Maxwell's Equations:

- when there are no free charges or currents, Maxwell's equations can be written as

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad -(1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad -(2)$$

$$\left.\right\} \text{S.I. units}$$

$$\nabla . \vec{D} = 0 \quad -(3)$$

$$\nabla . \vec{B} = 0 \quad -(4)$$

$\vec{D}$ = electric diplacement / electric flux density

$\vec{B}$ = magnetic induction / magnetic flux density

$\vec{E}$ = electric field

$\vec{H}$ = magnetic field

$\vec{E}/\vec{D}$ and $\vec{H}/\vec{B}$ are connected through "constitutive relations"

~ equations (1) & (2) are that a time-varying magnetic field produces a spatially varying electric field (& vice-versa)

~ equations (3) & (4) are consequences that the flow of electric/magnetic energy into a volume must equal the flow out

Constituitive Relations:

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ $\quad$ $\vec{P}$ = polarization (average dipole per unit volume)

now $\vec{P} = \epsilon_0 \chi \vec{E}$ where χ = electric susceptibility

$\Rightarrow \vec{D} = \epsilon_0 (1 + \chi) \vec{E} = \epsilon \vec{E}$ $\quad$ ϵ = electric permittivity

$\epsilon_r = \epsilon / \epsilon_0$ = dielectric constant

Likewise, $\vec{B} = \mu_0 \vec{H} + \mu_0 \vec{M}$ $\quad$ $\vec{M}$ = magnetization

$= \mu \vec{H}$ $\quad$ μ = magnetic permeability

~ in free space we simply have

$$\vec{D} = \epsilon_0 \vec{E} \quad \text{and} \quad \vec{B} = \mu_0 \vec{H}$$

So Maxwell's Equations are:

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial H}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = 0$$

not in free space, substitute ϵ for ϵ_0 (μ_0 stays the same for non-magnetic materials)

Maxwell's Equations are vector equations

$\nabla \cdot \vec{E}$ = divergence of $\vec{E}$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

scalar (just a #!)

$\nabla\times\vec{E}$ ~ curl of $\vec{E}$

$$\nabla\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} \quad \leftarrow \text{vector}$$

$$= \hat{i}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{j}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

Some useful vector relations:

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$\Rightarrow \nabla^2\vec{A}$ = vector

we also have (for later)

$$\nabla\cdot(\vec{A}\times\vec{B}) = \mathbf{B}\cdot\nabla\times\vec{A} - A\cdot\nabla\times\vec{B}$$

vector ⊥ to
$\vec{A}$ and $\vec{B}$

Look at $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ - (1)

$\nabla \times \vec{H} = +\frac{\partial \vec{D}}{\partial t}$ - (2)

~ take the curl of b.s. of (1)

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{B}$$

now $\vec{B} = \mu \vec{H}$ $\Rightarrow \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \nabla \times \vec{H}$

now use (2) $\Rightarrow \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial^2 \vec{D}}{\partial t^2}$

$$\Rightarrow \underbrace{\nabla \times \nabla \times \vec{E}} = -\mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Simplify this using

$$\nabla \times \nabla \times \vec{E} = \nabla(\underbrace{\nabla . \vec{E}}_{=0}) - \nabla^2 \vec{E}$$

$\Rightarrow \boxed{\nabla^2 \vec{E} = \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2}}$ $\hookleftarrow$ the wave equation

Example: plane wave

$$\vec{E} = \vec{E}_0 e^{i(k_x x - \omega t)}$$

$$\nabla^2 \vec{E} = \vec{E}_0 \frac{\partial^2}{\partial x^2} e^{i(k_x x - \omega t)} = -k_x^2 \vec{E}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}$$

$\Rightarrow \nabla^2 \vec{E} = \mu \epsilon \dfrac{\partial^2 \vec{E}}{\partial t^2}$ becomes $k^2 \vec{E} = \mu \epsilon \omega^2 \vec{E}$

