# Graduate Macro Theory II: Notes on The Zero Lower Bound (ZLB)

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In this set of notes, we are going to analyze the implications of the zero lower bound (ZLB, also sometimes called the effective lower bound, ELB) in the New Keynesian model. The basic NK model is comprised of the IS equation, (1), and the Phillips Curve, (2).

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f\right) \tag{1}$$

$$\pi_t = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} \tag{2}$$

I am ignoring a cost-push shock for this set of notes. To close the model, we need some description of policy. Outside of the ZLB, I'm going to assume that the central bank follows a strict inflation target:

$$\pi_t = 0 \tag{3}$$

(3) is consistent with optimal policy under either discretion or commitment, and, outside of the ZLB, will result in $x_t = 0$ and $i_t = r_t^f$ in equilibrium.

We shall assume that the natural rate of interest, $r_t^f$, obeys an exogenous process:

$$r_t^f = \rho_r r_{t-1}^f + s_r \varepsilon_{r,t} \tag{4}$$

For this set of notes, we are going to analyze a situation in which the interest rate, $i_t$, is unable to adjust for some period of time. What matters is not whether the threshold where it can adjust is literally zero or something else, but rather that the interest rate is *pegged* (i.e. fixed). The interest rate cannot be pegged forever – this would result in equilibrium indeterminacy. We are going to analyze situations in which the economy starts in a situation in which the interest rate is pegged (i.e. $i_t = 0$, so constant) for some period of time. The period of time will be either deterministic or stochastic, but in either event will be exogenous. Once the peg lifts, it is never expected to bind again. This means that we are ignoring "precautionary" behavior in which agents might alter their behavior in anticipation of the ZLB binding.

# 1 Deterministic Interest Rate Peg

With a deterministic peg, the interest rate is fixed for some known, finite duration, i.e. $i_{t+h} = 0$ for $h = 0, \dots H-1$. This means that the peg lasts for $H > 0$ periods. Once the peg lifts, the central bank sets policy according to (3). This makes finding a solution particularly straightforward – once the peg lifts, we have $\pi_{t+k} = x_{t+k} = 0$ for $k \geq H$.

To find the solution, we simply need to work backwards. This is a special case of how all DSGE models are solved – we have equilibrium conditions, we impose a terminal condition(s), and we find the initial condition for jump variables consistent with the terminal condition(s) holding. Suppose that the peg lasts for one period, $H = 1$, so $i_t = 0$ but thereafter $i_{t+k} = r^f_{t+k}$. We know that $\mathbb{E}_t x_{t+1} = \mathbb{E}_t \pi_{t+1} = 0$. Then, from (1), we have:

$$x_t = \frac{1}{\sigma} r^f_t \tag{5}$$

But then we can solve for period $t$ inflation from (2):

$$\pi_t = \frac{\gamma}{\sigma} r^f_t \tag{6}$$

Easy enough. Now, suppose that the interest rate peg will last for two periods, $H = 2$. We know that $\mathbb{E}_t x_{t+2} = \mathbb{E}_t \pi_{t+2} = 0$. Hence, from (1) led forward one period, we have:

$$\mathbb{E}_t x_{t+1} = \frac{1}{\sigma} \mathbb{E}_t r^f_{t+1} = \frac{\rho_r}{\sigma} r^f_t$$

Similarly, for inflation:

$$\mathbb{E}_t \pi_{t+1} = \frac{\rho_r \gamma}{\sigma} r^f_t$$

Now, go back to (1) in period $t$. Imposing the peg, we have:

$$x_t = \mathbb{E}_t x_{t+1} + \frac{1}{\sigma} \mathbb{E}_t \pi_{t+1} + \frac{1}{\sigma} r^f_t$$

Now, using the results from above, we have:

$$x_t = \frac{\rho_r}{\sigma} r^f_t + \frac{\rho_r \gamma}{\sigma^2} r^f_t + \frac{1}{\sigma} r^f_t = \frac{1}{\sigma}\left(1 + \rho_r\left(1 + \frac{\gamma}{\sigma}\right)\right) r^f_t$$

For inflation, we have:

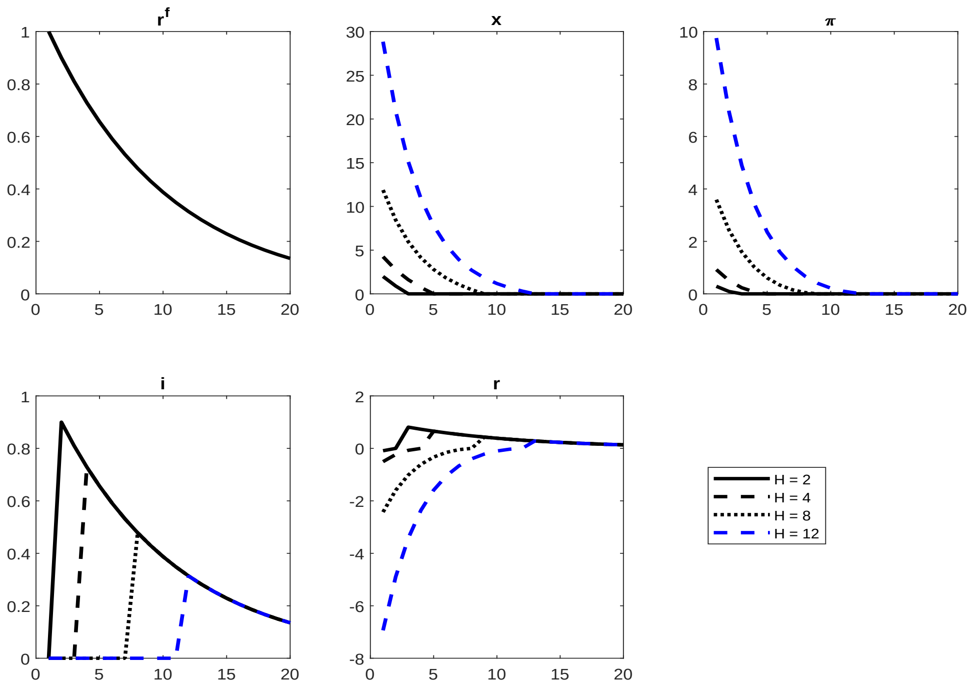
$$\pi_t = \frac{\gamma}{\sigma}\left(1 + \rho_r\left(1 + \frac{\gamma}{\sigma}\right)\right) r^f_t + \frac{\rho_r \beta \gamma}{\sigma} r^f_t = \frac{\gamma}{\sigma}\left(1 + \rho_r\left(1 + \frac{\gamma}{\sigma}\right) + \beta \rho_r\right) r^f_t$$

