Graduate Macro Theory II:
A New Keynesian Model with Wage and Price Stickiness

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1 Introduction

This set of notes lays out a New Keynesian model where both prices *and* nominal wages are sticky. The production side of the model is identical to what we encountered earlier. There is a competitive final good firm that combines differentiated intermediate outputs into a final output using a CES technology. This generates a downward-sloping demand curve for each intermediate variety, which gives producers price-setting power. Prices are sticky as in Calvo (1983).

The difference is on the household side of the model. There are a continuum of ex-ante identical households who supply differentiated labor. We assume there exists a competitive "labor packer" that plays an analogous role to the final good firm. The labor packer combines household labor input into a final labor input that is sold to firms. This gives market power to households who supply labor. We introduce a Calvo (1983) assumption on wage-setting on households. This gives rise to a wage Phillips Curve that looks a lot like the price Phillips Curve.

These notes will skip over details of the production side of the model and focus on what is new.

2 Production

The production side of the model is *identical* to the model with just price stickiness. There is a final good firm that combines intermediate outputs into a final output using a CES aggregate with elasticity of substitution ϵ_p (I'm going to index this elasticity by p to differentiate it from something similar for wages). Intermediate firms produce output according to a linear production technology in labor and productivity (the latter of which is taken as given). They hire labor at a common wage from a "labor packer" (more on this below). Intermediate firms are subject to Calvo (1983) price rigidity – each period, only a fraction $1 - \phi_p$ of firms can adjust their price (again, I'm indexing by p to differentiate this from a Calvo parameter for wages to come).

The optimality conditions, written in terms of the gross inflation rate, Π_t, and relative reset price, $p_t^{\#} = P_t^{\#}/P_t$, are given below:

$$p_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1}\frac{\widehat{X}_{1,t}}{\widehat{X}_{2,t}} \tag{1}$$

$$\widehat{X}_{1,t} = mc_t Y_t + \phi_p \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_p} \widehat{X}_{1,t+1} \tag{2}$$

$$\widehat{X}_{2,t} = Y_t + \phi_p \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_p - 1} \widehat{X}_{2,t+1} \tag{3}$$

$$mc_t = \frac{w_t}{A_t} \tag{4}$$

If prices were flexible (i.e. $\phi_p = 0$), then intermediate firms would all be identical and would set their price equal to a fixed markup over marginal cost, with the markup being given by $\frac{\epsilon_p}{\epsilon_p - 1}$.

3 Labor Packer

Before we get to the households, I wish to discuss the "labor packer," which plays a role analogous to the final good firm on the production side. Households are going to supply differentiated labor that is imperfectly substitutable across households. This gives them power in wage-setting. They will be subject to a Calvo-style friction.

The labor packer combines differentiated household labor into a final labor input that is then sold to intermediate firms. Let households be indexed by $h \in [0, 1]$. They each supply $N_t(h)$ units of labor at nominal wage $W_t(h)$. The labor packer combines the differentiated labor into a final labor input, N_t, that is then sold to intermediate firms at nominal wage W_t. The function mapping differentiated labor into final labor is CES:

$$N_t = \left(\int_0^1 N_t(h)^{\frac{\epsilon_w - 1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{5}$$

The problem of the labor packer is to pick labor of each variety, h, to maximize profit. The problem is:

$$\max_{N_t(h)} \quad P_t D_t^L = W_t N_t - \int_0^1 W_t(h) N_t(h) dh$$

This may be written:

$$\max_{N_t(h)} \quad W_t \left(\int_0^1 N_t(h)^{\frac{\epsilon_w - 1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_t(h) N_t(h) dh$$

The FOC is:

$$\frac{\epsilon_w}{\epsilon_w - 1} W_t \left(\int_0^1 N_t(h)^{\frac{\epsilon_w - 1}{\epsilon_w}} dh \right)^{\frac{\epsilon_w}{\epsilon_w - 1} - 1} \frac{\epsilon_w - 1}{\epsilon_w} N_t(h)^{\frac{\epsilon_w - 1}{\epsilon_w} - 1} = W_t(h)$$

This may be written:

$$\left(\int_0^1 N_t(h)^{\frac{\epsilon_w-1}{\epsilon_w}} dh\right)^{\frac{1}{\epsilon_w-1}} N_t(h)^{-\frac{1}{\epsilon_w}} = \frac{W_t(h)}{W_t}$$

Which can be written:

$$\left(\int_0^1 N_t(h)^{\frac{\epsilon_w-1}{\epsilon_w}} dh\right)^{-\frac{\epsilon_w}{\epsilon_w-1}} N_t(h) = \left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w}$$

Or, finally:

$$N_t(h) = \left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w} N_t \tag{6}$$

(6) is exactly analogous to the downward-sloping demand curve for each product variety. $\epsilon_w > 1$ is the elasticity of substitution. The aggregate wage index, W_t, is then implicitly defined by:

$$W_t N_t = \int_0^1 W_t(h) N_t(h) dh$$

Plugging in (6), we have:

$$W_t N_t = \int_0^1 W_t(h) \left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w} N_t$$

Which is:

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(h)^{1-\epsilon_w} dh$$

Or:

$$W_t = \left(\int_0^1 W_t(h)^{1-\epsilon_w} dh\right)^{\frac{1}{1-\epsilon_w}} \tag{7}$$

(7) is again analogous to the expression for the aggregate price index on the production side.

4 Households

There are a continuum of households, index by $h \in [0, 1]$. These households supply differentiated labor input, $N_t(h)$, to the labor packer at nominal wage $W_t(h)$. We abstract from money. Households have identical preferences. Flow utility for a household is given by:

$$u(C_t(h), N_t(h)) = \frac{C_t(h)^{1-\sigma}}{1-\sigma} - \theta \frac{N_t(h)^{1+\chi}}{1+\chi}$$

Households will choose $N_t(h)$ subject to the demand curve for labor described above, (6). As we will introduce below, households are subject to a Calvo-style wage rigidity. This presents a complication, because households won't earn the same income if they don't have the same wages.

But this will then spill over into heterogeneity in consumption, bond holdings, and the like. This becomes problematic. We will follow Erceg, Henderson, and Levin (2000) in supposing that, in the background, there are state-contingent securities that act as income insurance arising from wage rigidity. As long as preferences are additively separable between consumption and labor (which I have assumed), this will mean that households will be identical along all margins except the choice of labor and wage. Therefore, I shall henceforth abstract from h subscripts for all household-level variables with the exception of labor input and wages.

Taking this into account, the a household's nominal flow budget constraint is:

$$P_t C_t + B_t - B_{t-1} \leq W_t(h) N_t(h) + P_t D_t - P_t T_t + i_{t-1} B_{t-1}$$

The household chooses $N_t(h)$ (equivalently $W_t(h)$) to maximize the present discounted value of flow utility, subject to the budget constraint and the demand curve for labor, (6). I just plug in the demand curve and think about the household as choosing $W_t(h)$, but I will focus first on non-labor choices.

A Lagrangian for non-labor choices is:

$$\mathbb{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \theta \frac{\left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w(1+\chi)} N_t^{1+\chi}}{1+\chi} + \right.$$
$$\left. \lambda_t \left(\frac{W_t(h)^{1-\epsilon_w} W_t^{\epsilon_w} N_t}{P_t} + D_t + T_t + (1+i_{t-1}) \frac{B_{t-1}}{P_t} - C_t - \frac{B_t}{P_t} \right) \right]$$

Note that the household takes N_t (aggregate labor input as given), as well as W_t (the aggregate wage). The household gets to choose C_t and B_t, and, subject to a Calvo-restriction, $W_t(h)$. The FOC with respect to C_t and B_t are:

$$\frac{\partial \mathbb{L}}{\partial C_t} = 0 \Leftrightarrow C_t^{1-\sigma} = \lambda_t$$
$$\frac{\partial \mathbb{L}}{\partial B_t} = 0 \Leftrightarrow \frac{\lambda_t}{P_t} = \beta \mathbb{E}_t \lambda_{t+1} (1+i_t) P_{t+1}^{-1}$$

Define the stochastic discount factor as:

$$\Lambda_{t,t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \tag{8}$$

The Euler equation for bonds can then be written:

$$1 = \mathbb{E}_t \Lambda_{t,t+1} (1+i_t) \Pi_{t+1}^{-1} \tag{9}$$

Where $\Pi_t = P_t / P_{t-1}$, the gross inflation rate.

