Fourier Optics

~ start by looking at Fourier Transforms

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\, e^{i2\pi\nu t}\, d\nu$$
$$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\nu t}\, dt$$

FT pair

Some properties:

(1) Linear $\quad F_1 + F_2 = \int_{-\infty}^{\infty} (f_1 + f_2)\, e^{-i2\pi\nu t}\, dt$

(2) Translation:

Consider $F(\nu) = \int f(t)\, e^{-i2\pi\nu t}\, dt$

F.T. of $f(t-\tau)$ is $\int_{-\infty}^{\infty} f(t-\tau)\, e^{-i2\pi\nu t}\, dt$

write $t' = t - \tau \Rightarrow t = t' + \tau$

$$\int f(t-\tau)\, e^{-i2\pi\nu t}\, dt = \int f(t')\, e^{-i2\pi\nu(t'+\tau)}\, dt'$$
$$= e^{-i2\pi\nu\tau} F(\nu)$$

$\Rightarrow$ shifting in time by τ introduces a phase factor $e^{-i2\pi\nu\tau}$

likewise $\int_{-\infty}^{\infty} F(\nu-\nu_0)\, e^{i2\pi\nu t}\, d\nu$

$= e^{i2\pi\nu_0 t} \int F(\nu')\, e^{i2\pi\nu' t}\, d\nu'$

$= e^{i2\pi\nu_0 t} f(t)$

(3) Convolution

~ the convolution b/w 2 functions f_1 + f_2 is

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$$

in Fourier Space: $F(\nu) = F_1(\nu) \times F_2(\nu)$

F.T. of $f(t)$

F.T.s of $f_1(t)$ and $f_2(t)$

Examples: $f_1(t) = e^{-\gamma t}$ ← exponential decay

$f_2(t) = e^{-t^2/\sigma^2}$ ← Gaussian (laser pulse?)

$$F_2(\nu) = \int_{-\infty}^{\infty} f_2(t)\, e^{-i2\pi\nu t}\, dt$$

$$= \sigma\sqrt{\pi}\, e^{-\pi^2\sigma^2\nu^2}$$ ← F.T. of a Gaussian is a Gaussian

$$F_1(\nu) = \int_{-\infty}^{\infty} f_1(t)\, e^{-i2\pi\nu t}\, dt$$

$$= \int_{-\infty}^{0} f_1(-t)\, e^{-i2\pi\nu t}\, dt + \int_{0}^{\infty} f_1(t)\, e^{-i2\pi\nu t}\, dt$$

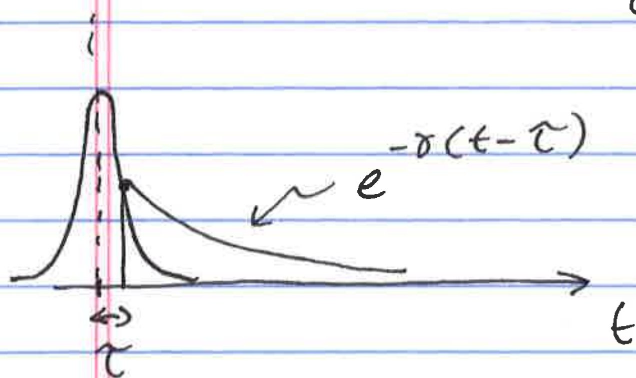
$$= \frac{1}{-2i\pi\nu + \gamma} + \frac{1}{2i\pi\nu + \gamma}$$

$$= \frac{2\gamma}{4\pi^2\nu^2 + \gamma^2}$$ ← F.T. of an exponential decay is a Lorentzian

Convolution of f_1 and f_2 ?

~ signal in T.R. experiments is a convolution of the IRF and decay

$$\underset{f_1(t)}{e^{-t^2/\sigma^2}} \qquad\qquad \underset{f_2(t)}{e^{-\gamma t}}$$



Total signal $f(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau$

$$= \frac{\sigma\sqrt{\pi}}{2}\, e^{-\gamma t} \times \left\{ 1 + \mathrm{Erf}[t/\sigma] \right\}$$

"Error Function"
~ not analytic

decay still $e^{-\gamma t}$ after pulse

~ we can also define spatial Fourier Transforms, i.e.

$$F(\nu_x) = \int_{-\infty}^{\infty} f(x)\, e^{i 2\pi \nu_x x}\, dx$$

$$f(x) = \int_{-\infty}^{\infty} F(\nu_x)\, e^{-i 2\pi \nu_x x}\, d\nu_x$$

. spatial frequency $\nu_x = k_x / 2\pi$

2D: $F(\nu_x, \nu_y) = \iint f(x,y)\, e^{i 2\pi (\nu_x x + \nu_y y)}\, dx\, dy$

seperates into 2 integrals if $f(x,y)$ is separable, otherwise you have "nested" integrals

Convolution in 2D:

$$f(x,y) = \iint f_1(x', y')\, f_2(x-x', y-y')\, dx'\, dy'$$

$$F(\nu_x, \nu_y) = F_1(\nu_x, \nu_y)\, F_2(\nu_x, \nu_y)$$

Back to optics...