$\Rightarrow k^2 = \mu \epsilon \omega^2$ or $\left(\omega / k\right)^2 = 1/\mu\epsilon$

now $c = \omega / k$ $\Rightarrow$ $c^2 = 1/\mu\epsilon$

or $c = 1/\sqrt{\mu\epsilon}$

~ in free space (vacuum) $c_0 = 1/\sqrt{\mu_0 \epsilon_0}$

~ in a dielectric $c = \dfrac{1}{\sqrt{\mu\epsilon}} = \dfrac{1}{\sqrt{\mu_0 \epsilon_r \epsilon_0}} = \dfrac{c_0}{\sqrt{\epsilon_r}}$

$\Rightarrow n = \sqrt{\epsilon_r}$

Go back to a plane wave

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}.\vec{r} - \omega t)} \longleftarrow \text{same form for } \vec{H}$$

$$\left.\begin{aligned} \nabla.\vec{E} &= i\vec{k}.\vec{E} = 0 \\ \& \ \nabla.\vec{H} &= i\vec{k}.\vec{H} = 0 \end{aligned}\right\} \Rightarrow \vec{k} \perp \text{ to } \vec{E} \ \& \ \vec{H}$$

Furthermore $\nabla \times \vec{E} = i\vec{k} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$$= i\mu\omega \vec{H}$$

$$\Rightarrow \vec{k} \times \vec{E} = \mu\omega \vec{H}$$

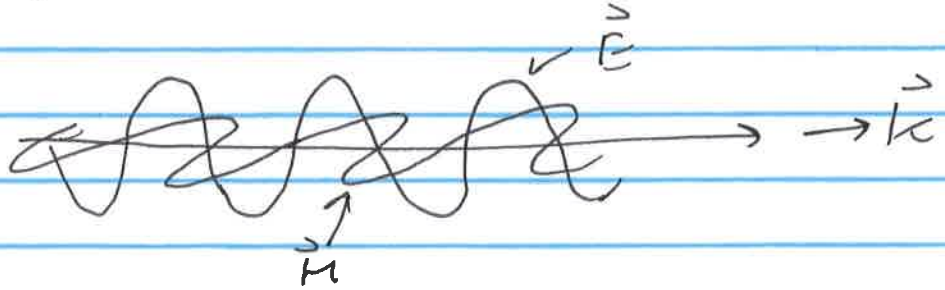
and $\nabla \times \vec{H} = i\vec{k} \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} = -i\epsilon\omega \vec{E}$

$$\Rightarrow \vec{k} \times \vec{H} = -\epsilon\omega \vec{E}$$

Collect together: $\vec{k}.\vec{E} = 0 \qquad \vec{k} \times \vec{E} = \mu\omega\vec{H}$

$\vec{k}.\vec{H} = 0 \qquad \vec{k} \times \vec{H} = -\epsilon\omega\vec{E}$

$\Rightarrow \vec{E}, \vec{H} \ \& \ \vec{k}$ are all $\perp$ to each other



Note: $|\vec{k} \times \vec{E}_0| = k|\vec{E}_0| = \omega\mu|\vec{H}_0|$

$$\Rightarrow \frac{|E_0|}{|H_0|} = \frac{\omega\mu}{k} = \frac{\mu}{\sqrt{\mu\epsilon}} = \sqrt{\frac{\mu}{\epsilon}}$$

magnetic & electric fields have a different magnitudes

In vacuum $\left|\frac{E_0}{H_0}\right| = \sqrt{\frac{\mu_0}{\epsilon_0}} = c_0\,\mu_0$

values $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$\mu_0 = 4\pi \times 10^{-7}$ H/m

$\Rightarrow \left|\frac{E_0}{H_0}\right| = 377\ \Omega$ ⇐ "Impedance of free space"

Energy contained in elec. & mag. fields

$u_e = \frac{1}{2}\vec{E}\cdot\vec{D} = \frac{\epsilon}{2}|E_0|^2$

$u_m = \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{\mu}{2}|H_0|^2$

Total Energy: $u = \frac{1}{2}\left(\epsilon|E_0|^2 + \mu|H_0|^2\right)$

Note: $$\frac{\frac{1}{2}\epsilon_0|E_0|^2}{\frac{1}{2}\mu_0|H_0|^2} = \frac{\epsilon_0}{\mu_0}\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^2 = 1$$

equal energy in the elec. & mag. fields even though $|E_0| >> |H_0|$

Poynting Vector

Consider $\vec{E}\times\vec{H}$ ← vector ⊥ $\vec{E}$ and $\vec{H}$

$$\nabla.(\vec{E}\times\vec{H}) = \vec{H}.\nabla\times\vec{E} - \vec{E}.\nabla\times\vec{H}$$ ← one of our vector relations.

