Graduate Macro Theory II: Notes on Determinacy with Interest Rate Rules

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1 Introduction

In models with nominal rigities, it is popular to characterize monetary policy with simple interest rate rules instead of exogenous money supply rules. Such rules focus in on the instrument central banks seem to care about (e.g. interest rates, not measures of the money supply), seem to fit the data reasonably well, and often have good normative properties.

A complicating factor with interest rate rules is that issues of determinacy arise. In general, interest rate rules must react sufficiently strongly to *endogenous* variables (like inflation and/or a measure of output) in order to guarantee a determinant rational expectations equilibrium. By "determinate" I mean "unique." If a rule does not respond aggressively enough to endogenous variables then there may be indeterminacy, which can give rise to non-fundamental "sun-spot" equilibria. If there is an indeterminate equilibrium in these models, then that means that there is no unique non-explosive value of current inflation that satisfies the equilibrium conditions of the model given the current state. In a model with no nominal rigidity this just means there is nominal indeterminacy. But if there is nominal rigidity (e.g. price stickiness), then nominal indeterminacy also gives rise to real indeterminacy in the sense that there may be non-fundamental fluctuations in real quantities. From a welfare perspective real indeterminacy is undesirable, so we'd like to understand the restrictions on policy rules giving rise to determinacy.

2 Taylor's Original Intuition

The "grandfather" of interest rate rules is widely considered to be John Taylor, after whom the "Taylor Rule" is named. His famous paper on the topic was Taylor (1993), "Discretion Versus Policy Rules in Practice," which appeared in the 1993 *Carnegie-Rochester Conference Series on Public Policy*. In this paper, he proposed a policy rule of the following form (omitting constants, so we can interpret all variables as deviations from trend/steady state):

$$i_t = \phi_\pi \pi_t + \phi_x x_t$$

Here π_t is inflation, x_t is the gap between actual and potential output, and i_t is the interest rate controlled by the central bank (e.g. the Fed Funds rate).[1] Taylor argued that values of $\phi_\pi = 1.5$ and $\phi_x = 0.5$ fit the data well. He argued that the coefficient on inflation needed to be greater than 1. This came to be known as the "Taylor principle."

Taylor's logic for this parameter restriction is loosely as follows. Total aggregate demand depends on the real interest rate, and inflation depends on aggregate demand. The real interest rate is approximately $r_t = i_t - \mathbb{E}_t \pi_{t+1}$. Even though the real rate depends on expected inflation, for sake of argument suppose that we have *adaptive expectations* so that $\mathbb{E}_t \pi_{t+1} = \pi_t$. Whenever inflation increases, if $\phi_\pi > 1$, the nominal interest rate increases by more. Under a backward-looking expectations model like this, this means the real interest rate increases whenever inflation increases (holding output fixed). A higher real interest rate depress aggregate demand, which brings inflation down. In contrast, suppose that $\phi_\pi < 1$. This means that whenever inflation increases the real interest rate declines. This decline in the real rate fuels more inflation by stimulating aggregate demand, and so inflation can "spiral" out of control.

This is *stabilizing* logic. Implicitly, it sounds like you need a sufficient reaction to inflation to generate a stable root to keep the system from exploding. Though a similar restriction obtains in a forward-looking New Keynesian model, such a restriction is not really about "stabilizing" per se. Rather, we need a sufficient response to endogenous variables in a policy rule to impart a sufficient number of *unstable* roots into the system. This makes the model equilibrium unique.

3 Determinacy in a Model with Flexible Prices

Suppose that we have a very simple model. There is no capital, so all output must be consumed. Prices are flexible, meaning that the classical dichotomy holds and there is no effect of nominal variables on real variables. Money demand is implicitly generated via money in the utility function, additively separable from consumption. The demand side of the economy is summarized by the Euler/IS equation (all variables are taken to be either percentage deviations from steady state or deviations from steady state):

$$y_t = \mathbb{E}_t y_{t+1} - (i_t - \mathbb{E}_t \pi_{t+1})$$

Here, I have implicitly assumed a unitary elasticity of intertemporal substitution. Suppose the policy rule just reacts to inflation with a random, mean zero shock, u_t:

$$i_t = \phi_\pi \pi_t + u_t$$

Suppose that u_t follows a stationary (e.g. $0 < \rho < 1$ AR(1) process):

$$u_t = \rho u_{t-1} + e_t$$

[1] Taylor's paper is empirical. He measured the "gap" with a measure of output less a statistical trend, which is different than the theoretical concept of the gap.