We could keep doing this for arbitrary peg lengths, $H$, but it gets laborious and algebraically messy. It is straightforward to solve this backwards via a loop in a program like Matlab. I did so. I assume that there is a one-unit (positive) shock to the natural rate of interest in the first period that decays with AR parameter $\rho_r = 0.9$. I assume that $\gamma = 0.1$, $\beta = 0.99$, and $\sigma = 1$. Here is my code, which generates impulse responses of variables to the natural rate shock for an arbitrary peg length, $H$. I include the real interest rate, $r_t = i_t - \mathbb{E}_t \pi_{t+1}$, as an endogenous variable when calculating these.

```matlab
% set parameters
beta = 0.99;
gamma = 0.1;
sigma = 1;
rhor = 0.9;


% pre-populate IRFs
N = 20; % number of periods to plot
H = 4; % length of peg

xirf = zeros(N,1);
piirf = zeros(N,1);
rfirf = zeros(N,1);
iirf = zeros(N,1);
rirf = zeros(N,1);

% natural rate irf
for j = 1:N
    rfirf(j,1) = rhor^(j-1);
end

% now do IRFs of the rest

for j = H:N
    xirf(j,1) = 0;
    piirf(j,1) = 0;
    iirf(j,1) = rfirf(j,1);
    rirf(j,1) = rfirf(j,1);
end

% now work backwards
for j = 1:H
    xirf(H-j+1,1) = xirf(H-j+2,1) - (1/sigma)*(-piirf(H-j+2,1) - rfirf(H-j+1,1));
    piirf(H-j+1,1) = gamma*xirf(H-j+1,1) + beta*piirf(H-j+2,1);
    rirf(H-j+1,1) = -piirf(H-j+2,1);
end
```

IRFs are shown in Figure 1. With no ZLB, the nominal rate would move one-for-one with the natural rate, inflation and the output gap would always be zero, and the real interest rate would move one-for-one with the natural rate (and with the nominal rate). I consider four different peg lengths: $H = 2$, $H = 4$, $H = 8$, and $H = 12$. When the interest rate is pegged for any duration, the output gap and inflation go up on impact. What is driving this is pretty simple. The real interest rate is $r_t = i_t - \mathbb{E}_t \pi_{t+1}$. If the nominal rate can't react, the real interest rate is equal to the negative of the expected inflation rate. As we saw above in the analytics of a one- or two-period peg, the inflation rate is positive in the period before the peg ends. Working backwards, this means that the real interest rate either doesn't react ($H = 1$) or falls on impact ($H > 1$). This stimulates the output gap. But a higher output gap stimulates inflation. The longer the peg lasts, the more exacerbated these effects are. When the interest rate is pegged for 12 periods, for example, both the output gap and inflation react quite significantly (and the real interest rate falls quite significantly).

Figure 1: IRFs to Natural Rate Shock, Deterministic Peg of Duration $H$




Note that, once the peg ends, the IRFs are all identical. This is true regardless of $H$. This is a consequence of having no endogenous state variables in the model.

## 2 Stochastic Interest Rate Peg

Next, consider a situation in which the length of the interest rate peg is stochastic rather than deterministic. In particular, assume that, in period $t$, the nominal interest rate is stuck at 0. Then, in each subsequent period there is a probability $1 - \alpha$ that the peg lifts (and, consequently, a probability $\alpha$ that it stays in place), with $0 \leq \alpha < 1$. The expected duration of the peg is therefore:

$$\mathbb{E}[\text{Duration}] = (1-\alpha) \times 1 + \alpha(1-\alpha) \times 2 + \alpha^2(1-\alpha) \times 3 + \alpha^3(1-\alpha) \times 3 + \dots$$

In this setup, with the interest rate fixed in the present, the probability of the peg lasting one period is $1-\alpha$: this is the probability the peg lifts for period $t+1$. The probability of the peg last two periods is $\alpha(1-\alpha)$: $\alpha$ is the probability it remains pegged for $t+1$ and $1-\alpha$ is the probability that it lifts in period $t+2$. Similarly, the probability of the peg lasting three periods is $\alpha^2(1-\alpha)$: $\alpha^2$ is the probability that the peg is in place for $t+1$ and $t+2$, and $1-\alpha$ is the probability that it lifts in $t+3$. And so on. We can therefore write the expected duration as:

$$\mathbb{E}[\text{Duration}] = (1-\alpha)\left[1 + 2\alpha + 3\alpha^2 + 4\alpha^3 + \dots\right]$$

Define:

$$S = 1 + 2\alpha + 3\alpha^2 + 4\alpha^3$$

So:

$$\alpha S = \alpha + 2\alpha^2 + 3\alpha^3 + \dots$$

Therefore:

$$(1-\alpha)S = 1 + \alpha + \alpha^2 + \alpha^3 + \dots$$

Since we know that the right hand side reduces to $(1-\alpha)^{-1}$, we have $S = \frac{1}{(1-\alpha)^2}$. Therefore, the expected duration is:

$$\mathbb{E}[\text{Duration}] = \frac{1}{1-\alpha}$$

So, if $\alpha = 1/2$, the ZLB is expected to be in place for two periods. If $\alpha = 3/4$, the expected duration is four periods. And so on.

