Problem Set 3

Graduate Macro II, Spring 2024
The University of Notre Dame
Professor Sims

Instructions: You may consult with other members of the class, but please make sure to turn in your own work. Where applicable, please print out figures and codes. When asked to solve a model in Dynare, unless otherwise instructed you (i) should solve via a first-order approximation, (ii) may give Dynare the equilibrium conditions and do not need to do the linearization by hand, (iii) should show impulse responses up to a horizon of 20 periods, and (iv) should plot impulse responses (or analyze moments) of logged variables. This problem set is due on Canvas by 5:00 pm on February 23.

1. Consider a real business cycle (RBC) model with *anticipated* productivity shocks. I will write this as a planner's problem:

$$\max_{C_t, N_t, K_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t + \theta \ln(1 - N_t)\right]$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t \leq A_t K_t^{\alpha} N_t^{1-\alpha}$$

Productivity, A_t, follows an AR(1) process in the log, but shocks are observed four periods prior to materializing:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t-4}$$

To be clear: agents (and the planner) observe $\varepsilon_{A,t}$ today, but it does not impact the level of A_t until four periods into the future.

   (a) Derive the first order conditions for the planner's problem.
   (b) Derive an expression for θ as a function of other variables that is consistent with steady state labor input being $N = 1/3$.
   (c) Assume $\beta = 0.99$, $\alpha = 1/3$, and $\delta = 0.025$. Let $\rho_A = 0.95$ and $s_A = 0.01$. Write a Dynare file to solve this model with a first-order approximation and show impulse responses to the anticipated shock. Note: in Dynare, you can just use the timing convention ($e(-4)$) in writing the productivity process. This will create additional state variables that keep track of the shock between when it is seen (t) and when it is realized ($t+4$).
   (d) Provide some written intuition for the effects of the shock.

2. Consider a real business cycle (RBC) model with one slight twist. A representative household solves the following problem:

$$\max_{C_t, K_{t+1}, N_t, B_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \theta N_t\right]$$

s.t.

$$C_t + K_{t+1} - (1-\delta)K_t + B_t \leq w_t N_t + R_t^k K_t + \Pi_t + (1 + r_{t-1})B_{t-1}$$

A representative firm produces output according to:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} + Z_t$$

Here, A_t is a conventional productivity shock. It follows a non-stochastic process with mean of one:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t}$$

Z_t is an additive productivity shock. It follows a stochastic process with non-stochastic mean of zero:

$$Z_t = \rho_Z Z_{t-1} + s_Z \varepsilon_{Z,t}$$

Both shocks are drawn from a standard normal distribution. The firm picks capital and labor to maximize:

$$\Pi_t = Y_t - w_t N_t - R_t^k K_t$$

Bonds are in zero supply, $B_t = 0$.

(a) Derive the first-order conditions necessary for the household problem.

(b) Derive the first-order conditions for the firm problem.

(c) Define a competitive equilibrium and derive the aggregate resource constraint.

(d) Solve for expressions of the non-stochastic steady state. In doing so, assume that $N = 1/3$; derive an expression for θ that is consistent with that normalization.

(e) Write a Dynare file to solve the model assuming the following parameter values: $\beta = 0.99$, $\alpha = 1/3$, $\delta = 0.025$, $\rho_A = \rho_Z = 0.95$, and $s_A = s_Z = 0.01$. Produce impulse responses of endogenous variables to both the multiplicative (A_t) and additive (Z_t) productivity shocks. Provide some written intuition for why the impulse responses do (or do not) differ.

(f) Re-do the last part, but this time assume $\rho_A = \rho_Z = 0.5$. Comment on how the impulse responses do or do not look different relative to the case with a higher persistence.