4.1 The Wage Decision

Households are subject to a Calvo updating probability. Each period, there is a probability, $1-\phi_w$, that a household can adjust its wage. Otherwise, it receives its most recently chosen wage with probability ϕ_w. This persists into the future, and the probability of updating is independent of when the last update occurred.

Consider a household that gets to choose its wage today. Call this $W_t^{\#}(h)$. The household will therefore have to supply $\left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w} N_t$ units of labor. With probability ϕ_w, the wage the household charges in $t+1$ will also be $W_t^{\#}(h)$, meaning it will have to supply $\left(\frac{W_t(h)}{W_{t+1}}\right)^{-\epsilon_w} N_{t+1}$ units of labor. And so on into the future: with probability ϕ_w^2 the wage chosen in t will still be in effect in $t+2$, etc.

We can therefore re-create just the part of the Lagrangian that is related to wage-setting (which, given our assumptions, is linear), noting that future values will be discounted by $(\beta\phi_w)^s$:

$$\mathbb{L} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left[-\theta \frac{(W_t(h))^{-\epsilon_w(1+\chi)} W_{t+s}^{\epsilon_w(1+\chi)} N_{t+s}^{1+\chi}}{1+\chi} + \lambda_{t+s} \left(\frac{W_t(h)^{1-\epsilon_w} W_{t+s}^{\epsilon_w} N_{t+s}}{P_{t+s}} \right) \right]$$

The first order condition is:

$$\frac{\partial \mathbb{L}}{\partial W_t(h)} = 0 \Leftrightarrow \epsilon_w W_t(h)^{-\epsilon_w(1+\chi)-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_W)^s W_{t+s}^{\epsilon_w(1+\chi)} \theta N_{t+s}^{1+\chi} = (\epsilon_w - 1) W_t(h)^{-\epsilon_w} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} \frac{W_{t+s}^{\epsilon_w} N_{t+s}}{P_{t+s}}$$

This may be re-written:

$$W_t(h)^{1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s W_{t+s}^{\epsilon_w(1+\chi)} \theta N_{t+s}^{1+\chi}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} \frac{W_{t+s}^{\epsilon_w} N_{t+s}}{P_{t+s}}}$$

Note that nothing on the right hand side depends on h. Concretely, this means that all updating households will choose the same wage, call it $W_t^{\#}$. We can therefore write this condition as:

$$\left(W_t^{\#}\right)^{1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{H_{1,t}}{H_{2,t}}$$

Where:

$$H_{1,t} = \theta N_t^{1+\chi} W_t^{\epsilon_w(1+\chi)} + \phi_w \beta \mathbb{E}_t H_{1,t+1}$$

$$H_{2,t} = \frac{\lambda_t W_t^{\epsilon_w} N_t}{P_t} + \phi_w \beta \mathbb{E}_t H_{2,t+1}$$

Now, a complication arises because P_t and nominal wages (both the aggregate wage and the rest wage) may be non-stationary. So, we need to re-write these conditions in terms of real wages. Accordingly, define:

$$h_{1,t} = H_{1,t}/P_t^{\epsilon_w(1+\chi)}$$

$$h_{2,t} = H_{2,t}/P_t^{\epsilon_w-1}$$

We therefore have:

$$h_{1,t} = \theta N_t^{1+\chi} w_t^{\epsilon_w(1+\chi)} + \phi_w \beta \mathbb{E}_t \frac{H_{1,t+1}}{P_t^{\epsilon_w(1+\chi)}}$$

Which may be written:

$$h_{1,t} = \theta N_t^{1+\chi} w_t^{\epsilon_w(1+\chi)} + \phi_w \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\chi)} h_{1,t+1} \tag{10}$$

And:

$$h_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \phi_w \beta \mathbb{E}_t \frac{H_{2,t+1}}{P_t^{\epsilon_w-1}}$$

Above, I substituted out $\lambda_t = C_t^{-\sigma}$. So:

$$h_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \phi_w \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_w-1} h_{2,t+1} \tag{11}$$

Going back to the optimal reset price condition, I therefore have:

$$\left(W_t^{\#}\right)^{1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}} \frac{P_t^{\epsilon_w(1+\chi)}}{P_t^{\epsilon_w-1}}$$

But on the right hand side we have $P_t^{1+\epsilon_w\chi}$. Then we have the real reset wage on the left hand side, with:

$$\left(w_t^{\#}\right)^{1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}} \tag{12}$$

To gain some intuition for this expression, it is useful to think about what would happen if $\phi_w = 0$ (so no wage stickiness). Since all updating households will choose the same wage, if everyone updates, then $w_t^{\#} = w_t$. This would give us:

$$w_t^{1+\epsilon_w\chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}}$$

With $\phi_w = 0$, the ratio $h_{1,t}/h_{2,t}$ can be written:

$$\frac{h_{1,t}}{h_{2,t}} = \frac{\theta N_t^{1+\chi} w_t^{\epsilon_w(1+\chi)}}{C_t^{-\sigma} N_t w_t^{\epsilon_w}} = \theta C_t^{\sigma} N_t^{\chi} w_t^{\epsilon_w\chi}$$

But then we have:

$$w_t = \frac{\epsilon_w}{\epsilon_w - 1}\theta C_t^{\sigma} N_t^{\chi}$$

Note that θN_t^{χ} is the (negative) marginal disutility of labor, since if everyone charges the same wage they all have the same labor supply, which is equal to aggregate labor supply, N_t. C_t^{σ} is the (inverse) marginal utility of consumption. Thus, $C_t^{\sigma}\theta N_t^{\chi}$ is the marginal rate of substitution between labor and consumption (i.e. $-u_N(C_t, N_t)/u_C(C_t, N_t)$). The overall optimality condition therefore says that, absent wage stickiness, the household wants to set the wage as a *markup*, $\frac{\epsilon_w}{\epsilon_w - 1}$, over the marginal rate of substitution. If $\epsilon_w \to \infty$, so that labor of different varieties are perfect substitutes and the optimal markup is therefore unity, we would have:

$$\theta N_t^{\chi} = C_t^{-\sigma} w_t$$

But this would be just be our "standard" intratemporal labor supply condition in a frictionless RBC model!

The more general condition is *exactly analogous* to the optimal price-setting condition under Calvo staggered contracts. The idea is that households want their wage to be a markup over the marginal rate of substitution. When given the chance to update a wage, they do so in a forward-looking way, incorporating the knowledge that they may not be able to adjust their wage for a while. Hence, the wage-setting problem is forward-looking in that households are, roughly speaking, trying to get the right wage markup on average.

5 Policy, Equilibrium, and Aggregation

We shall assume that the central bank sets policy according to a Taylor rule:

$$i_t = (1-\rho_i)i + \rho_i i_{t-1} + (1-\rho_i)\phi_\pi \left(\ln \Pi_t - \ln \Pi\right) + (1-\rho_i)\phi_\pi \left(\ln Y_t - \ln Y_t^f\right) s_i \varepsilon_{i,t} \tag{13}$$

I am ignoring money altogether; though it could be included in the households' problem in an additively separable way without changing anything. The nominal government budget constraint is:

$$P_t G_t + i_{t-1} B_{t-1}^G = P_t T_t + B_t^G - B_{t-1}^G$$

I will assume that government spending follows an AR(1) process:

$$\ln G_t = (1-\rho_G)\ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{14}$$

Ricardian Equivalence will hold, so I don't need to worry about the mix between taxes and government debt. The aggregate production function and associated price dispersion term are the same as we encountered earlier. These are:

$$Y_t v_t^p = A_t N_t \tag{15}$$

$$v_t^p = (1-\phi_p)\left(p_t^{\#}\right)^{-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p} v_{t-1}^p \tag{16}$$

The aggregate inflation rate evolves according to the same expression we had earlier:

$$1 = (1-\phi_p)\left(p_t^{\#}\right)^{1-\epsilon_p} + \phi_\pi \Pi_t^{\epsilon_p - 1} \tag{17}$$

