ACMS 22550 Applied Mathematics Method I tutorial
Week 2: Aug 30th - Sep 3rd

Ernie Tsybulnik[1] and Guoxiang Grayson Tong[1]

[1]*Ph.D. student, Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN, USA, 46556*

1 Review and extension

1.1 Useful tools

1. If sequences $\{a_n\}$ and $\{b_n\}$ are both convergent sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then:
   (a) $\lim_{n\to\infty}(a_n + b_n) = a + b$
   (b) $\lim_{n\to\infty}(a_n b_n) = ab$

1.2 The squeeze theorem

Theorem 1. *Suppose that for all $n > N$, $a_n \leq b_n \leq c_n$, and*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \alpha$$

then $\lim_{n\to\infty} b_n = \alpha$

1.3 Preliminary test

1. If $\lim_{n\to\infty} a_n \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges
2. If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ requires further testing
3. If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

1.4 The absolute convergence theorem

Theorem 2. *Absolute convergent series converges: If $\sum_{n}^{\infty} |a_n| < \infty$, then $\sum_{n}^{\infty} a_n < \infty$*

1.5 The absolute convergence

Definition 1. *The series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n| = a$, a should be definite, not changing, not ∞*

1.6 The conditional convergence

Definition 2. *The series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally if it converges but not absolutely.*

A classical example of the conditional convergent series: the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

Now prove it!

2 Exercises

1. The "shifting" in series:

$$\sum_{n=2}^{20} a_n = \sum_{k=0}^{18} ?$$

2. Evaluate $\sum_{k=0}^{n} 7^{-k}$, and what about $n \to \infty$

3. Partial sums $s_1 = a_1, s_2 = a_1 + a_2$, and so forth are known to be given by:

$$s_n = \frac{n}{n+2}$$

   Find a_1, a_2, a_n and $\sum_{n=1}^{\infty} a_n$

4. Suppose $\sum_{n=0}^{\infty} a_n$ is proved convergent by the ratio test, now prove $\sum_{n=0}^{\infty} na_n$ is also convergent

5. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by using the followings:

   (a) $\frac{1}{n^2} < \frac{1}{n(n-1)}$

   (b) $\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

3 Homework hints

1. P3: (14)

   How a broken computer program gives $\frac{1}{6} = \sum_{n=0}^{\infty}(-5)^n$?

   Hint: Is this a (convergent) geometric series?

2. P5: (7)

   Find $\lim_{n\to\infty}(1+n^2)^{\dfrac{1}{\log(n)}}$

   Hint: What is $e^{\log(n)}$?

3. P5: (8)

   Find $\lim_{n\to\infty}\frac{(n!)^2}{(2n)!}$

   Hint: Write out the factorial!

4. P8: (7)

   Find a_n, S_n, R_n as $n \to \infty$

   $\frac{3}{1\cdot 2} - \frac{5}{2\cdot 3} + \frac{7}{3\cdot 4} - \frac{9}{4\cdot 5} + \cdots$

   Hint: After having a_n, what kind of series is this? Is it convergent? Write out a few terms to find S_n

5. P11: (2)

   Show the harmonic series is divergent by the comparison test with $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots$

   Hint: Is the given series divergent? What is the sum of it? Then how to apply the comparison rule?

6. P11: (3)

   Prove the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2}$

   Hint: Again, by the comparison test, find some convergent series you know, then group the specific terms to compare with $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Very similar to P11(2).

7. P13:(9)

   Using integral test to find whether $\sum_{n=1}^{\infty}\frac{1}{n^2-4}$ converges?

   Hint: Apply "Partial-Fraction Decomposition", *i.e.*

$$\frac{1}{n^2-4} = \frac{1}{(n-2)(n+2)} = \frac{A}{(n-2)} + \frac{B}{(n+2)}$$

8. P13:(11)

   Using integral test to find whether $\sum_{n=1}^{\infty} \dfrac{1}{n(1+\log(n))^{1.5}}$ converges?

   Hint: Need at least two integration by substitutions.

9. P14:(30)

   Prove the ratio test

   Hint: Follow (exactly) the hints on the textbook. The key is the geometric series to be obtained. Another tools you may need:

   (a) The absolute convergence theorem
   (b) The preliminary test
   (c) The comparison test

Warning page!

4 Solutions for exercises

1.
$$\sum_{n=2}^{20} a_n = \sum_{k=0}^{18} a_{k+2}$$

2.
$$\frac{7}{6}(1 - \frac{1}{7^n}), \quad \frac{7}{6}$$

3.
$$a_1 = \frac{1}{3}, a_2 = \frac{1}{6}$$
$$a_n = s_n - s_{n-1} = \frac{n}{n+2} - \frac{n-1}{n+1}$$
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = 1$$

4. *Proof.* Set $b_n = na_n$, apply ratio test again, we get:
$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim_{n\to\infty} |(\frac{n+1}{n})\frac{a_{n+1}}{a_n}|$$
Since $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, so we have $\rho < 1$. □

5. *Proof.* All we need to do is to verify the series $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges, then we can use the comparison test. Note that
$$\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$$
Thus $\sum_{n=2}^{k} \frac{1}{n(n-1)} = 1 - \frac{1}{k}$, so converging as $k \to \infty$. □

5 Reference

1. *Calculus With Applications*, Peter D. Lax and Maria Shea Terrell, Undergraduate Texts in Mathematics book series (UTM), 2014

2. *Mathematical Methods in the Physical Sciences*, Mary L. Boas, Wiley, 3rd edition, 2005