3. Consider a real business cycle model, but with a separate production process for consumption goods and investment goods. A household accumulates physical capital and leases it to the production firm as usual. The household purchases new physical capital, $\widehat{I}_t$, from an investment goods firm at price P_t^k. Consumption goods are the numeraire. The household problem is:

$$\max_{C_t, B_t, N_t, \widehat{I}_t, K_{t+1}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \theta \frac{N_t^{1+\eta}}{1+\eta} \right]$$

s.t.

$$C_t + P_t^k \widehat{I}_t + B_t \leq w_t N_t + R_t^k K_t + (1 + r_{t-1}) B_{t-1} + \Pi_t + \Pi_t^k$$
$$K_{t+1} = \widehat{I}_t + (1 - \delta) K_t$$

Where Π_t is profit from the production firm and Π_t^k is profit from the capital goods firm. The production firm problem is straightforward:

$$\max_{N_t, K_t} \quad \Pi_t = A_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - R_t^k K_t$$

The problem of the capital goods firm is as follows. It produces new capital goods, $\widehat{I}_t$, using unconsumed output, I_t, according to the following:

$$\widehat{I}_t = \mu_t \left(1 - \frac{\kappa}{2} \left(\frac{I_t}{I_{t-1}} - 1 \right)^2 \right) I_t$$

Here, $\kappa \geq 0$ represents and adjustment cost – if the capital producing firm is changing its inputs at all, it loses some of these inputs in producing new capital. μ_t is an exogenous, stochastic process that measures the efficiency of transforming unconsumed output into new capital. The problem of the capital goods producing firm is to choose input, I_t, to maximize the present discounted value of profit. Because of the adjustment cost, the problem is dynamic – the choice of I_t today impacts the future. The capital producing firm discounts future flow profits by the by the stochastic discount factor of the household, which we can define as $\Lambda_{t,t+j} = \frac{\beta^j u'(C_{t+j})}{u'(C_t)}$ (i.e. the marginal rate of substitution between future and current consumption). The problem facing the capital producing firm is:

$$\max_{I_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_{0,t} \left[P_t^k \widehat{I}_t - I_t \right]$$

A_t and μ_t follow AR(1) processes with non-stochastic means normalize to unity.

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t}$$
$$\ln \mu_t = \rho_\mu \ln \mu_{t-1} + s_\mu \varepsilon_{\mu,t}$$

Market clearing requires that bonds are in zero supply, $B_t = 0$.

(a) Derive the optimality conditions for the household problem.

(b) Derive the optimality conditions for the production firm problem.

(c) Derive the optimality conditions for the capital producing firm problem.

(d) Derive the aggregate resource constraint.

(e) Argue that, if $\kappa = 0$ and $\mu_t = 1$, this problem is *identical* to the basic problem we have studied in class.

(f) Derive analytic expressions for the steady state of the model. Pick θ to be consistent with $N = 1/3$ in steady state, and show an expression for the required value of θ.

(g) Assume that the parameter values are as follows: $\beta = 0.99$, $\delta = 0.025$, $\alpha = 1/3$, θ consistent with $N = 1/3$ (as above), $\eta = 1$, $\rho_A = \rho_\mu = 0.95$, and $s_A = s_\mu = 0.01$. Consider two separate values of the parameter κ: 0 and 2. Produce graphs with impulse responses to both the neutral productivity shock ($\varepsilon_{A,t}$) and the investment shock ($\varepsilon_{\mu,t}$) for both values of κ. In doing so, show the responses of one variable to one shock for both values of κ. Comment on a couple of things:

   i. How the impulse responses to the productivity and investment shocks differ from one another (for any value of κ).

   ii. How the value of κ impacts the impulse responses.

(h) In the context of this model, can investment shocks (shocks to μ_t) be the major driver of business cycles if we want the model to be roughly consistent with observed data? Explain.

(i) In the data, the following are stylized empirical facts: the real interest rate is roughly acyclical, output growth (i.e. $\ln Y_t - \ln Y_{t-1}$) is positively autocorrelated, and labor input is procylical and about as volatile as output. Comment on how bigger values of κ help (or hurt) with this model qualitatively matching these three features of the data.