The aggregate nominal wage index is:

$$W_t^{1-\epsilon_w} = \int_0^1 W_t(h)^{1-\epsilon_w} dh$$

As we did with price rigidity, we make use of the Calvo assumption: $1-\phi_w$ of households choose the same reset wage, $W_t^{\#}$, and ϕ_w charge whatever they last charged. Since the fraction ϕ_w are randomly selected, that part of the integral is just proportional to the lagged aggregate wage. So we have:

$$W_t^{1-\epsilon_w} = (1-\phi_w)\left(W_t^{\#}\right)^{1-\epsilon_w} + \phi_w W_{t-1}^{1-\epsilon_w}$$

We need to write this in real terms, so divide both sides by $P_t^{1-\epsilon_w}$:

$$w_t^{1-\epsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \left(\frac{W_{t-1}}{P_t}\right)^{1-\epsilon_w}$$

Which may be written:

$$w_t^{1-\epsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \left(\frac{W_{t-1}}{P_{t-1}}\frac{P_{t-1}}{P_t}\right)^{1-\epsilon_w}$$

Or, finally:

$$w_t^{1-\epsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{18}$$

What about the aggregate resource constraint? Integrate the budget constraint across households at equality, imposing the demand curve for each variety of labor:

$$P_t C_t + B_t - B_{t-1} = \int_0^1 W_t(h)^{1-\epsilon} W_t^{\epsilon} N_t dh + P_t D_t - P_t T_t + i_{t-1} B_{t-1}$$

This can be written:

$$P_t C_t + B_t - B_{t-1} = W_t^{\epsilon_w} N_t \int_0^1 W_t(h)^{1-\epsilon_w} dh + P_t D_t - P_t T_t + i_{t-1} B_{t-1}$$

Now, note that $W_t^{1-\epsilon_w} = \int_0^1 W_t(h)^{1-\epsilon_w} dh$. But then we have:

$$P_t C_t + B_t - B_{t-1} = W_t N_t + P_t D_t - P_t T_t + i_{t-1} B_{t-1}$$

Plug in the government budget constraint for $P_t T_t$:

$$P_t C_t + B_t - B_{t-1} = W_t N_t + P_t D_t - \left(P_t G_t + i_{t-1} B_{t-1}^G - B_t^G + B_{t-1}^G\right) + i_{t-1} B_{t-1}$$

Market clearing for bonds requires $B_t = B_t^G$ in all periods. The bond terms therefore cancel:

$$P_t C_t + P_t G_t = W_t N_t + P_t D_t$$

There are two components to aggregate nominal dividends: the payout from the representative final good firm and the sum of profits from the intermediate producers:

$$P_t D_t = P_t Y_t - \int_0^1 P_t(j) Y_t(j) dj + \int_0^1 P_t(j) Y_t(j) dj - \int_0^1 W_t N_t(j) dj$$

This can be written:

$$P_t D_t = P_t Y_t - W_t \int_0^1 N_t(j) dj$$

Labor market clearing requires $\int_0^1 N_t(j) dj = N_t$. We therefore are left with a standard (real) resource constraint:

$$Y_t = C_t + G_t \tag{19}$$

As usual, I assume that aggregate productivity follows an AR(1) process in the log:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{20}$$

5.1 Aside: Labor Supply and Demand

The variable N_t above is labor "produced" (or compiled) by the labor packer. This does not necessarily correspond to labor supply. Labor supply is simply the sum of labor provided by households:

$$N_t^s = \int_0^1 N_t(h) dh$$

Plugging in the demand curve, this is:

$$N_t^s = \int_0^1 \left(\frac{W_t(h)}{W_t}\right)^{-\epsilon_w} N_t dh$$

Or:

$$N_t^s = N_t \int_0^1 \left(\frac{W_t(h)}{W_t} \right)^{-\epsilon_w} dh$$

The term $\int_0^1 \left(\frac{W_t(h)}{W_t} \right)^{-\epsilon_w}$ is a measure of wage dispersion – it is exactly analogous to price dispersion. It is not relevant for equilibrium allocations in this setup. But it is relevant for welfare, which is why wage inflation matters in a micro-founded loss function (see the end of these notes).

6 Full Set of Equilibrium Conditions

We have 17 equations and 17 variables (if I don't count Y_t^f – more on this below). The variables are $\Lambda_{t,t+1}$, C_t, i_t, Π_t, w_t, $w_t^{\#}$, $h_{1,t}$, $h_{2,t}$, $p_t^{\#}$, $\widehat{X}_{1,t}$, $\widehat{X}_{2,t}$, mc_t, Y_t, A_t, N_t, v_t^p, and G_t.

- Household

$$1 = \mathbb{E}_t \Lambda_{t,t+1}(1+i_t)\Pi_{t+1}^{-1} \tag{21}$$

$$\Lambda_{t-1,t} = \beta \left(\frac{C_t}{C_{t-1}} \right)^{-\sigma} \tag{22}$$

- Wage-setting

$$\left(w_t^{\#} \right)^{1+\epsilon_w \chi} = \frac{\epsilon_w}{\epsilon_w - 1} \frac{h_{1,t}}{h_{2,t}} \tag{23}$$

$$h_{1,t} = \theta N_t^{1+\chi} w_t^{\epsilon_w(1+\chi)} + \phi_w \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\chi)} h_{1,t+1} \tag{24}$$

$$h_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \phi_w \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_w - 1} h_{2,t+1} \tag{25}$$

- Production

$$p_t^{\#} = \frac{\epsilon_p}{\epsilon_p - 1} \frac{\widehat{X}_{1,t}}{\widehat{X}_{2,t}} \tag{26}$$

$$\widehat{X}_{1,t} = mc_t Y_t + \phi_p \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_p} \widehat{X}_{1,t+1} \tag{27}$$

$$\widehat{X}_{2,t} = Y_t + \phi_p \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon_p - 1} \widehat{X}_{2,t+1} \tag{28}$$

$$mc_t = \frac{w_t}{A_t} \tag{29}$$

- Policy

$$i_t = (1-\rho_i) i + \rho_i i_{t-1} + (1-\rho_i)\phi_\pi (\ln \Pi_t - \ln \Pi) + (1-\rho_i)\phi_x \left(\ln Y_t - \ln Y_t^f \right) s_i \varepsilon_{i,t} \tag{30}$$

- Aggregate conditions

$$Y_t v_t^p = A_t N_t \tag{31}$$

$$v_t^p = (1-\phi_p) \left(p_t^{\#} \right)^{-\epsilon_p} + \phi_p \Pi_t^{\epsilon_p} v_{t-1}^p \tag{32}$$

$$1 = (1-\phi_p) \left(p_t^{\#} \right)^{1-\epsilon_p} + \phi_\pi \Pi_t^{\epsilon_p - 1} \tag{33}$$

$$Y_t = C_t + G_t \tag{34}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w} \tag{35}$$

- Exogenous processes:

$$\ln A_t = \rho_A \ln A_{t-1} + s_A \varepsilon_{A,t} \tag{36}$$

$$\ln G_t = (1-\rho_G)\ln G + \rho_G \ln G_{t-1} + s_G \varepsilon_{G,t} \tag{37}$$

What is Y_t^f? If I don't have the central bank reacting to the output gap, then I don't need to worry about this. But if I want a response to this in the Taylor rule, I need to define Y_t^f and derive an expression for it. We will define potential output as the level of output that would obtain if *both* prices and wages were flexible. If prices are flexible, then we know that:

$$w_t^f = \frac{\epsilon_p - 1}{\epsilon_p} A_t$$

From the wage-setting condition, if wages are flexible we know that:

$$w_t^f = \frac{\epsilon_w}{\epsilon_w - 1}\theta \left(C_t^f\right)^\sigma \left(N_t^f\right)^\chi$$

Equate these two:

$$\frac{\epsilon_w}{\epsilon_w - 1}\theta \left(C_t^f\right)^\sigma \left(N_t^f\right)^\chi = \frac{\epsilon_p - 1}{\epsilon_p} A_t$$

