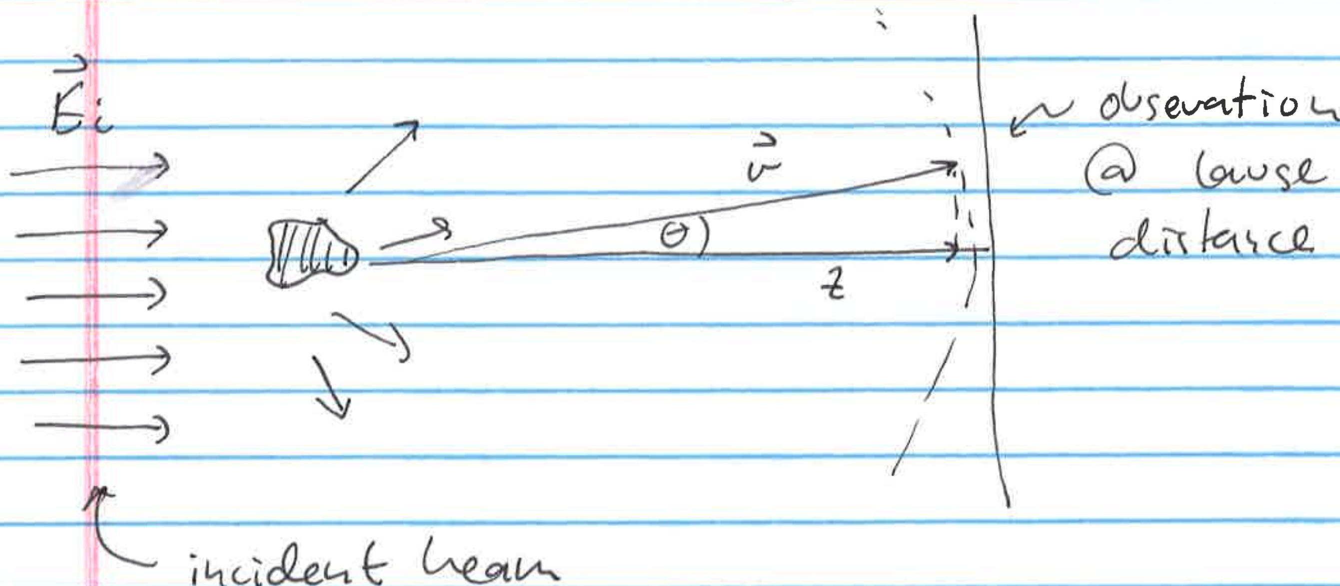
Optical theorem & extinction of small particles



$$\vec{E}_i = \vec{E}_0 \, e^{-ikz} \, e^{i\omega t}$$

scattered field = spherical wave modified by the scattering function for the particle $S(\theta,\phi)$

$$\Rightarrow \vec{E}_s = S(\theta,\phi) \, \frac{i e^{-ikr}}{\lambda r} \, e^{i\omega t} \, \vec{E}_0$$

$$\simeq S(\theta,\phi) \, \frac{i e^{-ikz}}{\lambda z} \, e^{-i(x^2+y^2)/2z} \, e^{i\omega t} \, \vec{E}_0$$

Fresnel approximation

$$\vec{E}_{tot} = \vec{E}_i + \vec{E}_s$$

$$= \left[1 + \frac{S(\theta,\phi)\, i\, e^{-i(x^2+y^2)/(2z)}}{\lambda z} \right] \vec{E}_0 e^{-ikz} e^{i\omega t}$$

$$I_{tot} = \vec{E}_{tot} \vec{E}_{tot}^*$$

$$= I_0 \left[1 + \frac{i S(\theta,\phi) e^{-i(\,)/(2z)}}{\lambda z} \right] \left[1 - \frac{i S(\theta,\phi)^* e^{i(\,)/(2z)}}{\lambda z} \right]$$

$$\simeq I_0 \left[1 + \frac{1}{\lambda z} \left(i S(\theta,\phi) e^{-i(\,)/(2z)} - i S(\theta,\phi) e^{i(\,)/(2z)} \right) \right]$$

dropped the term that goes as $S(\theta,\phi)^2$

$$\Rightarrow I_{tot} = I_0 \left[1 + \frac{2}{\lambda z} \mathrm{Re} \left[i S(\theta,\phi) e^{-i(\,)/(2z)} \right] \right]$$

need to integrate over the area of the detector

2nd term = extinction, light removed by interference b/w the incident and scattered beams

