Optics Course

lecture 2

Wave Optics

~ in classical optics light is described as a wave, and it satisfies the

wave equation

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

c = speed of light = c_0/n

E ~ electric field

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$ ~ Laplacian

Note: if $E_1(\vec{r},t)$ and $E_2(\vec{r},t)$ satisfy the wave equation, then

$$E(\vec{r},t) = E_1(\vec{r},t) + E_2(\vec{r},t)$$

also satisfies the wave eqn. (superposition)

~ different types of waves

Monochromatic waves

$$E(\vec{r},t) = E(\vec{r}) \cos(2\pi\nu t + \varphi(\vec{r}))$$

amplitude   phase

ν = frequency

$2\pi\nu = \omega$ = angular frequency

~ a convenient way of expressing $E(\vec{r},t)$ is to use complex notation

$$E(\vec{r},t) = \frac{E(\vec{r})}{2} \left\{ e^{i(2\pi\nu t + \phi(\vec{r}))} + e^{-i(2\pi\nu t + \varphi(\vec{r}))} \right\}$$

individual parts, as well as the sum, satisfy the wave equation

~ many cases it is sufficient to just use one part of the complex expression.

i.e $\quad E(\vec{r}, t) = E(r)\, e^{i(2\pi\nu t + \varphi(\vec{r}))}$

$$= \mathcal{E}(\vec{r})\, e^{i 2\pi\nu t}$$

where we have defined $\mathcal{E}(\vec{r}) = E(r)\, e^{i\varphi(r)}$

~ substitute $E(\vec{r}, t)$ into the wave equation

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

$$\Rightarrow \nabla^2 E = -\left(\frac{2\pi\nu}{c}\right)^2 E$$

$$\Rightarrow \nabla^2 \mathcal{E}(\vec{r})\, e^{i\omega t} = -\left(\frac{\omega}{c}\right)^2 \mathcal{E}(\vec{r})\, e^{i\omega t}$$

$$\Rightarrow \nabla^2 \mathcal{E}(\vec{r}) = -k^2 \mathcal{E}(\vec{r}) \qquad k = \frac{\omega}{c} = \frac{2\pi\nu}{c}$$

$$\Rightarrow (\nabla^2 + k^2)\, \mathcal{E}(\vec{r}) = 0$$ ~ Helmholtz Eq^{n}.

~ Helmholtz equation tells us how time harmonic waves propagate in space

k = wavenumber ~ units of reciprocal distance

~ 2 simple examples of waves are <u>plane waves</u> and <u>spherical waves</u>

<u>Plane Wave:</u>

~ wave with $\mathcal{E}(\vec{r}) = \mathcal{E}_0 e^{-i\vec{k}\cdot\vec{r}}$

$\mathcal{E}_0$ ~ constant

$\vec{k}$ = wavevector $= k_x\hat{i} + k_y\hat{j} + k_z\hat{k}$

Note: $\nabla^2 \mathcal{E}(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathcal{E}_0 e^{-i\vec{k}\cdot\vec{r}}$

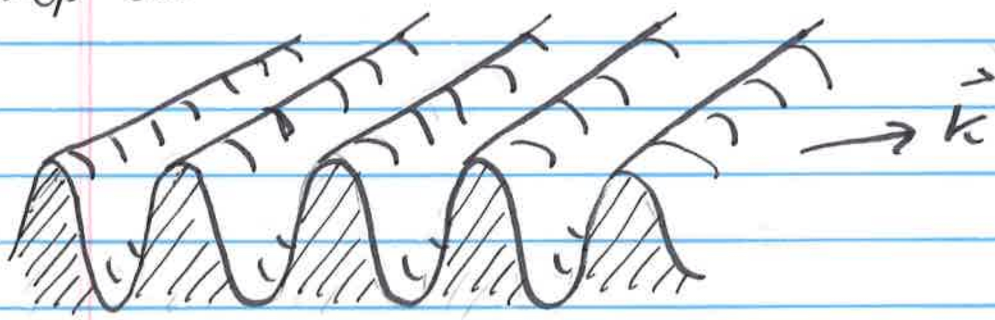
$= -(k_x^2 + k_y^2 + k_z^2)\,\mathcal{E}_0 e^{-i\vec{k}\cdot\vec{r}}$