We know further that $C_t^f = Y_t^f - G_t$, and $N_t^f = Y_t^f / A_t$. Hence:

$$\frac{\epsilon_w}{\epsilon_w - 1}\theta \left(Y_t^f - G_t\right)^\sigma \left(\frac{Y_t^f}{A_t}\right)^\chi = \frac{\epsilon_p - 1}{\epsilon_p} A_t \tag{38}$$

This equation *implicitly* defines Y_t^f as a function of two exogenous variables – A_t and G_t. We can simply include it as an equilibrium condition above. Suppose we are in the special case in which $G_t = 0$. Then we can get a closed-form solution. We would have:

$$\frac{\epsilon_w}{\epsilon_w - 1}\theta \left(Y_t^f\right)^\sigma \left(\frac{Y_t^f}{A_t}\right)^\chi = \frac{\epsilon_p - 1}{\epsilon_p} A_t$$

Or:

$$\left(Y_t^f\right)^{\sigma+\chi} = \frac{1}{\theta}\frac{\epsilon_p - 1}{\epsilon_p}\frac{\epsilon_w - 1}{\epsilon_w} A_t^{1+\chi}$$

If we log-linearized this, we would have:

$$y_t^f = \frac{1+\chi}{\sigma+\chi} a_t$$

This expression is *exactly* the same as what we had in the baseline New Keynesian model without wage stickiness. But one can see a difference – there is an additional distortion, given by $\frac{\epsilon_w - 1}{\epsilon_w} < 1$, relative to the efficient allocation.

7 Steady State

To make life as simple as possible, I am only going to compute the steady state in a zero net inflation steady state (so $\Pi = 1$). This will have the implication that $p^{\#} = 1$, $v^p = 1$, and $w^{\#} = w = mc$. The steady state on the household side of the model is straightforward. We have:

$$\Lambda = \beta$$

For price-setting, since $\Pi = 1$ and $p^{\#} = 1$, we get:

$$mc = w = \frac{\epsilon_p - 1}{\epsilon_p} = w^{\#}$$

The steady state level of output is the expression it would take in the flexible price/wage allocation. Since we are assuming $G = \psi Y$, we have:

$$\frac{\epsilon_w}{\epsilon_w - 1}\theta(1-\psi)^{\sigma} Y^{\sigma+\chi} = \frac{\epsilon_p - 1}{\epsilon_p}$$

Which means:

$$Y = \left(\frac{\epsilon_p - 1}{\epsilon_p}\frac{\epsilon_w - 1}{\epsilon_w}\theta^{-1}(1-\psi)^{-\sigma}\right)^{\frac{1}{\sigma+\chi}} = N$$

Once we know Y (and hence N), we have C and G as well:

$$C = (1-\psi)Y$$
$$G = (1-\psi)Y$$

We know that:

$$h_1 = \frac{\theta N^{1+\chi} w^{\epsilon_w(1+\chi)}}{1 - \phi_w \beta}$$
$$h_2 = \frac{C^{-\sigma} w^{\epsilon_w} N}{1 - \phi_w \beta}$$

Similarly:

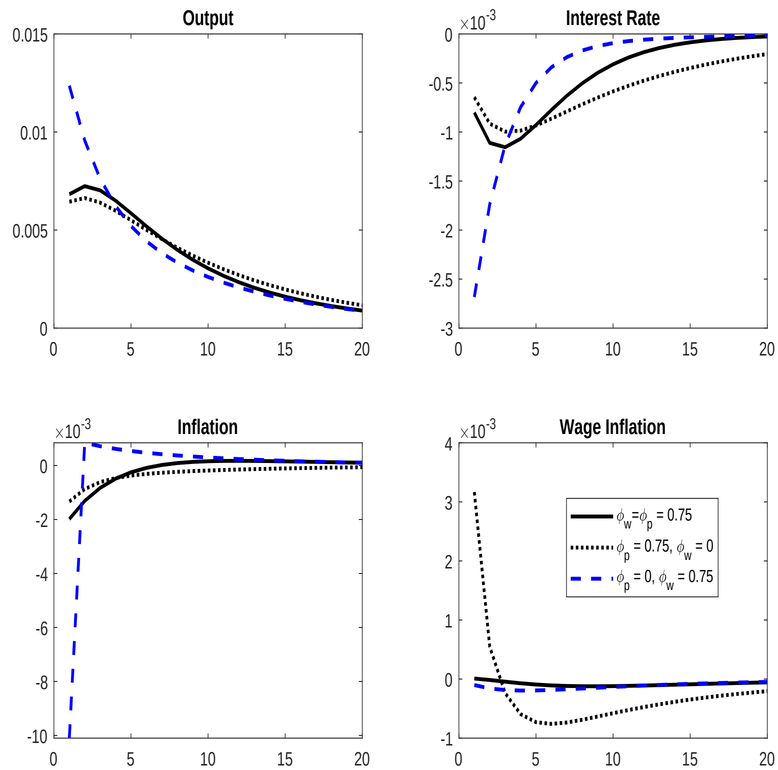
$$\widehat{X}_1 = \frac{mcY}{1 - \phi_p \beta}$$
$$\widehat{X}_2 = \frac{Y}{1 - \phi_p \beta}$$

8 Quantitative Analysis

To solve the model, I need to assume values of parameters. I will set $\beta = 0.99$, $\sigma = \chi = 1$, $\theta = 1$, $\epsilon_p = \epsilon_w = 11$, and $\psi = 0.2$. As noted above, I am linearizing about a zero inflation steady state, so $\Pi = 1$. For monetary policy, I assume $\rho_i = 0.8$, $\phi_\pi = 1.5$, and $\phi_x = 0.5$. For the government spending and productivity shocks, I assume $\rho_A = \rho_G = 0.9$. For shock standard deviations, I assume $s_i = 0.0025$ and $s_G = s_A = 0.01$. I consider different constellations of the price and wage rigidity parameters. I consider three cases: both prices and wages are sticky ($\phi_p = \phi_w = 0.75$), only prices are sticky ($\phi_p = 0.75$, $\phi_w = 0$), and only wages are sticky ($\phi_w = 0.75$, $\phi_p = 0$). The expected duration between wage changes takes the same formula as the expected duration between price changes (i.e. the expected duration of a wage chosen today is $(1 - \phi_w)^{-1}$).

I compute impulse responses of output, the nominal interest rate, the inflation rate, and wage inflation (defined as $\pi_t^w = \ln w_t - \ln w_{t-1} + \ln \Pi_t = \ln W_t - \ln W_{t-1}$ to each of the three shocks. When both prices and wages are sticky, responses are shown with the solid black line. The dotted black line coincides with just prices being sticky, whereas the case of just wages being sticky is shown with the dashed blue line. In all of these responses, when wages (prices) are flexible, wage inflation (price inflation) reacts quite strongly compared to when sticky. This is natural. With one exception, the responses of output are quite similar.

Figure 1: Productivity Shock



Consider first the productivity shock. The responses with both prices and wages sticky are very similar to the case when just prices are sticky – output undershoots its flexible price level. The main difference is that output overshoots its flexible price response when just wages are sticky. When just wages are sticky, the wage markup falls – some households would like to increase their wages but can't and hence end up working more than they would like, so the labor market becomes less distorted on average after a productivity shock. This is relatively expansionary for output – output reacts more to a productivity shock (relative to a flexible price/wage equilibrium) when just wages are sticky. Price stickiness works in the opposite direction. Firms would like to lower their prices after a positive productivity shock, but some can't. This means that price markups end up higher than optimal, so the economy becomes relatively more distorted. This accounts for why output undershoots its flexible price/wage level after a productivity shock when just prices are sticky. When both are sticky, at the same time, output reacts slightly more than if just prices are sticky, but the difference is small.

The next two figures show impulse responses to monetary policy and government spending shocks. In terms of output responses, these are all quite similar.