We can solve for analytic expressions for inflation and the output gap using the method of undetermined coefficients. Once the peg lifts, we know that both will equal zero. During the peg, guess that these are:

$$\pi_t = \theta_\pi r_t^f$$
$$x_t = \theta_x r_t^f$$

Plug these guesses into the Phillips Curve, (2):

$$\theta_\pi r_t^f = \gamma\theta_x r_t^f + \beta\alpha\theta_\pi \mathbb{E}_t r_{t+1}^f$$

In writing the above, note that with probability $1-\alpha$ we have $\mathbb{E}_t\pi_{t+1} = 0$; with probability $\alpha$, we will still be in the peg and hence $\mathbb{E}_t\pi_{t+1} = \theta_\pi \mathbb{E}_t r_{t+1}^f$. Since $\mathbb{E}_t r_{t+1}^f = \rho_r r_t^f$, we have:

$$\theta_\pi r_t^f = \gamma\theta_x r_t^f + \beta\alpha\theta_\pi\rho_r r_t^f$$

Which implies:

$$\theta_\pi = \frac{\gamma}{1-\alpha\rho_r\beta}\theta_x$$

Now go to the IS equation, (1), and plug in these guesses:

$$\theta_x r_t^f = \theta_x \alpha \mathbb{E}_t r_{t+1}^f + \frac{1}{\sigma}\alpha\theta_\pi \mathbb{E}_t r_{t+1}^f + \frac{1}{\sigma} r_t^f$$

In writing this out, note that with probability $1-\alpha$ we have $\mathbb{E}_t \pi_{t+1} = \mathbb{E}_t x_{t+1} = 0$; with probability $\alpha$ these take on the same policy functions as before. Since $\mathbb{E}_t r_{t+1}^f = \rho_r r_t^f$, we have:

$$\theta_x r_t^f = \theta_x \alpha \rho_r r_t^f + \frac{1}{\sigma}\alpha\theta_\pi \rho_r r_t^f + \frac{1}{\sigma} r_t^f$$

Or:

$$\theta_x \sigma\left(1-\alpha\rho_r\right) = \alpha\rho_r\theta_\pi + 1$$

Plug in for $\theta_\pi$ in terms of $\theta_x$ above:

$$\theta_x \sigma(1-\alpha\rho_r) = \frac{\gamma\alpha\rho_r}{1-\alpha\rho_r\beta}\theta_x + 1$$

Or:

$$\theta_x\sigma(1-\alpha\rho_r)(1-\alpha\rho_r\beta) = \gamma\alpha\rho_r\theta_x + (1-\alpha\rho_r\beta)$$

Solving for $\theta_x$, we have:

$$\theta_x = \frac{1-\alpha\rho_r\beta}{\sigma(1-\alpha\rho_r)(1-\alpha\rho_r\beta)-\alpha\rho_r\gamma} \tag{7}$$

Which then means:

$$\theta_\pi = \frac{\gamma}{\sigma(1-\alpha\rho_r)(1-\alpha\rho_r\beta)-\alpha\rho_r\gamma} \tag{8}$$

Note that there is a useful check on your math here. If $\alpha = 0$, so that the peg lasts one period with certainty, these answers should collapse to what we had in the deterministic peg case for $H = 1$. And they do. With $\alpha = 0$, (7) becomes $1/\sigma$ and (8) becomes $\gamma/\sigma$, same as (5) and (6).

In analyzing (7)-(8), we can see that $\theta_x$ and $\theta_\pi$ will be bigger the bigger is $\alpha$ (i.e. the longer is the expected duration of the ZLB) – but only up to a point. In particular, there is going to be a sign-flip at a big enough value of $\alpha$ – these coefficients will go from positive to negative. This sign flip occurs at:

$$\alpha\rho_r\gamma > \sigma(1-\alpha\rho_r)(1-\alpha\rho_r\beta)$$

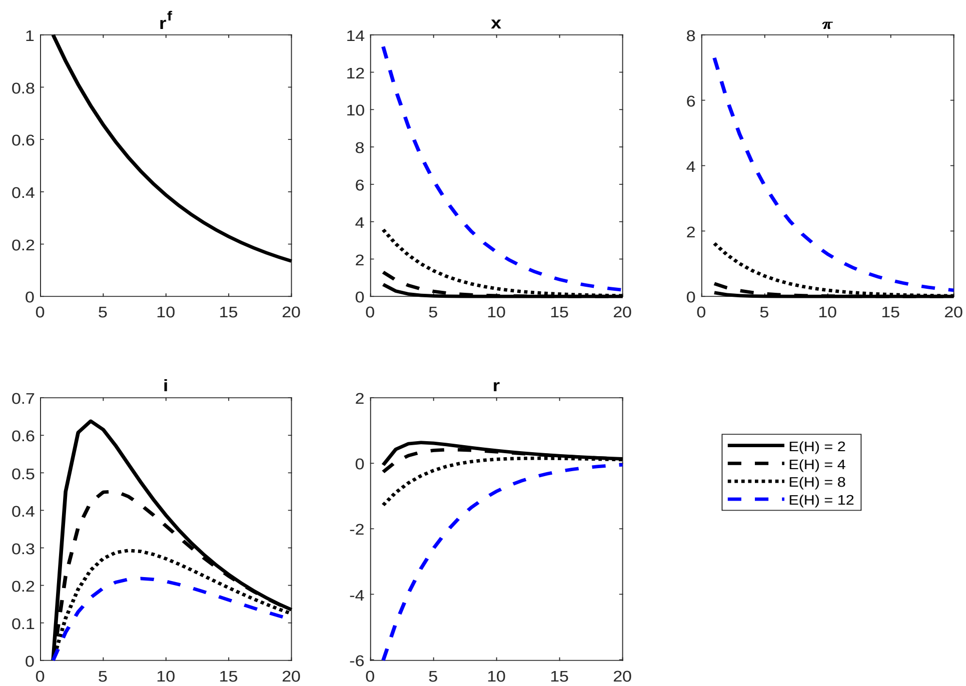
This sign-flip is a "pathological" feature of the basic NK model and I'm only going to consider values of $\alpha$ that are sufficiently small that the coefficients for the output gap and inflation are positive.

As I did in the deterministic case, I can construct impulse responses to a natural rate shock at the ZLB. There is one subtle issue here, however. We define an impulse response as how the *expected* future values of endogenous variables react to a shock. The impulse responses of inflation and the output gap are therefore going to be $\alpha^j \rho^j \theta_\pi$ and $\alpha^j \rho^j \theta_x$ for $j = 0, \ldots$. $\rho^j$ is the IRF of the natural rate to a one-unit shock in period $j$. $\theta_\pi$ and $\theta_x$ are the policy functions at the ZLB in that period. Since inflation and output equal zero if the ZLB lifts, we can ignore those parts.