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\nabla\times\vec{H} = \epsilon\frac{\partial\vec{E}}{\partial t}$$

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\mu\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} - \epsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\left(\frac{\mu\vec{H}\cdot\vec{H}}{2} + \frac{\epsilon\vec{E}\cdot\vec{E}}{2}\right)$$

$$\frac{\partial\vec{E}\cdot\vec{E}}{\partial t} = \vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\cdot\vec{E} = 2\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\partial}{\partial t}\underbrace{\left(\frac{\mu|H_0|^2}{2} + \frac{\epsilon|E_0|^2}{2}\right)}_{\text{total energy } u}$$

$$\Rightarrow \boxed{\nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\partial u}{\partial t}}$$

$\vec{E}\times\vec{H}$ is a vector that represents the flow of EM energy

Poynting vector $\vec{S} = \vec{E}\times\vec{H}$ ← $\vec{S} \perp \vec{E}$ & $\vec{H}$

& ∥ to $\vec{k}$

~ lets calculate the time averaged Poynting vector for harmonic fields

$\vec{E} = \vec{E}_0 e^{-i\omega t} + c.c.$ / $\vec{H} = \vec{H}_0 e^{-i\omega t} + c.c.$ ← $\vec{H}_0$ & $\vec{E}_0$ can be complex

$$\langle S \rangle = \langle Re(\vec{E}) \times Re(\vec{H}) \rangle$$

$$Re(\vec{E}) = \vec{E}_{0r} \cos\omega t - \vec{E}_{0i} \sin\omega t$$

$$Re(\vec{H}) = \vec{H}_{0r} \cos\omega t - \vec{H}_{0i} \sin\omega t$$

$$Re(\vec{E}) \times Re(\vec{H}) = \vec{E}_{0r} \times \vec{H}_{0r} \cos^2\omega t + \vec{E}_{0i} \times \vec{H}_{0i} \sin^2\omega t - \cos\omega t \sin\omega t \left(\vec{E}_{0r} \times \vec{H}_{0i} + \vec{E}_{0i} \times \vec{H}_{0r} \right)$$

Now $\langle \cos^2\omega t \rangle = \langle \sin^2\omega t \rangle = 1/2$

$\langle \cos\omega t \sin\omega t \rangle = 0$

$$\Rightarrow \langle Re(\vec{E}) \times Re(\vec{H}) \rangle = \frac{1}{2} \left(\vec{E}_{0r} \times \vec{H}_{0r} + \vec{E}_{0i} \times \vec{H}_{0i} \right)$$

Note: $$\mathrm{Re}(\vec{E}\times\vec{H}^*) = \mathrm{Re}\left((\vec{E}_{0r} + i\vec{E}_{0i}) \times (\vec{H}_{0r} - i\vec{H}_{0i})\right)$$
$$= \vec{E}_{0r}\times\vec{H}_{0r} + \vec{E}_{0i}\times\vec{H}_{0i}$$

$$\Rightarrow \langle S\rangle = \langle \mathrm{Re}(\vec{E})\times\mathrm{Re}(\vec{H})\rangle = \tfrac{1}{2}\,\mathrm{Re}\langle \vec{E}\times\vec{H}^*\rangle$$

This is how $\langle S\rangle$ is calculated in COMSOL (e.g.)

