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Stability and Large-Time Behavior on 3D Incompressible MHD Equations with Partial Dissipation Near a Background Magnetic Field

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Abstract

Physical experiments and numerical simulations have observed a remarkable stabilizing phenomenon: a background magnetic field stabilizes and dampens electrically conducting fluids. This paper intends to establish this phenomenon as a mathematically rigorous fact on a magnetohydrodynamic (MHD) system with anisotropic dissipation in $\mathbb{R}^3$. The velocity equation in this system is the 3D Navier–Stokes equation with dissipation only in the x_1-direction, while the magnetic field obeys the induction equation with magnetic diffusion in two horizontal directions. We establish that any perturbation near the background magnetic field (0, 1, 0) is globally stable in the Sobolev setting $H^3(\mathbb{R}^3)$. In addition, explicit decay rates in $H^2(\mathbb{R}^3)$ are also obtained. For when there is no presence of a magnetic field, the 3D anisotropic Navier–Stokes equation is not well understood and the small data global well-posedness in $\mathbb{R}^3$ remains an intriguing open problem. This paper reveals the mechanism of how the magnetic field generates enhanced dissipation and helps to stabilize the fluid.

Mathematics Subject Classification: 35A05 35Q35 76D03

1. Introduction

This paper deals with the stability and large-time behavior problem on a system of 3D anisotropic MHD equations near a background magnetic field. To shed some light on the potential difficulties of this problem, we briefly review several facts about the behavior of solutions to the Euler and the anisotropic Navier–Stokes equations.

It is well-known that solutions of the incompressible Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla P, \\ \nabla \cdot u = 0 \end{cases}$$

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can grow rather rapidly in time. In fact, Kiselev and Sverak were able to construct a vorticity solution of the 2D Euler equations in a disk whose gradient grows double exponentially in time [54]. In the periodic setting, an example of Zlatos shows that the vorticity gradient can grow at least exponentially [110]. Choi and Jeong obtained linear in time growth for the vorticity gradient for certain smooth and compactly supported initial vorticity in $\mathbb{R}^2$ [22]. Classical solutions to the 3D Euler equations could develop finite-time singularities [20,38]. Many more results in this direction can be found in a review paper by Drivas and Elgindi [35]. As a special consequence, perturbations governed by the Euler equations near the trivial solution are generally not stable. How much dissipation does one really need in order to achieve the stability? Adding the full Laplacian dissipation is certainly sufficient. As demonstrated by Schonbek and others (see, example, [75–77,88]), solutions of the Navier–Stokes equations

$$\begin{cases}\partial_t u + (u\cdot\nabla)u = -\nabla P + \mu\Delta u,\\ \nabla\cdot u = 0\end{cases}$$

are asymptotically stable and decay in time with explicit decay rates. When the dissipation is anisotropic and only in two directions, the Navier–Stokes equations become

$$\begin{cases}\partial_t u + (u\cdot\nabla)u = -\nabla P + \mu\Delta_h u,\\ \nabla\cdot u = 0,\end{cases} \tag{1.1}$$

where $\Delta_h = \partial_1^2 + \partial_2^2$ is the horizontal Laplacian. Due to its physical applications and special mathematical properties, (1.1) has attracted considerable interest and an array of beautiful small data global well-posedness results have been obtained (see, example, [18,19,50,65,69,70,103,104]). New approaches have very recently been developed to tackle the large-time behavior problem and explicit decay rates have been extracted for (1.1) (see [52,96]). If we further reduce the dissipation to be in just one direction, the resulting 3D anisotropic Navier–Stokes equations

$$\begin{cases}\partial_t u + (u\cdot\nabla)u = -\nabla P + \mu\partial_1^2 u, \quad x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u = 0\end{cases} \tag{1.2}$$

are not well-understood. In particular, the small data global well-posedness problem remains open. In addition, very little is known on the stability properties and the large-time behavior.