Things are a bit trickier in calculating the impulse response of the nominal interest rate. This will not just be zero until the peg lifts in expectation. Even with $\alpha$ very close to one, there is some probability that the peg will lift and the nominal rate will equal the natural rate. The probability that the interest rate is still at zero at horizon $j$ is $\alpha^j$. When $j = 0$, this is unity. But when $j > 0$, this will not be unity, and so the expected path of the nominal rate will be $\alpha^j \times 0 + (1 - \alpha^j) \times r^f_{t+j} = (1 - \alpha^j)\rho^j_r$, where $1 - \alpha^j$ is the probability that the peg has lifted by horizon $j$. Given impulse responses of inflation and the nominal rate, we can calculate a response of the nominal rate.

To avoid the "sign-flip" issue, I'm going to set $\sigma = 3$, keep the same values of $\beta$ and $\gamma$, and consider $\alpha$ consistent with expected duration of the ZLB of 2, 4, 8, and 12 quarters (corresponding to values of $\alpha$ of 0.5, 0.75, 0.875, and 0.9167, respectively).

Figure 2: IRFs to Natural Rate Shock, Stochastic Peg Duration



You can see here that similar stuff is at work as in the case of the deterministic peg. The longer the peg lasts (in expectation), the more inflation and the output gap react. When the expected duration is 12, for example, the output gap and inflation react quite a lot (and the real interest rate goes down, which is a consequence of expected inflation rising and the nominal interest rate not). Note that there are a couple of subtle differences relative to the deterministic case. First, the responses under pegs of different expected durations never exactly lie on top of the no peg case. This is because there is always some (small) probability of the peg lasting a very long time. Second, the impulse response of the interest rate is not zero during the period of the expected peg. This is because there is always some possibility that the peg lifts sooner than expected, and the impulse response function is plotting how the nominal rate evolves in expectation. What is the case is that, in expectation, the nominal rate is *under-reacting* relative to the natural rate, particularly during "early" periods. Monetary policy is "too loose" (in expectation) for the first several periods following the increase in the natural rate of interest. This is what generates the positive output gap and inflation responses.

# 3 Applications: Productivity and Government Spending Shocks

Instead of thinking about the consequences of a natural rate shock in the abstract, let's consider two possible sources of fluctuations in the natural rate – a positive shock to productivity and a positive shock to government spending. The former is pretty easy to analyze – it maps nicely into the basic three-equation model described above. In the case of the government spending shock, we can still write the three equation model in terms of $x_t$, $\pi_t$, and $i_t$, but we have to do some work to get there (and some of the coefficients look different) because we will no longer have $C_t = Y_t$ in equilibrium.

## 3.1 Productivity Shock

The key (non-linearized) equations of the model for understanding how this shock impacts the natural rate of interest are the labor supply condition, the consumption Euler equation, the relationship between the real wage and real marginal cost, the resource constraint, and the aggregate production function:

$$\theta N_t^{\chi} = C_t^{-\sigma} w_t$$
$$mc_t = \frac{w_t}{A_t}$$
$$Y_t = \frac{A_t N_t}{v_t^P}$$
$$Y_t = C_t$$
$$1 = \mathbb{E}_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} (1+i_t)\Pi_{t+1}^{-1} \right]$$

The price-setting conditions are all separated from this, and can be written in linearized form in terms of real marginal cost as:

$$\pi_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \widetilde{mc}_t + \beta \mathbb{E}_t \pi_{t+1}$$

Focusing on the non-linearized conditions, we can eliminate $w_t$ and $C_t$:

$$\theta N_t^{\chi} = Y_t^{-\sigma} mc_t A_t$$
$$Y_t = \frac{A_t N_t}{v_t^P}$$
$$1 = \mathbb{E}_t \left[ \beta \left( \frac{Y_{t+1}}{Y_t} \right)^{-\sigma} (1+i_t)\Pi_{t+1}^{-1} \right]$$

The linearized versions of these are (linearized about a zero-inflation steady state, so price dispersion is constant, with lowercase variables denoting log deviations where appropriate):

$$\widetilde{mc}_t = \chi n_t + \sigma y_t - a_t$$
$$y_t = a_t + n_t$$
$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1})$$

The flexible price equilibrium, denoted with a superscript $f$, is a situation in which real marginal cost is constant, so $\widetilde{mc}_t^f = 0$. Combining these together, we have:

$$a_t = \chi \left( y_t^f - a_t \right) + \sigma y_t^f$$

Hence:

$$y_t^f = \frac{1+\chi}{\sigma+\chi} a_t \tag{9}$$

The natural rate of interest is defined as the real interest rate that would be consistent with the IS equation with flexible prices, so:

$$r_t^f = \sigma \left[ \mathbb{E}_t y_{t+1}^f - y_t^f \right]$$

Combining the two, we have:

$$r_t^f = \frac{\sigma(1+\chi)}{\sigma+\chi} \left[ \mathbb{E}_t a_{t+1} - a_t \right]$$

Assume that productivity follows an AR(1), with autoregressive coefficient $\rho_A$. Then we have:

$$r_t^f = \frac{\sigma(1+\chi)(\rho_A - 1)}{\sigma+\chi} a_t$$

Now, since $a_t = \rho_A a_{t-1} + s_A \varepsilon_{A,t}$, we can write this as:

$$r_t^f = \frac{\sigma(1+\chi)(\rho_A - 1)}{\sigma+\chi} \rho_A a_{t-1} + \frac{\sigma(1+\chi)(\rho_A - 1)}{\sigma+\chi} s_A \varepsilon_{A,t}$$

But since $\frac{\sigma(1+\chi)(\rho_A-1)}{\sigma+\chi} a_{t-1} = r_{t-1}^f$, we have:

$$r_t^f = \rho_A r_{t-1}^f + \frac{\sigma(1+\chi)(\rho_A - 1)}{\sigma+\chi} s_A \varepsilon_{A,t} \tag{10}$$

Provided $\rho_A < 1$, what this tells us is that a positive shock to $\rho_A$ (say $s_A \varepsilon_{A,t} = 1$) causes the natural rate of interest to *decline*. Note that we can write the log-deviation of real marginal cost as:

$$\widetilde{mc}_t = \chi(y_t - a_t) + \sigma y_t - a_t = (\sigma+\chi) y_t - (1+\chi) a_t$$

