage deviation about the steady state, into actual levels so as to make the comparison appropriate. Comment on the quality of the linear approximation.

(i) Repeat the exercise in (h) for the following different values of σ: 5 and 10. How does the quality of the linear approximation vary with σ? Why does this make sense?

2. Consider a neoclassical growth model with government consumption, G_t. The time path of government consumption is exogenously given. We will first consider a decentralized version of the problem.

The household supplies labor inelastically ($N_t = 1$) and can save via physical capital or one-period bonds. The household pays a lump-sum tax to the government, T_t. It receives lump-sum profit, Π_t, from the firm, which it owns. The household problem is:

$$\max_{C_t, K_{t+1}, B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \ln C_t$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + B_t \leq w_t + R_t^k K_t + \Pi_t - T_t + (1 + r_{t-1})B_{t-1}$$

The firm rents capital and labor from the household to produce output:

$$Y_t = K_t^{\alpha} N_t^{1-\alpha}$$

Its problem is:

$$\max_{K_t, N_t} \quad \Pi_t = K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t^k K_t$$

The government chooses G_t exogenously and finances this with lump-sum taxes each period:

$$G_t = T_t$$

(a) Solve for the first-order conditions for the household.

(b) Solve for the first-order conditions for the firm.

(c) What is the asset market-clearing condition? Use this to derive the aggregate resource constraint.

(d) Write down a social planner's version of this economy, assuming that the planner has to take G_t as given, and show the that planning solution and the competitive equilibrium are one in the same.

(e) Suppose that, in steady state, the ratio of government spending to output, G/Y, is fixed at $g \in [0, 1)$. Use this to solve for analytical expressions for steady state values of C, K, R^k, r, w, and Y.

(f) Suppose that the household receives flow utility from government spending in a way that is additively separable:

$$u(C_t, G_t) = \ln C_t + \theta \ln G_t$$

i. Argue that having government consumption enter the household objective function in an additively separable way has *no impact* on the competitive equilibrium in which G_t is exogenous.

ii. Solve for the value of g that would maximize household welfare in steady state in the planner's version of the problem.

(g) Log-linearize the equilibrium conditions of the decentralized problem, taking g as given and supposed that G_t follows an exogenous AR(1) process:

$$\ln G_t = (1 - \rho) \ln(gY) + \rho \ln G_{t-1} + s_G \varepsilon_{G,t}$$

Where gY is the steady state value of government consumption.

(h) Before doing anything on a computer, provide some written intuition – based on your analysis to this point – for how you expect consumption and other variables to react to (i) a *transitory* change in government consumption (ρ close to zero) and (ii) a *permanent* change in government consumption ($\rho \to 1$).

(i) Use Dynare (or a similar program) to produce impulse responses of all variables in the model to a one-unit shock to government consumption for three different values of ρ: 0.05, 0.8, and 0.99. Use the following values for other parameters: $\beta = 0.99$, $\delta = 0.025$, $\alpha = 1/3$, $g = 0.2$.

3. Consider a representative agent economy. There is no capital (like an endowment economy), but the representative household endogenously supplies labor. The household is endowed with one unit of time each period; it choose to work N_t, with $1 - N_t$ is leisure. The household problem is:

$$\max_{C_t, N_t, B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1 - \sigma} + \theta \ln(1 - N_t) \right)$$

s.t.

$$C_t + B_t \leq w_t N_t + (1 + r_{t-1}) B_{t-1} + \Pi_t$$

A representative firm, owned by the household, produces output using the technology $Y_t = A_t N_t$. w_t is the real wage. The firm wants to pick N_t to maximize Π_t, profit remitted back to the household:

$$\max_{N_t} \quad \Pi_t = A_t N_t - w_t N_t$$

A_t follows an exogenous process, with non-stochastic steady state value of unity:

$$\ln A_t = \rho \ln A_{t-1} + s_A \varepsilon_{A,t}$$

(a) Find the first-order conditions characterizing the solution to the household's problem.

(b) Do the same for the firm.

(c) Impose the market-clearing condition that $B_t = 0$ and derive the aggregate resource constraint.

(d) The non-stochastic steady state value of $A = 1$. Assume that $N = 1/3$ (i.e. the household works one-third of its time endowment). Solve for the value of θ (expressed as a function of other parameters) that is consistent with $N = 1/3$.

(e) Log-linearize the first order conditions of your model. Derive a parameter restriction under which labor input is constant (and the model is therefore equivalent to an endowment economy model). Provide some intuition for what you find.