Note, when the detector is placed in the path of the incident beam we have

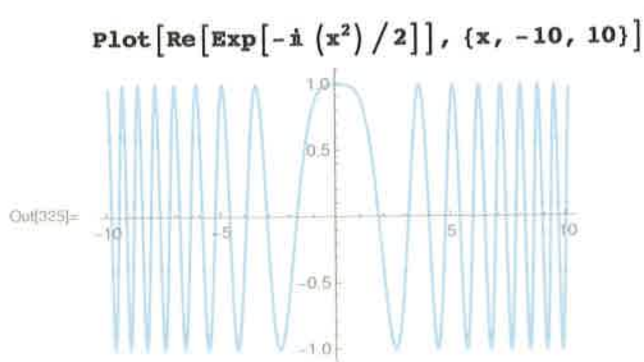
$\theta, \phi \simeq 0 \qquad S(\theta,\phi) = S(0)$ ← light scattered in the forward direction

$$\Rightarrow \frac{\Delta I}{I_0} = \frac{2}{\lambda z} \, Re\left(i S(0) \iint_D e^{-ik(x^2+y^2)/2z} \, dx\,dy \right)$$

integral over the area of the detector

~ assume detector is large enough that we can extend limits of integration to $\pm\infty$, but not too large that the Fresnel approx^n breaks down (?)

How?



oscillating part averages out

only central part contributes

$$\iint_{-\infty}^{\infty} e^{-ik(x^2+y^2)/(2z)}\, dx\,dy = \frac{2\pi z}{ik}$$

$$\Rightarrow \frac{\Delta I}{I_0} = \frac{2}{\lambda z} \mathrm{Re}\left[i S(0) \frac{2\pi z}{ik} \right] = 2\,\mathrm{Re}[S(0)]$$

~ now, compare the expression for the scattered field to the scattered wave from a radiating dipole

Dipole: $$E_s = \frac{k^2 p \cos\theta}{4\pi \varepsilon_m} \frac{e^{-ikr}}{r} e^{i\omega t}$$ ← far field

$$= \frac{\lambda k^2 p \cos\theta}{i 4\pi \varepsilon_m} \underbrace{\frac{i e^{-ikr}}{\lambda r}}_{\text{spherical wave}} e^{i\omega t}$$

$$\vec{p} = \varepsilon_m \alpha \vec{E}_0 \qquad \alpha = \text{polarizability}$$

~ for scattering in the forward direction

$$E_s = \frac{\lambda k^2 p}{i 4\pi \varepsilon_m} \frac{i e^{-ikr}}{\lambda r} e^{i\omega t}$$

$$\Rightarrow S(0) = \frac{\lambda k^4 (p/E_0)}{i 4\pi \varepsilon_m} = \frac{\lambda k^2 \alpha}{i 4\pi} = -i \frac{\lambda k^2 \alpha}{4\pi}$$

$$\frac{\Delta I}{I_0} = \underbrace{C_{ext}}_{\text{extinction cross-section}} = 2\,Re\{S(0)\} = -\frac{\lambda}{2\pi} k^2 \, Re\{i\alpha\}$$

$$\Rightarrow \boxed{C_{ext} = -k\, Im\{\alpha\}}$$

← this is the amt. of light extinguished from the incident beam

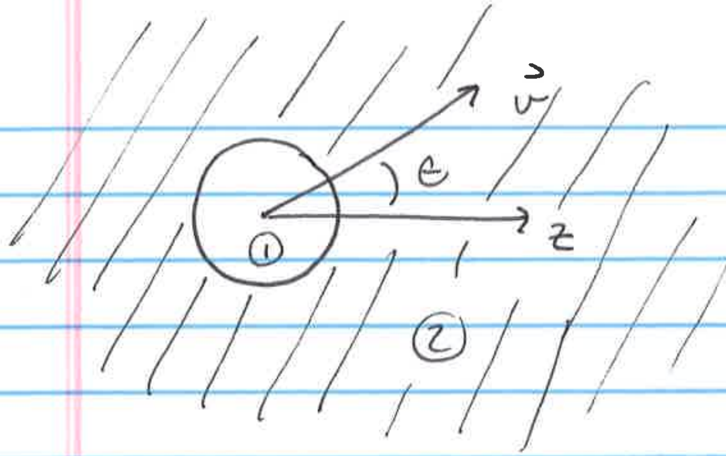
units are m^2

~ now all we have to do is work out an expression for the polarizability

~ consider a particle much smaller than the wavelength of light (radius $a \ll \lambda$)