To make life as simple as possible, suppose that real output is both exogenous and constant. This means that $y_t = \mathbb{E}_t y_{t+1} = 0$. The Euler equation then becomes:

$$i_t = \mathbb{E}_t \pi_{t+1}$$

If we combine this expression with the policy rule, we get:

$$\mathbb{E}_t \pi_{t+1} = \phi_\pi \pi_t + u_t$$

This is a forward-looking difference equation for which there exist many different solutions as a general matter. To get a solution we use the equivalent of a transversality condition, requiring that $\lim_{T \to \infty} \mathbb{E}_t \pi_{t+T} = 0$. For there to be a unique non-explosive solution, you need the difference equation to be explosive. Basically, this is a system of one forward-looking variable, π_t, and one state variable, u_t. The eigenvalue associated with u_t will be ρ, which is stable. For saddle point stability, we need an unstable eigenvalue associated with π_t. This eigenvalue is ϕ_π. If $\phi_\pi < 1$, then there is no unique solution – any value of π_t will have expected inflation go to zero in the limit for any u_t. To see this as cleanly as possible, suppose $u_t = 0$. Then, solving forward, we have:

$$\mathbb{E}_t \pi_{t+T} = \phi_\pi^T \pi_t$$

If $\phi_\pi < 1$ (ruling out negative values), then $\phi_\pi^T \to 0$ for T big. This means *any* value of π_t is consistent with inflation not exploding. In contrast, if $\phi_\pi > 1$, then $\phi_\pi^T \to \infty$. The only for inflation to not explode is then if $\pi_t = 0$; that would be the unique solution.

For the more general situation in which we allow $u_t \neq 0$, we can solve for the unique solution by guess that $\pi_t = a u_t$. Doing so, we get:

$$a \mathbb{E}_t u_{t+1} = \phi_\pi a u_t + u_t$$

Since $\mathbb{E}_t u_{t+1} = \rho u_t$, we have:

$$a(\rho - \phi_\pi) = 1$$

So we get $\pi_t = \frac{1}{\rho - \phi_\pi}$ as the solution (which is unique, provided $\phi_\pi > 1$).

4 Determinacy in a Basic New Keynesian Model

Consider a standard New Keynesian model. There is the Euler/IS equation, the Phillips Curve, and an exogenous process for the flexible price level of output (recall that there are multiple ways to write down the equilibrium conditions). Let all variables denote percentage (or actual) deviations from the non-stochastic steady state. The equations of the model are:

$$\pi_t = \gamma(y_t - y_t^f) + \beta \mathbb{E}_t \pi_{t+1}$$
$$y_t = \mathbb{E}_t y_{t+1} - (i_t - \mathbb{E}_t \pi_{t+1})$$
$$y_t^f = \rho y_{t-1}^f + s_y \varepsilon_t$$

The slope coefficient $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}(1+\chi)$. I am assuming that the inverse elasticity of intertemporal substitutions (the coefficient of relative risk aversion) is one; ϕ is the Calvo parameter, and χ is the inverse Frisch elasticity. I could have written the equilibrium conditions in terms of the output gap and the natural rate of interest; it doesn't really matter how I do it.

Suppose that the nominal interest rate (in deviations from steady state, so abstracting from constants) obeys a simple Taylor rule of the form:

$$i_t = \phi_\pi \pi_t + \phi_x (y_t - y_t^f)$$

Note that, for determinacy ay least, it does not matter if I have the Taylor rule reacting to the output gap, $x_t = y_t - y_t^f$, or to output itself (in deviation from steady state, y_t). This is because y_t^f is exogenous, and determinacy will depend on reactions of the interest rate to endogenous variables. It will matter for how the economy reacts to shocks, as well as for welfare, whether the central bank reacts to the output gap or to the output level, but it does not matter for determinacy.

We want to know the following: what restrictions on ϕ_π and ϕ_x must be made in order to ensure a determinate rational expectations equilibrium? To see this, eliminate i_t and form a three variable system. The Euler/IS equation becomes:

$$y_t = \mathbb{E}_t y_{t+1} - \phi_\pi \pi_t - \phi_x y_t + \phi_x y_t^f + \mathbb{E}_t \pi_{t+1}$$

This is a difference equation in y_t and π_t. The Phillips Curve is already that (since i_t doesn't enter directly into the Phillips Curve, we don't need to do any substitution into it). In essence, i_t is a static variable and we want to substitute it out in writing down a system of equations to solve. Given these two remaining equations, plus the exogenous process for y_t^f, we can form a vector system as follows:

$$\mathbb{E}_t \begin{bmatrix} \pi_{t+1} \\ y_{t+1} \\ y_{t+1}^f \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} & \frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} & -\frac{\gamma}{\beta} - \phi_x \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \\ y_t^f \end{bmatrix}$$

To get this, I isolate π_{t+1}, y_{t+1}, and y_{t+1}^f separately on the left hand sides of each equation. Our goal now is to find the eigenvalues of this system of equation. One of them is clearly ρ, which is than one, and hence stable. This is the eigenvalue associated with the (exogenous) state variable, y_t^f. For equilibrium determinacy, we need two *unstable* eigenvalues associate with the remaining jump variables, y_t and π_t.