Figure 2: Monetary Policy Shock

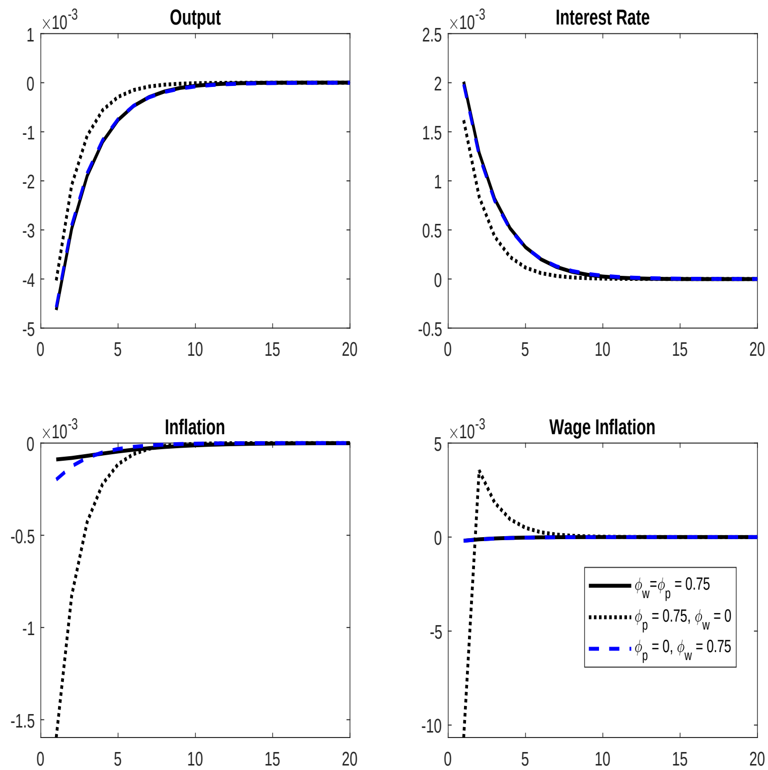
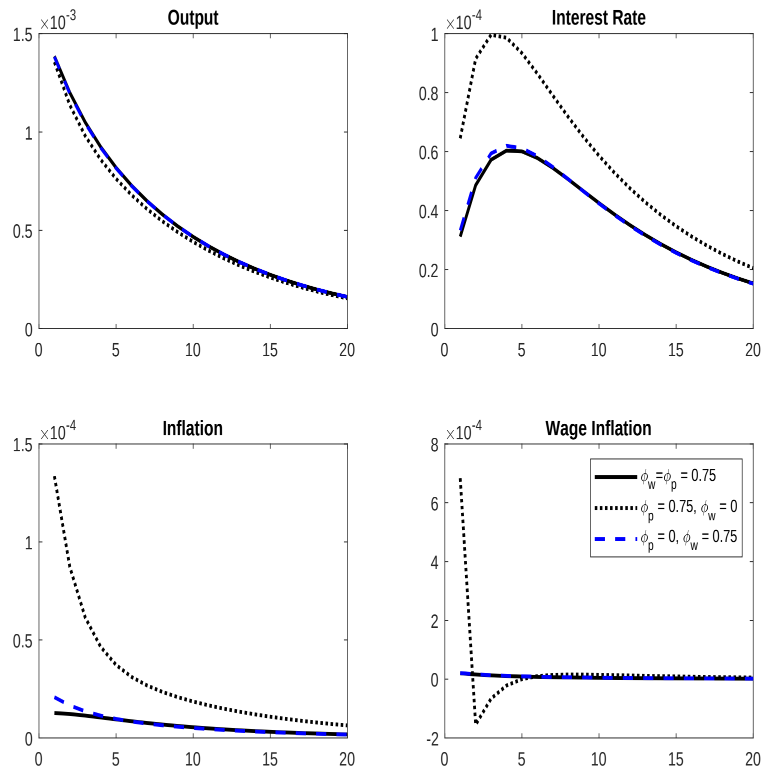


Figure 3: Government Spending Shock



9 Log-Linearization

It is helpful to log-linearize the model to gain intuition. As we have done before, we will log-linearize about a zero inflation steady state (so $\Pi = 1$). One other simplification I'm going to make: steady state government spending is zero, and government spending shocks are turned off. This means that the aggregate resource constraint is $Y_t = C_t$ and simplifies the analysis.

The linearization of the production side of the model is the same as earlier. We have a Phillips Curve, an equation describing real marginal cost, and the linearized production function:

$$\pi_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{mc}_t + \beta \mathbb{E}_t \pi_{t+1} \tag{39}$$

$$\widetilde{mc}_t = \widetilde{mc}_t - a_t \tag{40}$$

$$y_t = a_t + n_t \tag{41}$$

With the aggregate resource constraint implying $Y_t = C_t$ (with no government spending), we have a linearized IS equation:

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1}\right) \tag{42}$$

What we need to do is linearize the conditions related to wage-setting. First, start with the evolution of wages, (23)-(25) and (35). Start with (35), which describes the evolution of the real wage. Take logs and totally differentiate:

$$(1-\epsilon_w)\ln w_t = \ln\left[(1-\phi_w)\left(w_t^{\#}\right)^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w}\right]$$

$$(1-\epsilon_w)\frac{dw_t}{w} = \frac{1}{w^{1-\epsilon_w}}\left[(1-\epsilon_w)(1-\phi_w)\left(w^{\#}\right)^{-\epsilon_w} dw_t^{\#} + (\epsilon_w - 1)\phi_w w^{1-\epsilon_w} d\Pi_t + (1-\epsilon_w)\phi_w w^{-\epsilon_w} dw_{t-1}\right]$$

In taking total derivatives evaluated at the steady state, I have gone ahead and noted that $\Pi = 1$, which simplifies the analysis. When that is the case, we have $w^{\#} = w$. But this means we have:

$$(1-\epsilon_w)\frac{dw_t}{w} = (1-\epsilon_w)(1-\phi_w)\frac{dw_t^{\#}}{w^{\#}} - (1-\epsilon_w)\phi_w d\Pi_t + (1-\epsilon_w)\phi_w \frac{dw_{t-1}}{w}$$

The $1-\epsilon_w$ terms cancel, and we have:

$$\widetilde{w}_t = (1-\phi_w)\widetilde{w}_t^{\#} + \phi_w \widetilde{w}_{t-1} - \phi_w \pi_t \tag{43}$$

Where $\widetilde{w}_t = dw_t/w$ and $\pi_t = d\Pi_t$. This equation says that the linearized real wage is a convex combination of the real reset wage and the lagged real wage, with an adjustment for inflation (since inflation erodes the real wages of non-updating households, who are stuck with a given nominal wage).

Now, let's log-linearized (23), which is easy since it's just a product. We have:

$$(1+\epsilon_w \chi)\widetilde{w}_t^{\#} = \widetilde{h}_{1,t} - \widetilde{h}_{2,t} \tag{44}$$

Now we need to log-linearize the h terms. Start with (24):

$$\ln h_{1,t} = \ln\left[\theta N_t^{1+\chi} w_t^{\epsilon_w(1+\chi)} + \phi_w \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\chi)} h_{1,t+1}\right]$$

$$\frac{dh_{1,t}}{h_1} = \frac{1}{h_1}\left[(1+\chi)\theta N^{\chi} w^{\epsilon_w(1+\chi)} dN_t + \epsilon_w(1+\chi)\theta N^{1+\chi} w^{\epsilon_w(1+\chi)-1} dw_t + \epsilon_w(1+\chi)\phi_w \beta h_1 d\Pi_{t+1} + \phi_w \beta dh_{1,t+1}\right]$$

This looks a bit messy (note I have again gone ahead and imposed $\Pi = 1$ when evaluating derivatives at steady state). This may be written:

$$\frac{dh_{1,t}}{h_1} = \frac{1}{h_1}\left[(1+\chi)\theta N^{1+\chi} w^{\epsilon_w(1+\chi)} \frac{dN_t}{N} + \epsilon_w(1+\chi)\theta N^{1+\chi} w^{\epsilon_w(1+\chi)} \frac{dw_t}{w} + \epsilon_w(1+\chi)\phi_w \beta h_1 d\Pi_{t+1} + \phi_w \beta dh_{1,t+1}\right]$$

But now that steady state h_1 is:

$$h_1 = \frac{\theta N^{1+\chi} w^{\epsilon_w(1+\chi)}}{1-\phi_w\beta}$$

This simplifies the analysis. Making use of this, we have:

$$\widetilde{h}_{1,t} = (1-\phi_w\beta)(1+\chi)n_t + \epsilon_w(1+\chi)(1-\phi_w\beta)\widetilde{w}_t + \epsilon_w(1+\chi)\phi_w\beta\mathbb{E}\pi_{t+1} + \phi_w\beta\mathbb{E}_t\widetilde{h}_{1,t+1} \tag{45}$$