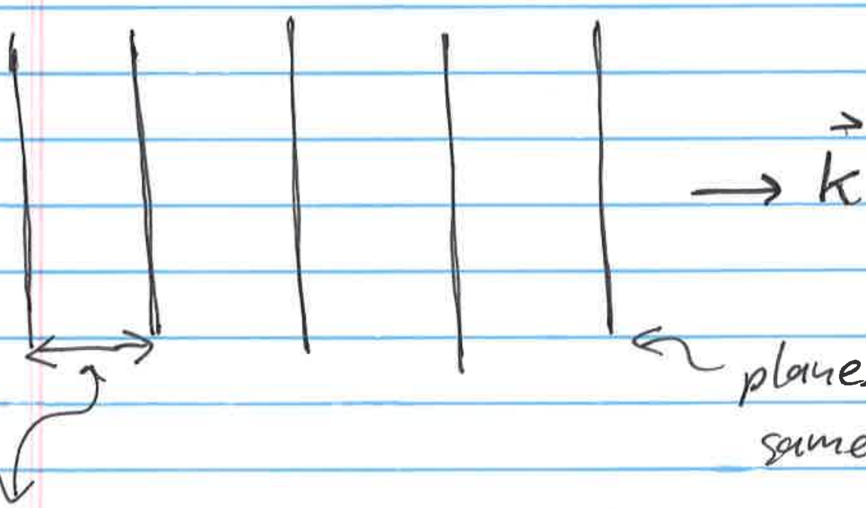
$\mathcal{E}(\vec{r}) = \mathcal{E}_0 e^{-i\vec{k}\cdot\vec{r}}$ satisfies the Helmholtz equation $\nabla^2 \mathcal{E}(\vec{r}) + k^2 \mathcal{E}(\vec{r}) = 0$

if $k_x^2 + k_y^2 + k_z^2 = k^2$

i.e. the magnitude of $\vec{k}$ is the wavenumber k

~ representations of $\mathcal{E}(\vec{r})$





planes w/ the same phase

sep. by a distance $\lambda = 2\pi/k$

Note: the intensity $I(r) = |\mathcal{E}(\vec{r})|^2 = |\mathcal{E}_0|^2$

constant in space

Note 2: $\lambda = \frac{2\pi}{k} = \frac{2\pi c}{2\pi \nu} = c/\nu$

For different media: $c = c_0/n$, $\lambda = \lambda_0/n$, $k = nk_0$

frequency remains the same, λ + k change

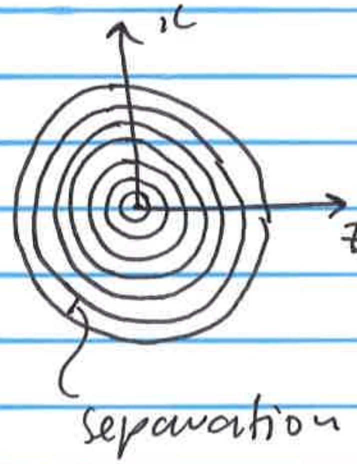
Spherical Waves:

$$\mathcal{E}(\vec{r}) = \frac{\mathcal{E}_0}{r} e^{-ikr}$$

$r \sim$ distance from the origin

~ intensity now decreases w/ r

$$I(\vec{r}) = |\mathcal{E}_0|^2 / r^2$$



wavefronts are now a collection of concentric spheres

Fresnel Approximation

~ imagine the case where we are close to the z-axis, but far away from the origin (x, y small c/f z)

$$\Rightarrow (x^2+y^2)^{1/2} \ll z$$

$$\Rightarrow r = (x^2+y^2+z^2)^{1/2} = z\left(1 + \frac{x^2+y^2}{z^2}\right)^{1/2}$$

$$\frac{x^2+y^2}{z^2} << 1$$

$$\Rightarrow \quad r \simeq z\left(1 + \frac{x^2+y^2}{2z^2}\right) = z + \frac{x^2+y^2}{2z}$$

$$\mathcal{E}(r) = \underbrace{\frac{\mathcal{E}_0}{z}\, e^{-ikz}\, e^{-ik\left(\frac{x^2+y^2}{2z}\right)}}_{\text{Fresnel approximation}}$$

~ very useful!