~ field is constant over the particle

"Quasi-Static" limit use electrostatics



Field inside

$\vec{E}_1 = -\nabla V_1$

Field outside

$\vec{E}_2 = -\nabla V_2$

V_1 & V_2 are potentials (scalar quantities)

~ at $r = a$ $\quad V_1 = V_2$ and $\epsilon_1 \dfrac{dV_1}{dr} = \epsilon_2 \dfrac{dV_2}{dr}$

continuity in V and $\vec{D} = \epsilon \vec{E}$

~ also have that $\displaystyle\lim_{r \to \infty} V_2 = -E_0 z = -E_0 r \cos\theta$

check

$E_z = -\dfrac{\partial V}{\partial z} = E_0$

Try: $V_1 = A E_0 r \cos\theta$

$V_2 = -E_0 r \cos\theta + \dfrac{B \cos\theta}{r^2}$

$\Rightarrow \dfrac{dV_1}{dr} = A E_0 \cos\theta$ & $\dfrac{dV_2}{dr} = -E_0 \cos\theta - \dfrac{2B\cos\theta}{r^3}$

at $r = a$ $\quad V_1 = V_2$

$$\Rightarrow A E_0 a \cos\theta = -E_0 a \cos\theta + \frac{B\cos\theta}{a^2}$$

$$\Rightarrow A E_0 = -E_0 + \frac{B}{a^3} \quad - (1)$$

Also have $\quad \epsilon_1 \frac{dV_1}{dr} = \epsilon_2 \frac{dV_2}{dr}$

$$\Rightarrow A \epsilon_1 E_0 \cos\theta = -\epsilon_2 E_0 \cos\theta - \frac{2B\cos\theta}{a^3}$$

$$\Rightarrow A \epsilon_1 E_0 = -\epsilon_2 E_0 - \frac{2\epsilon_2 B}{a^3} \quad - (2)$$

~substitute (1) into (2)

$$\epsilon_1\left(-E_0 + \frac{B}{a^3}\right) = -\epsilon_2 E_0 - \frac{2\epsilon_2 B}{a^3}$$

$$\Rightarrow B\left(\frac{\epsilon_1}{a^3} + \frac{2\epsilon_2}{a^3}\right) = (\epsilon_1 - \epsilon_2) E_0$$

$$\Rightarrow \boxed{B = \frac{a^3(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)} E_0}$$

now $\quad A = -1 + \dfrac{B}{E_0 a^3} = -1 + \dfrac{(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)}$

$$= \frac{-3\epsilon_2}{(\epsilon_1 + 2\epsilon_2)}$$

Sol^{n} (for V) $\quad V_1 = \dfrac{-3\epsilon_2}{(\epsilon_1 + 2\epsilon_2)} E_0 r \cos\theta$

$$V_2 = -E_0 r \cos\theta + \frac{a^3(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)} \times \frac{E_0 \cos\theta}{r^2}$$

First term is from the applied field

pot^{l} of a dipole

For a dipole: $\quad V = \dfrac{p\cos\theta}{4\pi\epsilon_2 r^2}$

$$\Rightarrow \quad \frac{p}{4\pi\epsilon_2} = \frac{a^3(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)} E_0$$

$$\Rightarrow \quad \vec{p} = 4\pi\epsilon_2 a^3 \frac{(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)} \vec{E}_0 = \epsilon_2 \alpha \vec{E}_0$$

~ so by comparing to the result for a dipole (again, far field)

$$\boxed{\alpha = 4\pi a^3 \frac{(\epsilon_1 - \epsilon_2)}{(\epsilon_1 + 2\epsilon_2)}}$$

polarizability of a sphere w/ $a \ll \lambda$

$$C_{ext} = k\, \mathrm{Im}\{\alpha\} = 4\pi a^3 k\, \mathrm{Im}\left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right]$$

ϵ_1 ~ dielectric constant of the sphere

ϵ_2 ~ dielectric constant of the surrounding medium

get a large polarizability (large extinction) when $\epsilon_1 \rightarrow -2\epsilon_2$

~ for a non-absorbing medium $\epsilon_2 > 0$

$\Rightarrow \epsilon_1 < 0$ ~ metal

~ look at the fields inside & outside the sphere

~ best to work in cartesian co-ords

$$V_1 = \frac{-3\epsilon_2}{\epsilon_1 + 2\epsilon_2} E_0 z$$

$$V_2 = -E_0 z + \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_1 + 2\epsilon_2} \frac{a^3 z E_0}{(x^2+y^2+z^2)^{3/2}}$$

