ACMS 22550 Applied Mathematics Method I tutorial
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1 Sequences

A sequence $\{a_n\}_1^\infty$ is convergent if $\lim_n a_n = a$, a should be definite, which means it can not change or be ∞

2 Infinite series

2.1 Geometric series

1. n-Sum of the geometric series with the first term a and the common ratio r:

$$S_n = \frac{a(1-r^n)}{1-r} \tag{1}$$

2. Convergence of a geometric series if $|r| < 1$:

$$S = \lim_{n\to\infty} S_n = \frac{a}{1-r} \tag{2}$$

2.2 The absolute convergence theorem

Theorem 1. *Absolute convergent series converges: If* $\sum_n^\infty |a_n| < \infty$*, then* $\sum_n^\infty a_n < \infty$

2.3 Preliminary test

Theorem 2. *If the terms of an infinite series do not converge to zero, the series diverges. If do, need further test.*

2.4 Convergence tests for series with positive terms

1. The comparison test:
   (a) If positive series $\sum_i^\infty m_i$ converges and $|a_n| \leq m_n$ for $n \geq N$, then a_n converges
   (b) If positive series $\sum_i^\infty d_i$ diverges and $|a_n| \geq d_n$ for $n \geq N$, then a_n diverges
2. The integral test: for monotonically decreasing positive series:
   (a) $0 \leq a_{n+1} \leq a_n$ for $n > N$, then $\sum_n a_n$ converges if $\int^\infty a_n(n)dn < \infty$
   (b) $0 \leq a_{n+1} \leq a_n$ for $n > N$, then $\sum_n a_n$ diverges if $\int^\infty a_n(n)dn = \infty$
3. The ratio test: Define $\rho_n = |\frac{a_{n+1}}{a_n}| = \rho_n$ and $\rho = \lim_n^\infty \rho_n$

(a) $\rho < 1$: Convergence

(b) $\rho = 1$: Need a different test

(c) $\rho > 1$: Divergence

4. Special comparison test

   (a) If $\sum_{n=1}^{\infty} b_n$ is positive convergent, $a_n \geq 0$ and $\lim_n \frac{a_n}{b_n}$ is finite, then $\sum_n^{\infty} a_n$ converges

   (b) If $\sum_{n=1}^{\infty} d_n$ is positive divergent, $a_n \geq 0$ and $\lim_n \frac{a_n}{b_n} \neq 0$ then $\sum_n^{\infty} a_n$ diverges

2.5 Alternating series

Theorem 3. *An alternating series converges if* $|a_{n+1}| \leq |a_n|$ *and* $\lim_{n\to\infty} a_n = 0$

3 Power series

General power series as a function of x goes like:

$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \tag{3}$$

Whether power series converges or not depends on the value x to be evaluated. we can use **ratio test** to determine the "**convergence interval**".

3.1 Exercises

1. Determine the convergence of the following sequences:

$$A: \left\{(-1)^n \frac{n^2-1}{2n^3+1}\right\}_{n=1}^{\infty} \quad B: \left\{e^{\frac{1}{n}}\right\}_{n=1}^{\infty} \quad C: \left\{\frac{n-1}{log(n)}\right\}_{n=1}^{\infty}$$

2. Determine the convergence of the following series:

$$A: \sum_{n=1}^{\infty} \frac{2n^2+7}{\sqrt{n^7+2}} \quad B: \sum_{n=2}^{\infty} \frac{2^{\frac{1}{n}}}{n} \quad C: \sum_{n=1}^{\infty} \frac{cos(n)}{n^2}$$

3. Find the sum of the following series:

$$\sum_{n=1}^{\infty} \left[\frac{n}{2^{n-1}} - \frac{n+1}{2^n}\right]$$

4. Which of the following series is **absolutely** convergent?

$$A: \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{n^3+1}\right] \quad B: \sum_{n=1}^{\infty} \left[\frac{(-1)^n n!}{n^3}\right] \quad C: \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}}{\sqrt{n}}\right]$$

$$D: \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1} log(n+1)}{n}\right] \quad E: \sum_{n=1}^{\infty} \left[\frac{(-1)^{n-1}\pi^n}{3^n}\right]$$

5. Find the sum of the following series (Hint: use known power series)

$$\sum_{n=0}^{\infty} \left[\frac{(-1)^n \pi^{2n}}{(2n)!}\right]$$

6. Find the power series representation of the following function centered at 0

$$f(x) = sin(2x^2)$$

7. Determine the convergence of the following series:

$$A: \sum_{n=2}^{\infty} \left[\frac{(-1)^n}{n}\right] \quad B: \sum_{n=2}^{\infty} \left[\frac{n^2}{log(n)}\right] \quad C: \sum_{n=1}^{\infty} \left[\frac{3^n}{2(n!)}\right]$$

8. Compute the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-1)^n$$

9. Find the power series expansion of the following expression:

$$F(x) = \int \frac{1}{\sqrt{1+x^2}} dx$$

Warning!

4 Solutions for exercises

1. A: Converge, B: Converge, C: Diverge
2. A: Converge, B: Diverge, C: Converge
3. 1
4. A
5. -1
6. $\sum_{n=0}^{\infty}(-1)^n 2^{2n+1}\frac{x^{4n+2}}{(2n+1)!}$
7. A: Converge, B: Diverge, C: Converge
8. 0.5
9. $x - \frac{x^3}{6} + \frac{3x^5}{40}$