$$\vec{S} = \tfrac{1}{2}\,\mathrm{Re}\{\vec{E}\times\vec{H}^*\}$$

now $\vec{k}\times\vec{E} = \omega\mu\vec{H}$

$$\Rightarrow \vec{S} = \tfrac{1}{2}\,\mathrm{Re}\left\{\frac{\vec{E}\times\vec{k}^*\times\vec{E}^*}{\omega\mu^*}\right\}$$

$$\vec{E}\times\vec{k}\times\vec{E}^* = (\vec{E}\cdot\vec{E}^*)\vec{k}^* - \underbrace{(\vec{E}\cdot\vec{k}^*)}_{0}\vec{E}^*$$

$$\Rightarrow \vec{S} = \mathrm{Re}\left\{\frac{|E_0|^2}{2\omega\mu^*}\vec{k}^*\right\}$$ $\longleftarrow$ if $\mu^* = \mu$

For a non-absorbing medium

$$\vec{k} = k\hat{e} = \omega\sqrt{\mu\epsilon}\,\hat{e}$$

$\hat{e}$ ~ unit vector along the direction of $\vec{k}$

$$\Rightarrow \vec{S} = \frac{1}{2} Re\left\{ \frac{|E_0|^2}{\mu\omega} \omega\sqrt{\epsilon\mu}\,\hat{e} \right\}$$

$$= \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\hat{e}$$ ~ $\vec{S}$ points along the direction of $\vec{k}$

Now, $\epsilon = \epsilon_r \epsilon_0$ ϵ_r = dielectric constant (relative permittivity)

$\mu = \mu_r \mu_0$ μ_r = relative permeability

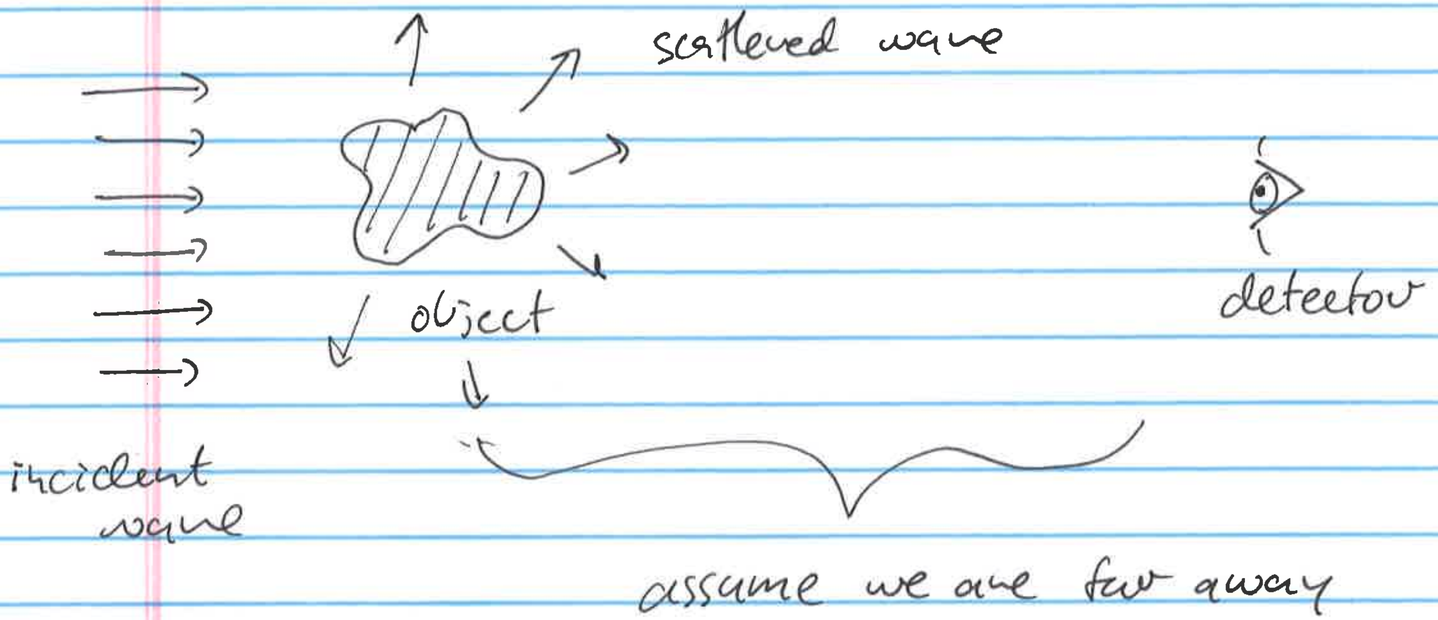
~ for non-magnetic materials $\mu_r = 1$