But, from above, we can write $(1+\chi)a_t = (\sigma+\chi)y_t^f$, so we have:

$$\widetilde{mc}_t = (\sigma+\chi)\left(y_t - y_t^f\right) = (\sigma+\chi) x_t$$

This means we can write the Phillips Curve as:

$$\pi_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}(\sigma+\chi) x_t + \beta \mathbb{E}_t \pi_{t+1} = \gamma x_t + \beta \mathbb{E}_t \pi_{t+1} \tag{11}$$

Then, in the IS equation, add and subtract $y_t^f$ and $\mathbb{E}_t y_{t+1}^f$ from both sides:

$$y_t - y_t^f = -y_t^f + \mathbb{E}_t y_{t+1} - \mathbb{E}_t y_{t+1}^f + \mathbb{E}_t y_{t+1}^f - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

But this can be written:

$$x_t = \mathbb{E}_t x_{t+1} + \mathbb{E}_t y_{t+1}^f - y_t^f - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

But since $\mathbb{E}_t y_{t+1}^f - y_t^f = \frac{1}{\sigma} r_t^f$, we can write this as:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f\right) \tag{12}$$

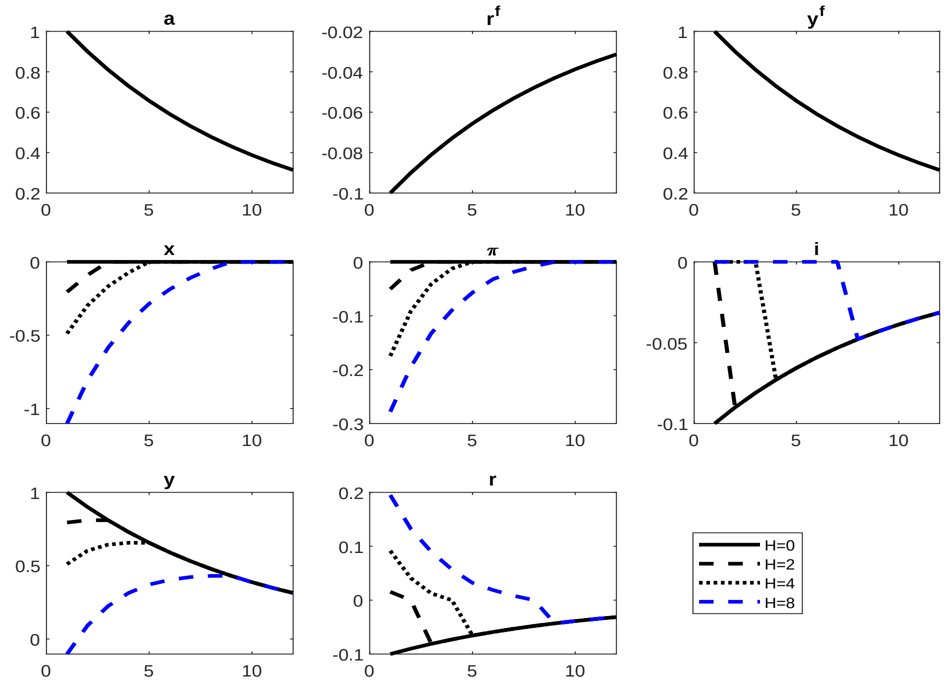
Note that (12) is identical to (1), and (11) is identical to (2). We can use (10) as an exogenous process for the natural rate of interest, though that is really being driven by the productivity shock. Then we can use (9) to back out $y_t^f$, but then once we know $x_t$ we can determine $y_t$ (and any of the other variables we subbed out).

#### 3.1.1 Impulse Responses

I'm going to take this model, summarized by (12) and (11), and solve it for different deterministic interest rate peg durations: $H = 0$, $H = 2$, $H = 4$, and $H = 8$. I could also do this using a stochastic duration, but that is subject to the sign-flip pathology for a sufficiently long peg. Outside of the ZLB, I assume that the central bank targets inflation: $\pi_t = 0$, which implies $x_t = 0$ and $i_t = r_t^f$. But during the peg, $i_t = 0$ (i.e. is constant).

I need to specify values of parameters. I assume that $\beta = 0.99$, $\chi = 1$, $\sigma = 1$, $\rho_A = 0.9$, and $\phi = 0.75$. This implies that the value of $\gamma = 0.17$. I assume that there is a one-unit shock to productivity. In terms of the productivity shock, the process for $r_t^f$ is given by (10). The model, as written, determines $x_t$. I can back out $y_t = x_t + y_t^f$, where $y_t^f = \frac{1+\chi}{\sigma+\chi} a_t$. Given that I assumed $\sigma = 1$, we just have $y_t^f = a_t$. I can back out the real interest rate from the Fisher relationship: $r_t = i_t - \mathbb{E}_t \pi_{t+1}$. Impulse responses are shown below.

Figure 3: IRFs to Productivity Shock, Deterministic Peg of Duration $H$




When $H = 0$, so there is no constraint on monetary policy, we have $x_t = \pi_t = 0$ at all horizons, which implies $i_t = r_t = r_t^f$. A positive productivity shock causes $r_t^f$ to fall.

What happens when the interest rate is pegged for some period of time? Thanks to the lack of endogenous state variables, once the peg lifts, inflation and the output gap are zero, and the nominal rate equals the real rate which equals the natural rate.

But not during the peg. During the peg, the productivity shock (negative shock to $r_t^f$) causes inflation to fall. With the nominal rate pegged, lower expected inflation causes the real interest rate to *rise*. The higher real interest rate causes the output gap to fall, which means that output, $y_t$, goes up by less than potential, $y_t^f$. *The longer the peg, the more exacerbated this gets.* As $H$ gets bigger, inflation falls more, which causes the real rate to rise more, which causes output, $y_t$, to rise less. In fact, when $H = 8$, for example, $y_t$ actually falls on impact (even though $y_t^f$ has risen).

Without a peg/ZLB, the central bank would like to cut the nominal rate when productivity improves. The inability to do so means that monetary policy is "too tight," which causes deflation ($\pi_t$ declining) and output rising by too little.