Now let's go to the expression for $h_{2,t}$. Start by taking logs:

$$\ln h_{2,t} = \ln\left[C_t^{-\sigma} w_t^{\epsilon_w} N_t + \phi_w\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon_w-1} h_{2,t+1}\right]$$

Totally differentiate about the steady state:

$$\frac{dh_{2,t}}{h_2} = \frac{1}{h_2}\left[C^{-\sigma}w^{\epsilon_w}dN_t - \sigma C^{-\sigma-1}w^{\epsilon_w}NdC_t + \epsilon_w C^{-\sigma}w^{\epsilon_w-1}Ndw_t + (\epsilon_w-1)\phi_w\beta h_2 d\Pi_{t+1} + \phi_w\beta dh_{2,t+1}\right]$$

This may be written:

$$\frac{dh_{2,t}}{h_2} = \frac{1}{h_2}C^{-\sigma}w^{\epsilon_w}N\left(\frac{dN_t}{N} - \sigma\frac{dC_t}{C} + \epsilon_w\frac{dw_t}{w}\right) + (\epsilon_w-1)\phi_w\beta d\Pi_{t+1} + \phi_w\beta\frac{dh_{2,t+1}}{h_2}$$

Note that:

$$h_2 = \frac{C^{-\sigma}w^{\epsilon_w}N}{1-\phi_w\beta}$$

But this means we can write the above:

$$\widetilde{h}_{2,t} = (1-\phi_w\beta)n_t - \sigma(1-\phi_w\beta)y_t + \epsilon_w(1-\phi_w\beta)\widetilde{w}_t + (\epsilon_w-1)\phi_w\beta\mathbb{E}_t\pi_{t+1} + \phi_w\beta\mathbb{E}_t\widetilde{h}_{2,t+1} \tag{46}$$

In writing the above, I have gone ahead and made the substitution that $y_t = c_t$ (note I am assuming no government spending for the linearization part). Now take the difference between (45) and (46). We have:

$$\widetilde{h}_{1,t}-\widetilde{h}_{2,t} = (1-\phi_w\beta)\chi n_t+\sigma(1-\phi_w\beta)y_t+(1-\phi_w\beta)\epsilon_w\chi\widetilde{w}_t+\phi_w\beta(1+\epsilon_w\chi)\mathbb{E}_t\pi_{t+1}+\phi_w\beta\mathbb{E}_t\left(\widetilde{h}_{1,t+1}-\widetilde{h}_{2,t+1}\right)$$

Now, let's introduce some terms to ease our analysis. The marginal rate of substitution (MRS) between labor and consumption is defined as:

$$MRS_t = \frac{-u_N(C_t, N_t)}{u_C(C_t, N_t)} = \theta N_t^{\chi} C_t^{\sigma}$$

In log-linear terms, where $mrs_t = d \ln MRS_t$, we have:

$$mrs_t = \chi n_t + \sigma y_t \tag{47}$$

Where above, I have gone ahead and imposed $y_t = c_t$ again. This means we can write:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = (1 - \phi_w \beta) mrs_t + (1 - \phi_w \beta)\epsilon_w \chi \widetilde{w}_t + \phi_w \beta (1 + \epsilon_w \chi) \mathbb{E}_t \pi_{t+1} + \phi_w \beta \mathbb{E}_t \left(\widetilde{h}_{1,t+1} - \widetilde{h}_{2,t+1} \right)$$

Let's define μ_t as the difference in log deviations of the marginal rate of substitution and the real wage. If wages were flexible, this would be constant, because households would charge a constant markup of the wage over the marginal rate of substitution:

$$\mu_t = mrs_t - w_t \tag{48}$$

This means we have:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = (1 - \phi_w \beta)\mu_t + (1 - \phi_w \beta)(1 + \epsilon_w \chi)\widetilde{w}_t + \phi_w \beta (1 + \epsilon_w \chi) \mathbb{E}_t \pi_{t+1} + \phi_w \beta \mathbb{E}_t \left(\widetilde{h}_{1,t+1} - \widetilde{h}_{2,t+1} \right) \tag{49}$$

Now, return to (43). We can write this:

$$\widetilde{w}_t^{\#} = \frac{1}{1 - \phi_w}\widetilde{w}_t - \frac{\phi_w}{1 - \phi_w}\widetilde{w}_{t-1} + \frac{\phi_w}{1 - \phi_w}\pi_t$$

Since $(1 - \epsilon_w \chi)\widetilde{w}_t^{\#} = \widetilde{h}_{1,t} - \widetilde{h}_{2,t}$, this means:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = \frac{1 + \epsilon_w \chi}{1 - \phi_w}\widetilde{w}_t - \frac{(1 + \epsilon_w \chi)\phi_w}{1 - \phi_w}\widetilde{w}_{t-1} + \frac{(1 + \epsilon_w \chi)\phi_w}{1 - \phi_w}\pi_t$$

It is helpful to re-write this in terms of nominal wages, where $\widetilde{W}_t = \widetilde{w}_t + \widetilde{P}_t$. Doing so, we have:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = \frac{1 + \epsilon_w \chi}{1 - \phi_w}\left(\widetilde{W}_t - \widetilde{P}_t\right) - \frac{(1 + \epsilon_w \chi)\phi_w}{1 - \phi_w}\left(\widetilde{W}_{t-1} - \widetilde{P}_{t-1}\right) + \frac{(1 + \epsilon_w \chi)\phi_w}{1 - \phi_w}\pi_t$$

This can be rewritten:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = \frac{1 + \epsilon_w \chi}{1 - \phi_w}\left[\widetilde{W}_t - \widetilde{W}_{t-1} + (1 - \phi_w)\widetilde{W}_{t-1} - \left(\widetilde{P}_t - \widetilde{P}_{t-1}\right) - (1 - \phi_w)\widetilde{P}_{t-1} + \phi_w \pi_t\right]$$

Now, define nominal wage inflation as:

$$\pi_t^w = \widetilde{W}_t - \widetilde{W}_{t-1} \tag{50}$$

In gross terms, we would have $\Pi_t^w = \frac{W_t}{W_{t-1}}$. Net price inflation is $\pi_t = \widetilde{P}_t - \widetilde{P}_{t-1}$. Hence, this expression works out to:

$$\widetilde{h}_{1,t} - \widetilde{h}_{2,t} = \frac{1+\epsilon_w \chi}{1-\phi_w}\pi_t^w + (1+\epsilon_w\chi)\widetilde{w}_{t-1} - (1+\epsilon_w\chi)\pi_t \tag{51}$$

Now use this, (51), with (49) to substitute out $\widetilde{h}_{1,t} - \widetilde{h}_{2,t}$. We have:

$$\frac{\pi_t^w}{1-\phi_w} + \widetilde{w}_{t-1} - \pi_t = \frac{1-\phi_w\beta}{1+\epsilon_w\chi}\mu_t + (1-\phi_w\beta)\widetilde{w}_t + \phi_w\beta\mathbb{E}_t\pi_{t+1} + \dots$$
$$\frac{\phi_w\beta}{1+\epsilon_w\chi}\left[\frac{1+\epsilon_w\chi}{1-\phi_w}\mathbb{E}_t\pi_{t+1}^w + (1+\epsilon_w\chi)\widetilde{w}_t - (1+\epsilon_w\chi)\mathbb{E}_t\pi_{t+1}\right]$$

Distributing, in the last part, we can write this as:

$$\frac{\pi_t^w}{1-\phi_w} + \widetilde{w}_{t-1} - \pi_t = \frac{1-\phi_w\beta}{1+\epsilon_w\chi}\mu_t + (1-\phi_w\beta)\widetilde{w}_t + \phi_w\beta\mathbb{E}_t\pi_{t+1} + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\pi_{t+1}^w + \phi_w\beta\widetilde{w}_t - \phi_w\beta\mathbb{E}_t\pi_{t+1}$$