Interference

~ if we have two monochromatic waves $E_1(\vec r)$ and $E_2(\vec r)$ then we add the fields to get the total field

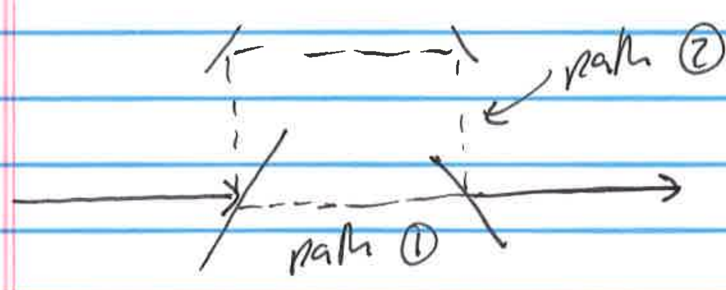
$$\mathcal{E}(\vec r) = \mathcal{E}_1(\vec r) + \mathcal{E}_2(\vec r)$$

$$I(\vec r) = \mathcal{E}(\vec r)\,\mathcal{E}(\vec r)^* = |\mathcal{E}_1(\vec r)|^2 + |\mathcal{E}_2(\vec r)|^2$$

$$+ \mathcal{E}_1(\vec r)\,\mathcal{E}_2^*(\vec r) + \mathcal{E}_1^*(\vec r)\,\mathcal{E}_2^*(\vec r)$$

$$\Rightarrow I(\vec{r}) = I_1(\vec{r}) + I_2(\vec{r}) + \mathcal{E}_1(\vec{r})\mathcal{E}_2^*(\vec{r}) + \mathcal{E}_1^*(\vec{r})\mathcal{E}_2(\vec{r})$$

Example: Mach-Zehnder Interferometer



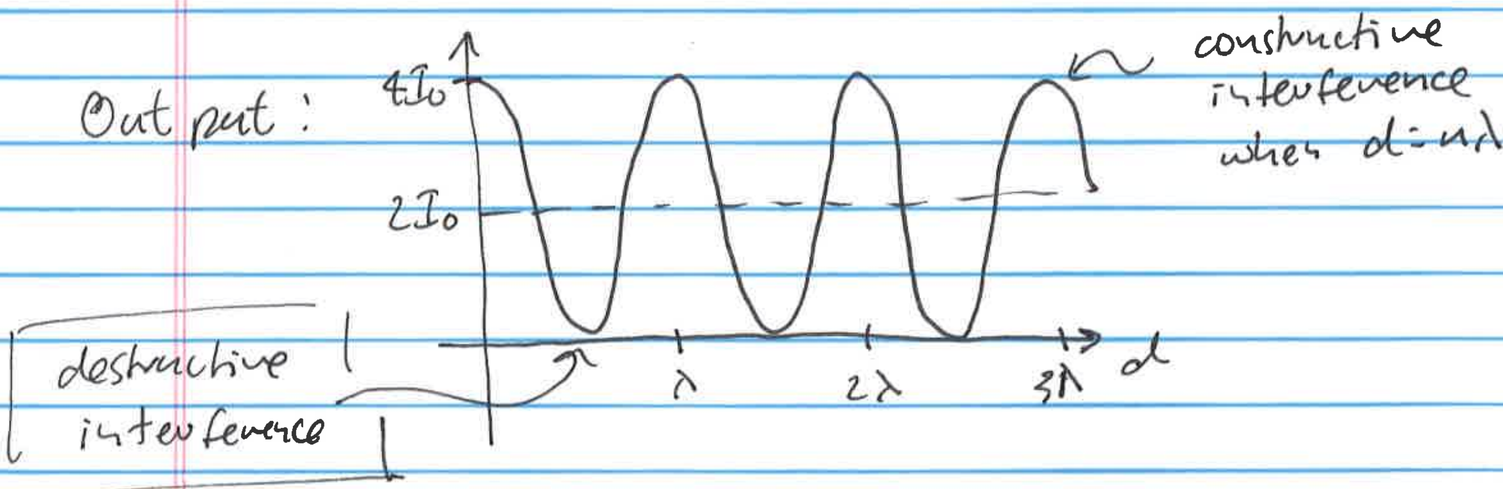
$$\mathcal{E}_1 = I_0^{1/2} e^{-ikz}, \quad \mathcal{E}_2 = I_0^{1/2} e^{-ik(z+d)}$$