## 3.2 Government Spending Shock

The key (non-linearized) equations of the model for understanding how a government spending impacts the natural rate of interest are the labor supply condition, the consumption Euler equation, the relationship between the real wage and real marginal cost, the resource constraint, and the aggregate production function:

$$\theta N_t^\chi = C_t^{-\sigma} w_t$$
$$mc_t = \frac{w_t}{A_t}$$
$$Y_t = \frac{A_t N_t}{v_t^P}$$
$$Y_t = C_t + G_t$$
$$1 = \mathbb{E}_t \left[ \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\sigma} (1+i_t)\Pi_{t+1}^{-1} \right]$$
$$\ln G_t = (1-\rho_G)\ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t}$$

The price-setting conditions are all separated from this, and can be written in linearized form in terms of real marginal cost as:

$$\pi_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widetilde{mc}_t + \beta \mathbb{E}_t \pi_{t+1}$$

The linearized production function is:

$$y_t = a_t + n_t$$

Let $\omega = G/Y$ as the steady state government spending share of output. The linearized resource constraint is:

$$y_t = (1-\omega)c_t + \omega g_t$$

Linearized real marginal cost is the same as before:

$$\widetilde{mc}_t = \widetilde{w}_t - a_t$$

Combining this with the linearized labor supply condition, we have:

$$\chi n_t = -\sigma c_t + \widetilde{mc}_t + a_t$$

Now eliminate $n_t$ and $c_t$ from the production function and resource constraint:

$$\chi(y_t - a_t) = -\sigma \left( \frac{y_t}{1-\omega} - \frac{\omega}{1-\omega} g_t \right) + \widetilde{mc}_t + a_t$$

Solve for $\widetilde{mc}_t$ and group like terms:

$$\widetilde{mc}_t = \left( \chi + \frac{\sigma}{1-\omega} \right) y_t - \frac{\sigma\omega}{1-\omega} g_t - (1+\chi)a_t$$

Which may be written:

$$\widetilde{mc}_t = \left(\frac{\chi(1-\omega)+\sigma}{1-\omega}\right) y_t - \frac{\sigma\omega}{1-\omega} g_t - (1+\chi) a_t$$

Now, to think about the flexible price equilibrium, note that $\widetilde{mc}_t^f = 0$. This means that:

$$\left(\frac{\chi(1-\omega)+\sigma}{1-\omega}\right) y_t^f = \frac{\sigma\omega}{1-\omega} g_t + (1+\chi) a_t$$

But this means:

$$\widetilde{mc}_t = \left(\frac{\chi(1-\omega)+\sigma}{1-\omega}\right) x_t$$

Where $x_t = y_t - y_t^f$. But this means we can write the NKPC as:

$$\pi_t = \frac{(1-\phi)(1-\phi\beta)}{\phi} \left(\frac{\chi(1-\omega)+\sigma}{1-\omega}\right) x_t + \beta \mathbb{E}_t \pi_{t+1} \tag{13}$$

This is the same form of the expression we had earlier, but the coefficient $\gamma = \frac{(1-\phi)(1-\phi\beta)}{\phi}\left(\frac{\chi(1-\omega)+\sigma}{1-\omega}\right)$ on the output gap is slightly different due to $\omega > 0$ (which means $c_t \neq y_t$).

Now, let's go to the Euler equation. Linearized, we have:

$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

Now plug in the linearized resource constraint:

$$\frac{1}{1-\omega} y_t - \frac{\omega}{1-\omega} g_t = \frac{1}{1-\omega} \mathbb{E}_t y_{t+1} - \frac{\omega}{1-\omega} \mathbb{E}_t g_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

This can be written:

$$y_t = \mathbb{E}_t y_{t+1} - \omega\left(\mathbb{E}_t g_{t+1} - g_t\right) - \frac{1-\omega}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

Now, add and subtract $y_t^f$ and $\mathbb{E}_t y_{t+1}^f$ from both sides:

$$y_t - y_t^f + y_t^f = \mathbb{E}_t y_{t+1} - \mathbb{E}_t y_{t+1}^f + \mathbb{E}_t y_{t+1}^f - \omega\left(\mathbb{E}_t g_{t+1} - g_t\right) - \frac{1-\omega}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

This may be written in terms of the gap as:

$$x_t = \mathbb{E}_t x_{t+1} + \mathbb{E}_t y_{t+1}^f - y_t^f - \omega\left(\mathbb{E}_t g_{t+1} - g_t\right) - \frac{1-\omega}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right)$$

Now, let's think about what $r_t^f$. It is the real interest rate where the IS equation holds with flexible prices. In particular:

$$y_t^f = \mathbb{E}_t y_{t+1}^f - \omega\left(\mathbb{E}_t g_{t+1} - g_t\right) - \frac{1-\omega}{\sigma} r_t^f$$

Therefore:

$$\frac{1-\omega}{\sigma} r_t^f = \mathbb{E}_t y_{t+1}^f - y_t^f - \omega\left(\mathbb{E}_t g_{t+1} - g_t\right)$$

But this means we can write the IS equation as:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1-\omega}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f\right) \tag{14}$$

Note that (13) and (14) are *exactly* the same form as in the basic three-equation model without government spending; it's just that the slope coefficient in the NKPC is slightly different (it now depends on $\omega$) and the response of the output gap to the interest rate gap is slightly different (it also depends on $\omega$).

Now, to complete the three-equation representation of the model, we need to map $g_t$ into $r_t^f$. From above, we have:

$$r_t^f = \frac{\sigma}{1-\omega}\left(\mathbb{E}_t y_{t+1}^f - y_t^f\right) - \frac{\sigma\omega}{1-\omega}\left(\mathbb{E}_t g_{t+1} - g_t\right)$$

Given that we are assuming $g_t$ obeys an AR(1) process, this can be written:

$$r_t^f = \frac{\sigma}{1-\omega}\left(\mathbb{E}_t y_{t+1}^f - y_t^f\right) + \frac{\sigma\omega(1-\rho_G)}{1-\omega} g_t$$

From above, we know that:

$$y_t^f = \frac{\sigma\omega}{\chi(1-\omega)+\sigma} g_t + \frac{(1+\chi)(1-\omega)}{\chi(1-\omega)+\sigma} a_t$$

Given that both $g_t$ and $a_t$ follow AR(1) processes, we know:

$$\mathbb{E}_t y_{t+1}^f = \frac{\sigma\omega\rho_G}{\chi(1-\omega)+\sigma} g_t + \frac{(1+\chi)(1-\omega)\rho_A}{\chi(1-\omega)+\sigma} a_t$$