Now, the expected price inflation term on the RHS cancels, and the term involving $\widetilde{w}_t$ on the RHS simplifies. So we can write:

$$\frac{\pi_t^w}{1-\phi_w} - (\widetilde{w}_t - \widetilde{w}_{t-1} + \pi_t) = \frac{1-\phi_w\beta}{1+\epsilon_w\chi}\mu_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\pi_{t+1}^w$$

But now we have $\pi_t^w = \widetilde{w}_t - \widetilde{w}_{t-1} + \pi_t$, so we have:

$$\frac{\phi_w}{1-\phi_w}\pi_t^w = \frac{1-\phi_w\beta}{1+\epsilon_w\chi}\mu_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\pi_{t+1}^w$$

Which can be simplified to:

$$\pi_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}\mu_t + \beta\mathbb{E}_t\pi_{t+1}^w \tag{52}$$

(52) is the wage Phillips Curve. It looks almost identical to the price Phillips curve (when that is written in terms of real marginal cost), though the coefficient on μ_t is slightly different in that it depends on ϵ_w (elasticity of substitution across labor types) and χ (inverse Frisch elasticity). Recall what μ_t measures – it is the difference between the (log) marginal rate of substitution and the wage. Households desire the wage to be a markup over the marginal rate of substitution. When $\mu_t > 0$, the wage is lower than households would like. Given the opportunity to adjust (a fraction $1-\phi_w$ adjust each period), households will raise wages, and we will see wage inflation.

In terms of linearized conditions, without government spending we can express the equilibrium of the model with the following conditions:

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1}) \tag{53}$$

$$\pi_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{mc}_t + \mathbb{E}_t\pi_{t+1} \tag{54}$$

$$\widetilde{mc}_t = \widetilde{w}_t - a_t \tag{55}$$

$$y_t = a_t + n_t \tag{56}$$

$$mrs_t = \chi n_t + \sigma y_t \tag{57}$$

$$\mu_t = mrs_t - w_t \tag{58}$$

$$\pi_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}\mu_t + \beta\mathbb{E}_t\pi_{t+1}^w \tag{59}$$

$$a_t = \rho_A a_{t-1} + s_A\varepsilon_{A,t} \tag{60}$$

$$x_t = y_t - y_t^f \tag{61}$$

$$y_t^f = \frac{1+\chi}{\sigma+\chi}a_t \tag{62}$$

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left[\phi_\pi\pi_t + \phi_x x_t\right] + s_i\varepsilon_{i,t} \tag{63}$$

$$\pi_t^w = \widetilde{w}_t - \widetilde{w}_{t-1} + \pi_t \tag{64}$$

This is 12 variables – y_t, i_t, π_t, $\widetilde{mc}_t$, $\widetilde{w}_t$, a_t, n_t, mrs_t, μ_t, π_t^w, x_t, y_t^f – and 12 equations. (53) is the IS equation (note I have already imposed the resource constraint that $C_t = Y_t$; recall I am omitting government spending from the linearization). (54) is the price Phillips Curve, and (55) defines real marginal cost as the log difference between the real wage and the marginal product of labor (which is just $a_t = d\ln A_t$ with linear production). (56) is the production function (around a zero-inflation steady state, price dispersion is constant). (57) defines the marginal rate of substitution between labor and consumption, and (58) defines the "gap" variable μ_t. The wage Phillips Curve is (59). Productivity follows an exogenous process, (59). (61) defines the output gap, and (62) expresses the equilibrium level of output when *both* prices and wages are flexible as a function of a_t. The policy rule is given by (63). (64) defines wage inflation in terms of real wage growth and price inflation.

9.1 A Gap Formulation

As we did before, it would be nice if we could get rid of some redundant variables (like n_t) and potential write this model in terms of gaps. We can, but not quite all the way.

Note that we can write μ_t by eliminating mrs_t and n_t:

$$\mu_t = \chi(y_t - a_t) + \sigma y_t - w_t$$

By adding and subtracting a_t, this can be written:

$$\mu_t = (\sigma+\chi)y_t - (1+\chi)a_t - (\widetilde{w}_t - a_t)$$

Note that $\widetilde{w}_t - a_t = \widetilde{mc}_t$. Note also that $(1+\chi)a_t = (\sigma+\chi)y_t^f$. So, we have:

$$\mu_t = (\sigma+\chi)x_t - \widetilde{mc}_t$$

This means that we can write the wage Phillips Curve as:

$$\pi_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}(\sigma+\chi)x_t - \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}\widetilde{mc}_t + \beta\mathbb{E}_t\pi_{t+1}^w \tag{65}$$

The IS equation, (53), can be re-written in terms of the output gap and the natural rate of interest as earlier:

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - r_t^f\right) \tag{66}$$

The natural rate of interest is defined by:

$$r_t^f = \sigma\left(\mathbb{E}_t y_{t+1}^f - y_t^f\right)$$

But, in terms of a_t, this is:

$$r_t^f = \frac{\sigma(1+\chi)}{\sigma+\chi}(\rho_A - 1)a_t \tag{67}$$

Which we could just as easily write as an exogenous process without reference to a_t at all:

$$r_t^f = \rho_A r_{t-1}^f + \frac{\sigma(1+\chi)}{\sigma+\chi}(\rho_A - 1)s_A\varepsilon_{A,t} \tag{68}$$

We can eliminate $\widetilde{w}_t$ from (64) by adding and subtracting a_t:

$$\pi_t^w = \widetilde{w}_t - a_t - (\widetilde{w}_{t-1} - a_{t-1}) + a_t - a_{t-1} + \pi_t$$

But $\widetilde{w}_t - a_t = \widetilde{mc}_t$, and we can write:

$$a_t - a_{t-1} = \frac{\sigma+\chi}{\sigma(1+\chi)(\rho_A-1)}\left(r_t^f - r_{t-1}^f\right)$$

Therefore:

$$\pi_t^w = \widetilde{mc}_t - \widetilde{mc}_{t-1} + \frac{\sigma+\chi}{\sigma(1+\chi)(\rho_A-1)}\left(r_t^f - r_{t-1}^f\right) + \pi_t \tag{69}$$

Hence, a reduced linear system in the variables x_t, π_t, π_t^w, mc_t, r_t^f, and i_t is (we have eliminated μ_t, mrs_t, y_t, a_t, y_t^f and n_t):

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - r_t^f\right) \tag{70}$$

$$\pi_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widetilde{mc}_t + \mathbb{E}_t\pi_{t+1} \tag{71}$$

$$\pi_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}(\sigma+\chi)x_t - \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}\widetilde{mc}_t + \beta\mathbb{E}_t\pi_{t+1}^w \tag{72}$$

$$\pi_t^w = \widetilde{mc}_t - \widetilde{mc}_{t-1} + \frac{\sigma+\chi}{\sigma(1+\chi)(\rho_A-1)}\left(r_t^f - r_{t-1}^f\right) + \pi_t \tag{73}$$

$$r_t^f = \frac{\sigma(1+\chi)}{\sigma+\chi}(\rho_A-1)a_t \tag{74}$$

$$i_t = \rho_i i_{t-1} + (1-\rho_i)\left[\phi_\pi\pi_t + \phi_x x_t\right] + s_i\varepsilon_{i,t} \tag{75}$$

This is six equations and six variables.

9.2 Breakdown of the Divine Coincidence

When both prices and wages are sticky, the Divine Coincidence breaks down. Put differently, it is not possible to achieve both $\pi_t = 0$ and $x_t = 0$.

The proof of this is straightforward. Suppose that a central bank implements a strict inflation target, so $\pi_t = 0$. From (71), this requires $\widetilde{mc}_t = 0$. With constant price inflation and constant real marginal cost, (73) would tell us that wage inflation is:

$$\pi_t^w = \frac{\sigma+\chi}{\sigma(1+\chi)(\rho_A-1)}\left(r_t^f - r_{t-1}^f\right)$$

In general, this will be non-zero since r_t^f will jump around due to productivity shocks. From (72), if wage inflation is not constant, even if $mc_t = 0$ we will have to have $x_t \neq 0$. We can't simultaneously achieve $\pi_t = 0$ and $x_t = 0$.