~ look @ the z-component

$$E_{1z} = \frac{+3\epsilon_2 E_0}{\epsilon_1 + 2\epsilon_2} \Rightarrow \frac{E_{1z}}{E_0} = \frac{+3\epsilon_2}{\epsilon_1 + 2\epsilon_2}$$

$$\frac{E_{2z}}{E_0} = +1 - \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_1 + 2\epsilon_2} \frac{a^3}{(x^2+y^2+z^2)^{3/2}}$$

$$+ \frac{3(\epsilon_1 - \epsilon_2)}{\epsilon_1 + 2\epsilon_2} \frac{a^3}{(x^2+y^2+z^2)^{3/2}} \frac{z^2}{(x^2+y^2+z^2)}$$

fields enhanced at $\epsilon_1 \to -2\epsilon_2$

$$\frac{E_{zz}}{E_0} = 1 - \frac{(\epsilon_1 - \epsilon_2)}{\epsilon_1 + 2\epsilon_2} \frac{a^3}{(x^2+y^2+z^2)^{3/2}} \left(1 - \frac{3z^2}{(x^2+y^2+z^2)}\right)$$

$\sim \left(\frac{a}{r}\right)^3 \Rightarrow$ field enhancement decreases w/ distance, decay scales as <u>radius</u>

Back to the cross-section

write $C_{ext} = (\pi a^2)\, 4ka\, \mathrm{Im}\left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right]$

$\pi a^2 =$ area

$\Rightarrow$ efficiency $Q = \frac{C_{ext}}{\pi a^2} = 4ka\, \mathrm{Im}\left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right]$

~ for a metal ϵ_1 has real & imaginary parts $\epsilon_1 = \epsilon_1' + i\epsilon_1''$

$$\Rightarrow \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} = \frac{\epsilon_1' + i\epsilon_1'' - \epsilon_2}{\epsilon_1' + i\epsilon_1'' + 2\epsilon_2}$$

~ multiply top & bottom by c.c. of denominator

$$\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2} = \frac{\epsilon_1' + i\epsilon_1'' - \epsilon_2}{\epsilon_1' + 2\epsilon_2 + i\epsilon_1''} \times \frac{\epsilon_1' + 2\epsilon_2 - i\epsilon_1''}{\epsilon_1' + 2\epsilon_2 - i\epsilon_1''}$$

$$= \frac{(\epsilon_1' - \epsilon_2)(\epsilon_1' + 2\epsilon_2) + \epsilon_1''^2 - i\epsilon_1''(\epsilon_1' - \epsilon_2 - \epsilon_1' - 2\epsilon_2)}{(\epsilon_1' + 2\epsilon_2)^2 + \epsilon_1''^2}$$

$$\mathrm{Im}\left[\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + 2\epsilon_2}\right] = \frac{3\epsilon_2 \epsilon_1''}{(\epsilon_1' + 2\epsilon_2)^2 + \epsilon_1''^2}$$

$$\Rightarrow C_{ext} = 4\pi a^3 k \, 3\epsilon_2 \frac{\epsilon_1''}{(\epsilon_1' + 2\epsilon_2)^2 + \epsilon_1''^2}$$

use $V = \frac{4\pi a^3}{3}$ $\quad k = \frac{2\pi n}{\lambda} = \frac{2\pi\sqrt{\epsilon_2}}{\lambda}$

$$\Rightarrow C_{ext} = \frac{18\pi V}{\lambda} \epsilon_2^{3/2} \frac{\epsilon_1''}{(\epsilon_1' + 2\epsilon_2)^2 + \epsilon_1''^2}$$

Mie theory expression for extinction in the "Quasi-Static" limit ($a \ll \lambda$)

cf $\quad C_{ext} = \frac{18\pi V}{\lambda} \epsilon_m^{3/2} \frac{\epsilon_i''}{(\epsilon_i' + 2\epsilon_m)^2 + \epsilon_i''^2}$

to general Lorentzian $\quad \frac{\Gamma/2}{(\omega-\omega_0)^2 + (\Gamma/2)^2}$

ω_0 ~ resonance frequency

Γ = FWHM = decay rate

Mie theory: resonance when $\epsilon_i' = -2\epsilon_m$

& linewidth determined by ϵ_i''

⇒ ϵ_i' determines the position of the "plasmon resonance"

ϵ_i'' determines the width

__Note__: In the quasi-static limit extinction = absorp^n

$C_{ext} \propto V$ ← magnitude of the ext^n cross-section ↑ w/ V, but shape of spectrum doesn't change