$$\Rightarrow \vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon_r\epsilon_0}{\mu_0}}\,|E_0|^2\,\hat{e}$$

or $|S| = \frac{1}{2}\frac{|E_0|^2}{Z_0}$

$Z_0 = 377\Omega$

Finally, lets consider the following situation



$$\underset{\substack{\uparrow \\ \text{total} \\ \text{field}}}{\vec{E}_{tot}} = \underset{\substack{\uparrow \\ \text{incident} \\ \text{field}}}{\vec{E}_i} + \underset{\substack{\nwarrow \\ \text{scattered} \\ \text{field}}}{\vec{E}_s}$$

$$\vec{S}_{tot} = \frac{1}{2} Re\left(\vec{E}_{tot} \times \vec{H}_{tot}^{*} \right)$$

$$= \frac{1}{2} Re\left((\vec{E}_i + \vec{E}_s) \times (\vec{H}_i^{*} + \vec{H}_s^{*}) \right)$$

$$= \frac{1}{2} Re\left[\vec{E}_i \times \vec{H}_i^{*} + \vec{E}_s \times \vec{H}_s^{*} + \vec{E}_i \times \vec{H}_s^{*} + \vec{E}_s \times \vec{H}_i^{*} \right]$$

~ write this as: $\vec{S}_{tot} = \vec{S}_i + \vec{S}_s + \vec{S}_{ext}$

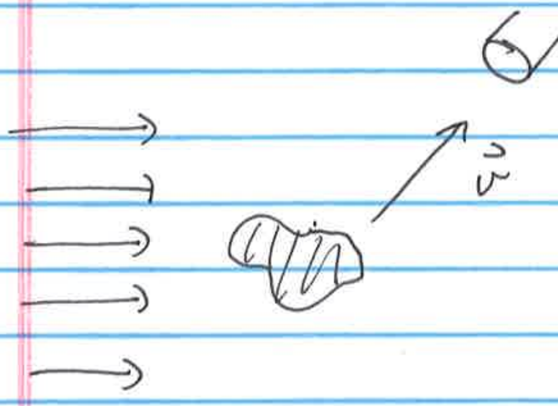
$$\vec{S}_i = \frac{1}{2} Re(\vec{E}_i \times \vec{H}_i^*)$$ ← incident field

$$\vec{S}_s = \frac{1}{2} Re(\vec{E}_s \times \vec{H}_s^*)$$ ← scattered field

$$\vec{S}_{ext} = \frac{1}{2} Re(\underbrace{\vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^*}_{})$$

cross-term b/w the incident & scattered fields

~ consider a detector in the direction $\vec{v}$ from the sample



← power detected is given by the integral of $\vec{S}$ over the surface of the detector

$$\mathcal{U} = \int_D \vec{S}_{tot} \cdot \hat{e}_v \, dA$$

$\vec{S}\cdot\hat{e}_v$ ~ component of $\vec{S}$ in the direction $\vec{v}$ ($\hat{e}_v$ unit vector $\parallel \vec{v}$)

$$U = \int_D \vec{S}_i\cdot\hat{e}_v\, dA + \int_D \vec{S}_s\cdot\vec{e}_v\, dA + \int_D \vec{S}_{ext}\cdot\hat{e}_v\, dA$$

$$= U_i + U_s + U_{ext}$$

~ position detector at an angle s.t. $E_i, H_i \simeq 0$

$\Rightarrow U = U_s$ ← only detect scattered light

~ position detector in the incident beam

$\Rightarrow U = U_i + U_s + U_{ext}$

note $E_i, H_i >> E_s, H_s$ $\Rightarrow U \simeq U_i + U_{ext}$

$$U_{ext} = \int_D \frac{1}{2} Re\left\{ \vec{E}_i \times \vec{H}_s^* + \vec{E}_s \times \vec{H}_i^* \right\}\cdot\hat{e}_r\, dA$$

"Optical theorem" ~ extinction arises from interference b/w incident light and scattered wave