The intuition for why the Divine Coincidence breaks down is actually quite straightforward. To implement the flexible price allocation (i.e. to have $x_t = 0$), the real wage has to move around. But if nominal wages are sticky, the nominal wage can't move enough to get the "right" real wage unless price inflation also moves around.

9.3 Optimal Policy

One can set up an optimal monetary policy problem in the model with sticky prices and wages. A micro-founded welfare loss function will be proportional to:

$$\mathcal{L} = \mathbb{E}_t\sum_{j=0}^{\infty}\beta^j\left[\frac{\pi_{t+j}^2}{2} + \frac{\omega_x x_{t+j}^2}{2} + \frac{\omega_w\left(\pi_{t+j}^w\right)^2}{2}\right]$$

Here, in addition to price inflation and the output gap, the policymaker also cares directly about wage inflation (with relative weight ω_w. The reason the policymaker ought to care about wage inflation in isolation is because of something mentioned above – wage inflation causes wage dispersion, and wage dispersion throws a wedge between labor supplied by households and labor used in production. This wedge isn't relevant in the equilibrium allocations, but is relevant when thinking about welfare.

I will characterize optimal policy under discretion. I'm not going to do the commitment case, though it is doable. The problem, as written, is quite a bit more complicated than the baseline sticky-price model – there are four equilibrium conditions that must be taken as constraints (the exogenous process for r_t^f and the policy rule aren't include), and the policymaker can effectively choose four variables – π_t, π_t^w, x_t, and $\widetilde{mc}_t$. Doing the problem under commitment amplifies the complexity. But it's reasonably straightforward under discretion. The optimality condition is nice, and reduces to what we had earlier in the situation in which wages are flexible (so $\phi_w = 0$).

To ease up on notation a bit, define auxiliary parameters:

$$\gamma_1 = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}$$
$$\gamma_2 = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}(\sigma+\chi)$$
$$\gamma_3 = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\chi)}$$

A Lagrangian is:

$$\mathbb{L} = \frac{\pi_t^2}{2} + \frac{\omega_x x_t^2}{2} + \frac{\omega_w\left(\pi_t^w\right)^2}{2} + \psi_{1,t}\left(\gamma_1\widetilde{mc}_t + \mathbb{E}_t\pi_{t+1} - \pi_t\right) + \psi_{2,t}\left(\gamma_2 x_t - \gamma_3\widetilde{mc}_t + \beta\mathbb{E}_t\pi_{t+1}^w - \pi_t^w\right) +$$
$$\psi_{3,t}\left(\widetilde{mc}_t - \widetilde{mc}_{t-1} + \frac{\sigma+\chi}{\sigma(1+\chi)(\rho_A-1)}\left(r_t^f - r_{t-1}^f\right) + \pi_t - \pi_t^w\right) + \psi_{4,t}\left(\mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - r_t^f\right) - x_t\right)$$

When taking the FOC under discretion, we will ignore the fact that the choice of $\widetilde{mc}_t$ today impacts the future constraint on the definition of wage inflation because the policymaker treats the future as given. Accordingly, the FOC are:

$$\frac{\partial\mathbb{L}}{\partial\pi_t} = \pi_t - \psi_{1,t} + \psi_{3,t} = 0$$
$$\frac{\partial\mathbb{L}}{\partial x_t} = \omega_x x_t + \gamma_2\psi_{2,t} - \psi_{4,t} = 0$$
$$\frac{\partial\mathbb{L}}{\partial\pi_t^w} = \omega_w\pi_t^w - \psi_{2,t} - \psi_{3,t} = 0$$
$$\frac{\partial\mathbb{L}}{\partial\widetilde{mc}_t} = \gamma_1\psi_{1,t} - \gamma_3\psi_{2,t} + \psi_{3,t} = 0$$
$$\frac{\partial\mathbb{L}}{\partial i_t} = -\frac{1}{\sigma}\psi_{4,t} = 0$$

From these, we see that $\psi_{4,t} = 0$. This means that:

$$\omega_x x_t = -\gamma_2\psi_{2,t}$$

From the first, we also have:

$$\psi_{1,t} = \pi_t + \psi_{3,t}$$

Plug these all into the FOC for marginal cost:

$$\gamma_1 \pi_t + \gamma_1 \psi_{3,t} + \frac{\gamma_3}{\gamma_2}\omega_x x_t + \psi_{3,t} = 0$$

This means:

$$\psi_{3,t} = -\frac{\gamma_1}{1+\gamma_1}\pi_t - \frac{\gamma_3}{(1+\gamma_1)\gamma_2}\omega_x x_t$$

Now go to the FOC for wage inflation. We know have expressions for $\psi_{3,t}$ and $\psi_{2,t}$, so:

$$\omega_w \pi_t^w + \frac{\omega_x}{\gamma_2}x_t + \frac{\gamma_1}{1+\gamma_1}\pi_t + \frac{\gamma_3}{(1+\gamma_1)\gamma_2}\omega_x x_t = 0$$

Or:

$$(1+\gamma_1)\gamma_2\omega_w\pi_t^w + (1+\gamma_1)\omega_x x_t + \gamma_1\gamma_2\pi_t + \gamma_3\omega_x x_t = 0$$

Which means:

$$x_t = -\frac{\gamma_2(1+\gamma_1)}{1+\gamma_1+\gamma_3}\frac{\omega_w}{\omega_x}\pi_t^w - \frac{\gamma_1\gamma_2}{1+\gamma_1+\gamma_3}\frac{1}{\omega_x}\pi_t \tag{76}$$

Which can also be written:

$$x_t = -\frac{\gamma_2}{1+\gamma_1+\gamma_3}\left[(1+\gamma_1)\frac{\omega_w}{\omega_x}\pi_t^w + \gamma_1\frac{1}{\omega_x}\pi_t\right] \tag{77}$$

(77) is a modified lean-against-the-wind condition. Rather than a negative relationship between the output gap and price inflation, it is a negative relationship between the output gap and a *weighted sum* of wage and price inflation. The bigger is ω_w, the bigger the weight on wage inflation relative to price inflation.

It isn't immediately obvious how to see it, but not that this condition reverts to the lean-against-the-wind condition we encountered earlier when wages are flexible. If wages are flexible, we should have $\omega_w = 0$ – i.e. the policy maker doesn't care about wage inflation per se. With that, the condition can be written:

$$x_t = -\frac{\gamma_1\gamma_2}{1+\gamma_1+\gamma_3}\frac{1}{\omega_x}\pi_t$$

Note that $\gamma_2 = (\sigma+\chi)\gamma_3$, so this can be written:

$$x_t = -\frac{\gamma_1\gamma_3(\sigma+\chi)}{1+\gamma_1+\gamma_3}\frac{1}{\omega_x}\pi_t$$

Which can be written:

$$x_t = -\frac{\gamma_3}{1+\gamma_1+\gamma_3}\frac{\gamma_1(\sigma+\chi)}{\omega_x}\pi_t$$

When wages are flexible, $\phi_w = 0$, which means $\gamma_3 \to \infty$. But this mean that $\frac{\gamma_3}{1+\gamma_1+\gamma_3} \to 1$. Since $\gamma_1(\sigma+\chi) = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}(\sigma+\gamma) = \gamma$ in our Phillips Curve when just prices were sticky, we get:

$$x_t = -\frac{\gamma}{\omega_x}\pi_t$$

Which is exactly the same condition we had earlier for optimal monetary policy under discretion!

Conversely, if prices were flexible ($\phi_p = 0$, we would want zero weight on price inflation in the objective function. As written, I've normalized the relative weight on inflation to be unity. With a zero weight on inflation, I would need ω_x and $\omega_w \to \infty$. In that case, the optimality condition would reduce to:

$$x_t = -\frac{1+\gamma_1}{1+\gamma_1+\gamma_3}\gamma_2\frac{\omega_w}{\omega_x}\pi_t^w$$

But if prices were flexible, we would have $\gamma_1 \to \infty$, and therefore:

$$\frac{1+\gamma_1}{1+\gamma_1+\gamma_3} \to 1$$

Hence, we'd have:

$$x_t = -\gamma_2\frac{\omega_w}{\omega_x}\pi_t^w$$

This would just be a "lean-against-the-wind" condition between wage inflation and the output gap.