$$I(r) = 2I_0 + I_0\left(e^{-ikz} e^{ik(z+d)} + e^{ikz} e^{-ik(z+d)}\right)$$

$$= 2I_0 + I_0\left(e^{ikd} + e^{-ikd}\right)$$

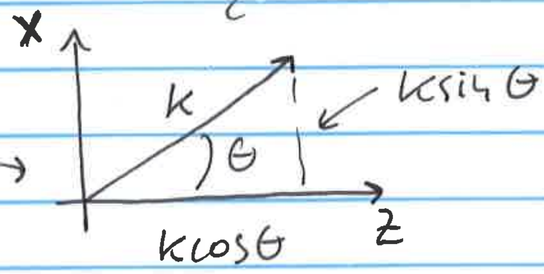
$$= 2I_0(1 + \cos kd) = 2I_0\left(1 + \cos\left(\frac{2\pi d}{\lambda}\right)\right)$$

Output:



Interference b/w a plane wave propagating along the z-axis & one at an angle θ to z

$$I_0^{\frac{1}{2}} e^{-ikz}$$



$$I_0^{\frac{1}{2}} e^{-i(k\sin\theta\, x + k\cos\theta\, z)}$$

~at $z=0$ (for example) the intensity pattern is $I = 2I_0(1 + \cos(k\sin\theta\, x))$

intensity varies along x-axis w/ period Λ where $\frac{2\pi}{\Lambda} = k\sin\theta$

$\Rightarrow \Lambda = 2\pi / k\sin\theta = \lambda / \sin\theta$

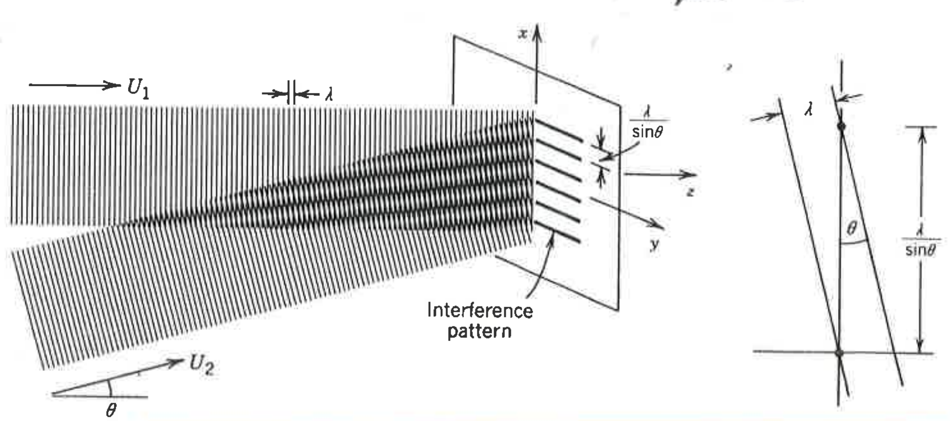


Figure 2.5-4

~ an interesting consequence of this type of periodic pattern is Bragg Reflection