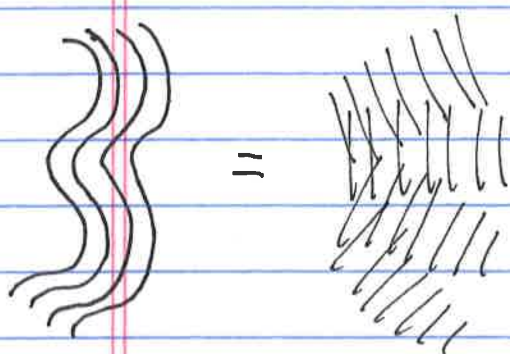
$k = \frac{2\pi}{\lambda} = (k_x^2 + k_y^2 + k_z^2)^{1/2}$

plane or "harmonic" waves

$$u(x,y,z) = A e^{-i(k_x x + k_y y + k_z z)}$$

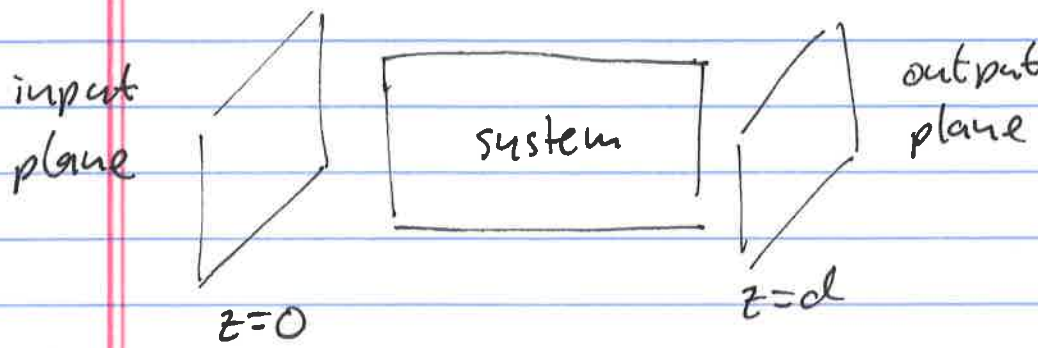
$$= A e^{-i2\pi(\nu_x x + \nu_y y + \nu_z z)}$$

~ any wave can be written as a sum of harmonic waves w/ different spatial frequencies



just need to find the coefficients "A_i" for each combination of frequencies

Goal: calculate how a wave changes as it travels through a system or propagates in free space



$u(x,y,0) = f(x,y)$ $\qquad$ $u(x,y,d) = g(x,y)$

~ we want to find $g(x,y)$ given $f(x,y)$

Note: $f(x,y) = \sum_i A_i e^{-i2\pi(\nu_x^i x + \nu_y^i y)}$

$$= \iint F(\nu_x, \nu_y)\, e^{-i2\pi(\nu_x x + \nu_y y)}\, d\nu_x\, d\nu_y$$

$f(x,y)$ & $F(\nu_x, \nu_y)$ ~ F.T. pair

tells us the amplitude of
the different spatial frequencies
that make up $u(x,y,0)$

~ calculate how $f(x,y)$ is transformed into $g(x,y)$ using either impulse-response function or transfer function

~ impulse-response function $h(x,y)$

$$\Rightarrow \quad g(x,y) = \iint_{-\infty}^{\infty} h(x-x', y-y')\, f(x',y')\, dx'\,dy'$$