To find the other two eigenvalues we just need to find the eigenvalues of the upper 2×2 block of the coefficient matrix. That is, we need to find the λ which makes:

$$\det \begin{bmatrix} \frac{1}{\beta} - \lambda & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} - \lambda \end{bmatrix} = 0$$

The determinant of a 2×2 matrix is just the difference of the product of the diagonals:

$$\left(\frac{1}{\beta} - \lambda\right)\left(1 + \phi_x + \frac{\gamma}{\beta} - \lambda\right) + \frac{\gamma}{\beta}\left(\phi_\pi - \frac{1}{\beta}\right) = 0$$

Now, two useful facts about eigenvalues and determinants. First, the product of the eigenvalues is just equal to the determinant of the matrix. Second, the sum of the eigenvalues is equal to the trace of the matrix. The determinant and trace of the upper 2×2 matrix are:

$$\lambda_1\lambda_2 = \det \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} \end{bmatrix} = \frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\gamma\phi_\pi}{\beta}$$

$$\lambda_1 + \lambda_2 = \text{trace} \begin{bmatrix} \frac{1}{\beta} & -\frac{\gamma}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\gamma}{\beta} \end{bmatrix} = \frac{1}{\beta} + 1 + \phi_x + \frac{\gamma}{\beta}$$

Since both the determinant and trace must be positive given standard assumptions on parameter values, we know that both eigenvalues must be positive as well.

For a unique equilibrium, we need both of these eigenvalues to be explosive (we already have one stable root, and we have two jump variables). Since we know from above that both these eigenvalues must be positive, then (ignoring complex roots), the necessary condition for stability is that:

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0$$

Multiply this out:

$$\lambda_1\lambda_2 - (\lambda_1 + \lambda_2) + 1 > 0$$
$$\lambda_1\lambda_2 - (\lambda_1 + \lambda_2) > -1$$

Plug in our expressions from above and simplify:

$$\frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\gamma\phi_\pi}{\beta} - \left(\frac{1}{\beta} + 1 + \phi_x + \frac{\gamma}{\beta}\right) > -1$$
$$\phi_x\left(\frac{1}{\beta} - 1\right) + \frac{\gamma\phi_\pi}{\beta} - \frac{\gamma}{\beta} > 0$$
$$\phi_x(1 - \beta) + \gamma\phi_\pi - \gamma > 0$$

The last line follows from multiplying both sides by β. Now divide both sides by γ and simplify:

$$\phi_x \frac{1-\beta}{\gamma} + \phi_\pi > 1$$

This is the condition that must be satisfied for there to exist a determinate equilibrium. We can see that $\phi_\pi > 1$ is slightly too strong of a restriction – determinacy also depends on the response to the output gap in the policy rule. But if $\beta \approx 1$, then unless γ is very small the determinacy condition is still roughly $\phi_\pi > 1$.

You can trick up the model along a number of dimensions but something like this basic condition usually emerges. It is very common to write down policy rules with explicit interest rate smoothing:

$$i_t = \rho_i i_{t-1} + \phi_\pi \pi_t + \phi_x (y_t - y_t^f)$$

With the interest rate now an endogenous state variable, the calculation of the eigenvalues of the system is more complicated. But the determinacy condition ends up being:

$$\phi_x \frac{1-\beta}{\gamma} + \phi_\pi > 1 - \rho_i$$

There is a useful interpretation of this condition. It effectively says that the *long-run* response of the interest rate to target variables must exceed one. How can we see this? Iterate the interest rate forward one period, assuming that all future inflation and output gaps are equal to the period t levels:

$$i_{t+1} = \rho_i i_t + \phi_\pi \pi_t + \phi_x x_t$$

Where, to ease notation, I have just written $y_t - y_t^f = x_t$. Note again I'm assuming $\pi_{t+1} = \pi_t$ and $x_{t+1} = x_t$. Plugging in for the lagged nominal rate, we have:

$$i_{t+1} = \rho_i^2 i_{t-1} + \rho_i \phi_\pi \pi_t + \rho_i \phi_x x_t + \phi_\pi \pi_t + \phi_x x_t$$

If we keep doing this, we get:

$$i_{t+T} = \rho_i^{T+1} i_{t-1} + \left(1 + \rho_i + \rho_i^2 + \ldots \rho_i^T\right) \phi_\pi \pi_t + \left(1 + \rho_i + \rho_i^2 + \ldots \rho_i^T\right) \phi_x x_t$$

If we take the limit as $T \to \infty$, we get:

$$\lim_{T\to\infty} \; i_{t+T} = \frac{\phi_\pi}{1-\rho_i} \pi_t + \frac{\phi_x}{1-\rho_i} x_t$$

The determinacy condition, in terms of long-run coefficients, is the same as without smoothing:

$$\frac{\phi_x}{1-\rho_i} \frac{1-\beta}{\gamma} + \frac{\phi_\pi}{1-\rho_i} > 1$$

For this reason, it is very common to write a Taylor rule with smoothing as:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) \left(\phi'_\pi \pi_t + \phi'_x x_t \right)$$

With the coefficients scaled in this way, we get the determinacy condition being:

$$\phi'_x \frac{1 - \beta}{\gamma} + \phi'_\pi > 1$$

An interest rate rule written this way is nice because it has a flavor of partial adjustment. In particular, we could write it as:

$$i_t = \rho_i i_{t-1} + (1 - \rho_i) i_t^*$$

I am calling i_t^* the target interest rate. Here, the actual interest rate is a convex combination of the lagged rate and the target rate, where the target rate follows a Taylor rule without smoothing:

$$i_t^* = \phi'_\pi \pi_t + \phi'_x x_t$$