Hence:

$$\mathbb{E}_t y_{t+1}^f - y_t^f = \frac{\sigma\omega(\rho_G-1)}{\chi(1-\omega)+\sigma} g_t + \frac{(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma} a_t$$

We can therefore write:

$$r_t^f = \frac{\sigma}{1-\omega}\frac{\sigma\omega(\rho_G-1)}{\chi(1-\omega)+\sigma} g_t + \frac{\sigma}{1-\omega}\frac{(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma} a_t + \frac{\sigma\omega(1-\rho_G)}{1-\omega} g_t$$

This can be written:

$$r_t^f = \frac{\sigma\omega(1-\rho_G)}{1-\omega}\left(1 - \frac{\sigma}{\chi(1-\omega)+\sigma}\right) g_t + \frac{\sigma}{1-\omega}\frac{(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma} a_t$$

Which can be simplified further:

$$r_t^f = \frac{\sigma\omega(1-\rho_G)}{1-\omega}\left(\frac{\chi(1-\omega)}{\chi(1-\omega)+\sigma}\right)g_t + \frac{\sigma}{1-\omega}\frac{(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma}a_t$$

$$r_t^f = \frac{\sigma\omega\chi(1-\rho_G)}{\chi(1-\omega)+\sigma}g_t + \frac{\sigma(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma}a_t$$

It is helpful to define:

$$r_{G,t}^f = \frac{\sigma\omega\chi(1-\rho_G)}{\chi(1-\omega)+\sigma}g_t$$
$$r_{A,t}^f = \frac{\sigma(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma}a_t$$

So:

$$r_t^f = r_{G,t}^f + r_{A,t}^f \tag{15}$$

With:

$$r_{G,t}^f = \rho_G r_{G,t-1}^f + \frac{\sigma\omega\chi(1-\rho_G)}{\chi(1-\omega)+\sigma}s_G\varepsilon_{G,t} \tag{16}$$

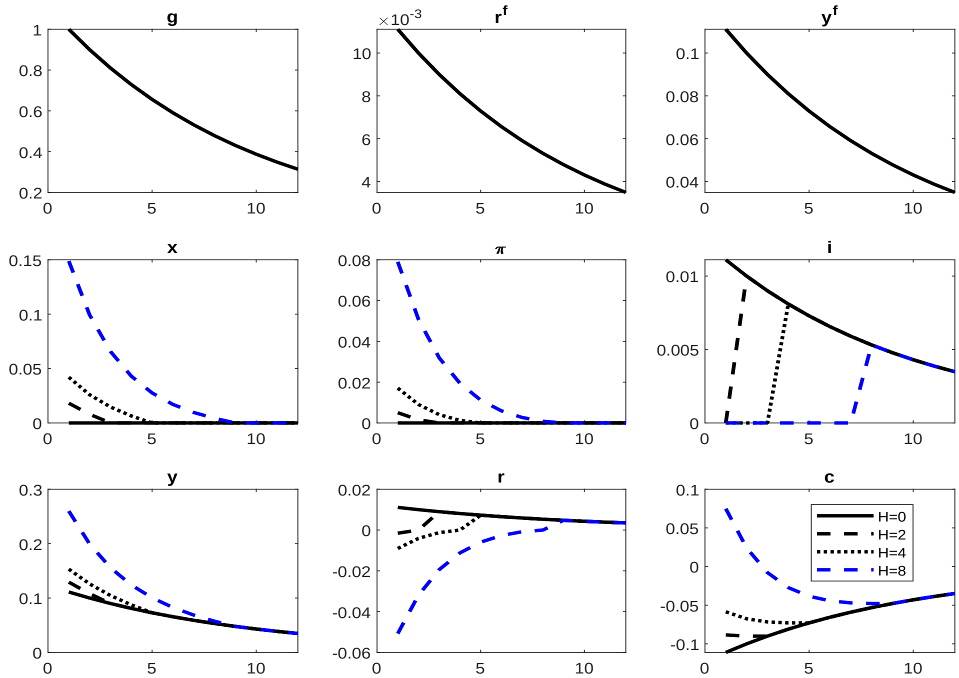
$$r_{A,t}^f = \rho_A r_{A,t-1}^f + \frac{\sigma(1+\chi)(1-\omega)(\rho_A-1)}{\chi(1-\omega)+\sigma}s_A\varepsilon_{A,t} \tag{17}$$

We see here that positive shocks to government spending raise the natural rate of interest (as long as $\rho_G < 1$), and positive shocks to productivity lower the natural rate of interest (as long as $\rho_A < 1$). Note that (17) is identical to (10) when $\omega = 0$.

### 3.2.1 Impulse Responses

I'm going to focus on impulse responses to a government spending shock for different peg values, just as I did before. I again need to parameterize the model, but there are no more parameters to specify. The model is characterized by an exogenous process for the natural rate, (16) – given linearity, I do not need to worry about the productivity shock. I assume that $\rho_G = 0.9$ and consider a one-unit shock to government spending. I assume the same parameter values as above: $\beta = 0.99$, $\phi = 0.75$, and $\chi = \sigma = 1$. I need to take a stand on the steady-state share of government spending in output, $\omega$. I assume this is $\omega = 0.20$. Note that, due to the presence of $\omega \neq 0$, the slope on the Phillips Curve is actually slightly different than in the productivity shock case ($\gamma = 0.19$).

Figure 4: IRFs to Government Spending Shock, Deterministic Peg of Duration $H$




The figure above shows IRFs. $y_t^f$ and $r_t^f$ are written as functions of exogenous government spending. The model, as specified, finds solutions for $x_t$ and $\pi_t$. I can back out $y_t = x_t + y_t^f$, and then also $c_t = \frac{y_t}{1-\omega} - \frac{\omega}{1-\omega} g_t$. The real interest rate is again backed out using the Fisher relationship.

When $H = 0$, the central bank is implementing a strict inflation target. This ensures that $x_t = \pi_t = 0$ at all horizons, so $i_t = r_t^f$. The government spending shock raises $r_t^f$. When $H = 0$, $y_t$ increases because $y_t^f$ increases, but $c_t$ declines.