$g(x,y)$ is a convolution of $h(x,y)$ + $f(x,y)$

$$\therefore \quad G(\nu_x, \nu_y) = H(\nu_x, \nu_y)\, F(\nu_x, \nu_y)$$

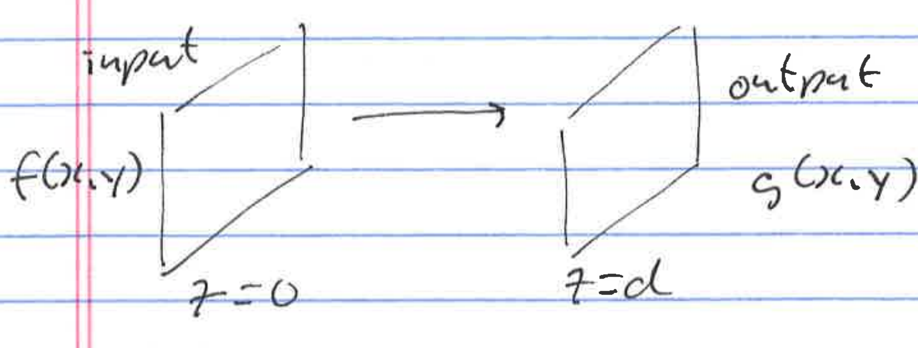
transfer function

$$= \int h(x,y)\, e^{i2\pi(\nu_x x + \nu_y y)}\, dx\,dy$$

~ for harmonic waves

$$g(x,y) = H(\nu_x, \nu_y)\, f(x,y)$$

~ look at the propagation of a plane wave in free space



@ $z=0$ $\quad f(x,y) = u(x,y,0) = A\, e^{-i2\pi(\nu_x x + \nu_y y)}$

$$= A\, e^{-i(k_x x + k_y y)}$$

@ $z=d$ $\quad g(x,y) = u(x,y,d) = A\, e^{-i(k_x x + k_y y + k_z d)}$

where $k_z = (k^2 - k_x^2 - k_y^2)^{1/2}$

$$k = 2\pi/\lambda \,, \quad k_x = 2\pi\nu_x \,, \quad k_y = 2\pi\nu_y$$

$$\Rightarrow k_z = 2\pi\left(1/\lambda^2 - \nu_x^2 - \nu_y^2\right)^{1/2}$$

Transfer Function: $H(\nu_x,\nu_y) = g(x,y)/f(x,y)$

$$= e^{-ik_z d}$$

$$\Rightarrow \boxed{H(\nu_x,\nu_y) = e^{-i2\pi(1/\lambda^2 - \nu_x^2 - \nu_y^2)^{1/2} d}}$$

2 regimes:

(i) $\nu_x^2 + \nu_y^2 \le 1/\lambda^2$

$\Rightarrow H(\nu_x,\nu_y) \sim$ complex, oscillating function of d

$\Rightarrow |H(\nu_x, \nu_y)|^2 = 1$

$\Rightarrow$ amplitude doesn't decay

c(ii) $\nu_x^2 + \nu_y^2 > 1/\lambda^2$ ~ high spatial frequencies

$$H(\nu_x, \nu_y) = e^{-i2\pi \underbrace{(1/\lambda^2 - \nu_x^2 - \nu_y^2)^{1/2}}_{\text{imaginary}} d}$$

$\Rightarrow H(\nu_x, \nu_y)$ decays w/ distance

"evanescent" waves

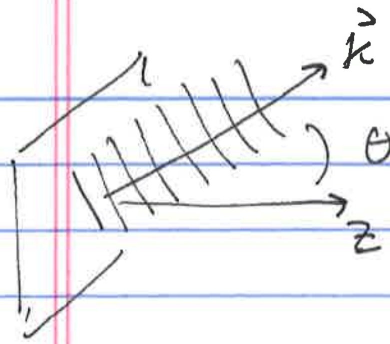
~ high spatial frequencies $\Leftrightarrow$ small distances

$\Rightarrow$ if we could collect high ν components we could improve spatial resolution (e.g. NSOM)

Fresnel Approximation:

$\nu_x^2 + \nu_y^2 \ll 1/\lambda^2$

~ waves that make small angles w.r.t z-axis



$\theta_x = \sin^{-1}(k_x/k)$

~ small angles $\quad \theta_x \simeq k_x/k = \dfrac{k_x\lambda}{2\pi} = \nu_x\lambda \ll 1$