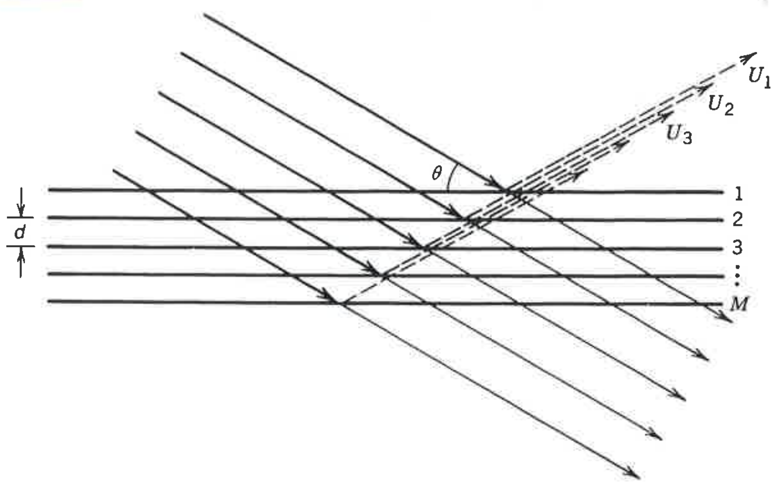
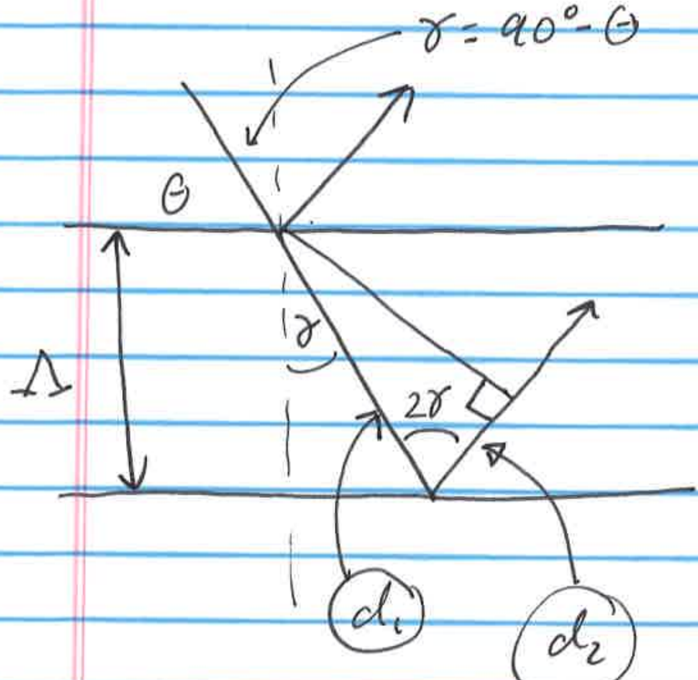


Figure 2.5-8 Reflection of a plane wave from M planes separated from each other by a distance d. The reflected waves interfere constructively and yield maximum intensity when the angle θ is the Bragg angle.

~ draw this in a different way



calculate the extra distance that the wavefronts have to travel:

$$d = d_1 + d_2$$

~ from our diagram

$$\frac{\Lambda}{d_1} = \cos\gamma \qquad \Rightarrow \qquad d_1 = \frac{\Lambda}{\cos\gamma}$$

and $$\frac{d_2}{d_1} = \cos 2\gamma \qquad \Rightarrow \qquad d_2 = d_1 \cos 2\gamma$$

$$= \frac{\Lambda \cos 2\gamma}{\cos\gamma}$$

~ total delay $d = d_1 + d_2$

$$= \frac{\Lambda}{\cos\gamma} + \frac{\Lambda \cos 2\gamma}{\cos\gamma}$$

$$= \frac{\Lambda(1+\cos 2\gamma)}{\cos\gamma} = \frac{\Lambda\, 2\cos^2\gamma}{\cos\gamma}$$

$$\Rightarrow \quad d = 2\Lambda\cos\gamma = 2\Lambda\sin\theta$$

~ constructive interference when $2\Lambda\sin\theta = \lambda$

$$\Rightarrow \boxed{\sin\theta = \lambda/2\Lambda}$$ Bragg angle