When $H > 0$, we see something similar as in the case of the productivity shock, but sort of in reverse given that the government spending shock moves the natural rate of interest in the opposite direction. The central bank would like to raise the nominal rate to implement the strict inflation target. But, because of the peg/ZLB, it cannot. This makes monetary policy "too loose" following the shock. Inflation raises, which causes the real interest rate to decline (rather than rise). This stimulates consumption (it falls less than it would when $H = 0$) and output rises more. These effects are, again, exacerbated the longer is the peg – the longer the ZLB lasts, the more inflation rises, the more the real interest rate declines, and therefore the more output goes up. Indeed, for $H = 8$, for example, consumption actually rises (rather than declines).

We can calculate the fiscal multiplier from these impulse responses. Recall the fiscal multiplier is defined (on impact) as $dY_t/dG_t$ (or the sum of this ratio over some horizon). The impulse responses as shown are in logs – $d\ln Y_t/d\ln G_t = dy_t/dg_t$. The ratio of impulse responses in the logs has the interpretation of an elasticity. To put it in "levels," we can post-multiply by the ratio of output to government spending in steady state, which is $1/\omega$.

| | ZLB Length | | | |
|---|---|---|---|---|
| | 0 | 2 | 4 | 8 |
| Impact Multiplier | 0.56 | 0.65 | 0.77 | 1.3 |
| Sum Multiplier (20 periods) | 0.56 | 0.57 | 0.62 | 0.85 |

The table above calculates the multiplier (either the impact or sum multiplier) for different peg lengths. Under the inflation target with no peg, the impact and cumulative sum multipliers are the same at 0.56. Consumption is crowded out, so output rises by less than government spending does. When the peg gets longer, output reacts more and more. For a long enough duration, the multiplier can exceed one (so consumption is crowded in, as shown in the impulse response graphs).

### 3.3 The Costs of the ZLB

As we saw in our discussion of optimal monetary policy, a central bank is interested in *stabilizing* fluctuations in both the output gap and inflation. The ZLB takes away the central bank's principal policy instrument (the short-term nominal interest rate) and makes it harder to do this. In fact, as we see in the exercises above, when the nominal interest rate cannot react for a while, both inflation and the output gap react more to exogenous shocks than when the central bank is unconstrained. This is bad.

In reality, what causes the ZLB to bind (i.e. for the interest rate to be pegged) is the fact that nominal interest rates cannot go (much) below zero. This fact has led some to call for a higher average inflation target. The idea is simple – the steady state nominal interest rate is increasing one-to-one in the steady state inflation rate. A higher steady state inflation rate gives a central bank "more wiggle room" before hitting zero. But raising the inflation target is not a free lunch – it moves the economy further away from the Friedman rule, and creates first-order price dispersion effects that might be undesirable.

Policymakers have thought about other ways to combat the problem of the ZLB rather than just raising the inflation target. One we studies above – use fiscal policy, rather than monetary policy, at the ZLB. As we showed above, fiscal expansion might be particularly effective at stimulating output at the ZLB. Another option is to use different kinds of tools. Previously "unconventional" monetary policy tools include *forward guidance* and *quantitative easing* (QE). Forward guidance essentially involves promising *lower* interest rates *after* the ZLB has lifted. It is not hard to see why forward guidance might be effective in this environment. Solving the IS equation forward, we have:

$$x_t = -\frac{1}{\sigma}\mathbb{E}_t \sum_{s=0}^{\infty} \left( r_{t+s} - r^f_{t+s} \right)$$

In the exercises we did above, we have $r_{t+k} = r^f_{t+k}$ once the ZLB lifts. The reason the ZLB is costly is that monetary policy is either "too tight" (real interest rate too high relative to the natural rate during the ZLB, so $x_t < 0$) or "too loose" (real rate too low relative to the natural rate during the ZLB, so $x_t > 0$). The most plausible reason the ZLB would bind is that $r^f_t$ is low (meaning the central bank would like to push $i_t < 0$). We will end up with policy being "too tight." In principle, this could be counteracted by promising to set $r_{t+k} < r^f_{t+k}$ for some period of time *after the ZLB has lifted*. This "eases" policy in the future, to counteract the tight policy in the present due to the ZLB. This is what forward guidance entails – promising "loose" policy *beyond* the ZLB period to attempt to counteract the adverse effects of the ZLB on the output gap and inflation. In principle, forward guidance can work extremely well in the basic NK model. The output gap only depends on the sum of real interest rate gaps. A policymaker can counteract the tight policy due to the ZLB by promising loose policy in the future. Of course, this requires commitment, and is subject to the time inconsistency problem. It's also the case that most people feel like forward guidance is *too* effective in this model – the so-called "forward guidance puzzle."[1]

As written, it is not possible to use the New Keynesian model to think about quantitative easing. QE involves purchasing *long-term* bonds with the attempt to push up their price (and push down long-term interest rates). To think sensibly about QE, we need a model with short- and long-term bonds, and we need some kind of friction to allow QE to influence long-term interest rates independent of the path of short-term interest rates. I have a paper that does just this along with Cynthia Wu.[2]

### 3.4 Alternative Solution Methodologies

What I have done to account for the ZLB is to use two (relatively) simple assumptions about the nominal interest rate being pegged in a linearized model. This is useful for intuition, but it has some obvious drawbacks. In either case, because of linearity, we are ruling out "precautionary" behavior wherein people change their actions in anticipation of the economy possibly hitting the ZLB. We are also ruling out concerns about the ZLB ever binding again in the future once it lifts.

A slightly more sophisticated approach (which does not deal with either of these issues) is to the use the "occbin" package (Guerrieri and Iacoviello (2015), "OccBin: A Toolkit for Solving Dynamic Models with Occasionally Binding Constraints Easily" *Journal of Monetary Economics* 70: 22-38). This procedures is compatible with Dynare, and gives a piecewise linear solution. This is easier to work with and more adaptable than what I did above, but it's not fundamentally different – what I

[1] See Del Negro, Giannoni, and Patterson (2023). "The Forward Guidance Puzzle." *Journal of Political Economy: Macroeconomics* 1(1): 43-79.

[2] See Sims, Wu, and Zhang (2023): "The Four-Equation New Keynesian Model." *Review of Economics and Statistics* 105(4): 931-947.

did above (in the stochastic peg case) is also a piecewise linear solution. To capture precautionary effects and the like, one needs to use a non-linear solution methodology to solve the model (which is feasible for a small-scale model but numerically challenging when and if the model gets more complicated).