"Paraxial Rays"

$$\Rightarrow \quad 2\pi\left(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2\right)^{1/2} d = \frac{2\pi d}{\lambda}\left(1 - \theta^2\right)^{1/2}$$

where $\quad \theta^2 = \lambda^2(\nu_x^2 + \nu_y^2) \ll 1$

$$\Rightarrow \quad \frac{2\pi d}{\lambda}\left(1 - \theta^2\right)^{1/2} \simeq \frac{2\pi d}{\lambda}\left(1 - \frac{\theta^2}{2}\right)$$

and $\quad H(\nu_x, \nu_y) = e^{-i\frac{2\pi d}{\lambda}(1-\theta^2/2)}$

$$= H_0\, e^{+i\frac{\pi d}{\lambda}\theta^2}$$

$$= H_0\, e^{i\pi\lambda d(\nu_x^2+\nu_y^2)}$$

$$e^{-i2\pi d/\lambda} = e^{-ikd}$$

Transfer function of free space, in the Fresnel approximation, is

$$H(\nu_x, \nu_y) = H_0\, e^{i\pi\lambda d(\nu_x^2 + \nu_y^2)}$$

impulse-response function

$$h(x,y) = \iint H(\nu_x, \nu_y)\, e^{-i2\pi(\nu_x x + \nu_y y)}\, d\nu_x\, d\nu_y$$

$$= h_0\, e^{-ik(x^2+y^2)/2d}$$

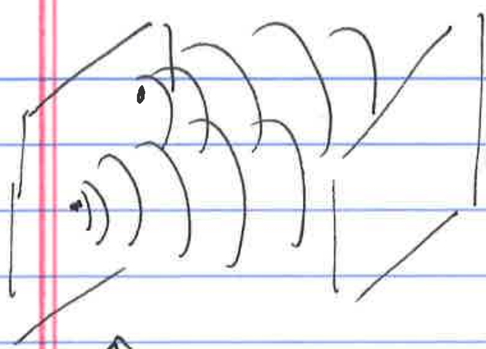
$$h_0 = \frac{i}{\lambda d}\, e^{-ikd}$$

"Paraboloid" wave

write an expression for $g(x,y)$ using $h(x,y)$

$$g(x,y) = \iint f(x',y')\, h(x-x', y-y')\, dx'\, dy'$$

Huygen's Principle: each pt. on the input plane generates a spherical wave, we add them all up at the output plane to get the total field



Fresnel approx = treat spherical waves (e^{-ikr}/r) as paraboloid waves

$$e^{-i\pi(x^2+y^2)/\lambda d}/d$$

Example: propagation of a Gaussian

$$f(x,y) = A e^{-(x^2+y^2)/w_0^2}$$

$$g(x,y) = h_0 \iint f(x',y') e^{\frac{-ik((x-x')^2+(y-y')^2)}{2z}} dx'dy'$$

$$h_0 = \frac{i}{\lambda z} e^{-ikz}$$

$$\int f(x') e^{-ik(x-x')^2/2z} dx' = \frac{\sqrt{2\pi}\, e^{-kx^2/(kw_0^2+2iz)}}{\sqrt{\frac{2}{w_0^2}+\frac{ik}{z}}}$$

Integral over y gives same term

$$\underbrace{\frac{i}{\lambda z}\frac{2\pi}{\left(\frac{2}{w_0^2}+\frac{ik}{z}\right)}}\; e^{-(x^2+y^2)\frac{k}{kw_0^2-2iz}}\; e^{-ikz}$$

$z_0 = kw_0^2/2$

$$\frac{ik}{\left(\frac{2z}{w_0^2}+ik\right)} = \frac{i(kw_0^2/2)}{(z+ikw_0^2/2)} = \frac{iz_0}{(z+iz_0)}$$

c/f 3.1-5

argument of exponential

$$\frac{-\rho^2 k}{kw_0^2-2iz} = \frac{-\rho^2 k}{k^2w_0^4+4z^2}*(kw_0^2+2iz)$$

Real part: $\dfrac{-\rho^2 k^2 w_0^2}{k^2w_0^4+4z^2} = \dfrac{-\rho^2}{w_0^2\left(1+\frac{4z^2}{k^2w_0^4}\right)}$

$$= \frac{-\rho^2}{w_0^2(1+(z/z_0)^2)}$$

Imaginary part: $\dfrac{-\rho^2 2izk}{k^2w_0^4+4z^2} = \dfrac{-ik\rho^2}{2z\left(\frac{k^2w_0^4}{4z^2}+1\right)}$

$$= -ik\rho^2 \Big/ 2z\left(1+\left(\tfrac{z_0}{z}\right)^2\right)$$

~ expression derived from convolution w/ $h(x,y)$ is equivalent to the Gaussian beam formula

i.e. $$\frac{i z_0}{z + i z_0} \, e^{-\rho^2/w(z)^2} \, e^{-ikz - ik\rho^2/2R(z)}$$

where $$w(z)^2 = w_0^2\left(1 + (z/z_0)^2\right)$$
$$R(z) = z\left(1 + (z_0/z)^2\right)$$