This paper focuses on a system of the 3D MHD equations with anisotropic dissipation

$$\begin{cases}\partial_t u + (u\cdot\nabla)u = -\nabla P + \mu\partial_1^2 u + (B\cdot\nabla)B, \quad x\in\mathbb{R}^3,\ t>0,\\[2ex] \partial_t B + (u\cdot\nabla)B = \eta\,\Delta_h B + (B\cdot\nabla)u,\\[2ex] \nabla\cdot u = \nabla\cdot B = 0,\end{cases} \tag{1.3}$$

with the initial data

$$u(x,0) = u_0, \quad B(x,0) = B_0.$$

Here $u = (u_1, u_2, u_3)^\top$, $B = (B_1, B_2, B_3)^\top$ and P represent the velocity field of the fluid, the magnetic field and the scalar pressure, respectively. The constants $\mu > 0$ and $\eta > 0$ are the viscosity coefficient and the magnetic diffusivity. The MHD system (1.3) focused here is relevant in the modeling of reconnecting plasmas (see, example, [23–25,72]). In fusion plasmas there is extreme anisotropy due to the high temperature and large magnetic field strength. This causes diffusive processes, heat diffusion and energy/momentum loss due to viscous friction, to effectively be aligned with the magnetic field lines. This alignment leads to different values for the respective diffusive coefficients in the magnetic field (see, example, [31,46,66,84]).

The motivation for studying (1.3) comes from two distinct sources. The first is the stabilizing phenomenon observed in physical experiments involving electrically conducting fluids. The experiments exhibit a remarkable phenomenon: a background magnetic field actually stabilizes and damps turbulent MHD fluids (see, example, [2–4,26–28,43,44]). We intend to establish this phenomenon as a mathematically rigorous fact on (1.3). The second is to initiate new strategies and develop innovative tools for stability and large-time behavior problems on anisotropic models.

We remark that anisotropic diffusion is a common physical phenomenon and describes processes where the diffusion is directionally dependent. Anisotropic diffusive processes occur in Darcy's flow for porous media, large scale turbulence where turbulence scales are anisotropic in size, and heat conduction and momentum dissipation in fusion plasmas. Many mathematically rigorous studies have been devoted to understanding such anisotropic flows. For example, there is a very large literature on the primitive and the Boussinesq equations with anisotropic dissipation. Various partial dissipation cases on the primitive equations have been examined by Cao, Li and Titi (see, example, [11–13]). Their main focus has been on the global existence and regularity problem. It may be interesting to understand the stability of perturbations near physically relevant steady states such as shear flows and hydrostatic balance.

To understand the stabilizing mechanism of a background magnetic field

$$u^{(0)} \equiv 0, \quad B^{(0)} \equiv e_2 := (0, 1, 0),$$

which is obviously a steady-state of (1.3), we study the dynamics of the perturbation (u, b) with $b = B - B^{(0)}$. Clearly, (u, b) satisfies the MHD equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla P + \mu \partial_1^2 u + (b \cdot \nabla) b + \partial_2 b, \quad x \in \mathbb{R}^3, \ t > 0, \\ \partial_t b + (u \cdot \nabla) b = \eta \, \Delta_h b + (b \cdot \nabla) u + \partial_2 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x). \end{cases} \tag{1.4}$$

Our main result asserts the global well-posedness and stability of (u, b), and provides precise decay rates for various Sobolev norms of (u, b). The precise statement of these results is presented in the next theorem. To simplify the notation, we use $\|f\|_{L^r L^q L^p}$ for the norm $\Big\| \big\| \|f\|_{L^p(\mathbb{R})} \big\|_{L^q(\mathbb{R})} \Big\|_{L^r(\mathbb{R})}$, and $\|f\|_{L^q_{x_3} L^p_{x_1x_2}}$ for $\big\| \|f\|_{L^p_{x_1x_2}(\mathbb{R}^2)} \big\|_{L^q_{x_3}(\mathbb{R})}$.

Theorem 1.1. *Assume that* $(u_0, b_0) \in H^3(\mathbb{R}^3)$ *with* $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$ *satisfies*

$$(u_0, b_0),\ (\partial_3 u_0, \partial_3 b_0),\ (\partial_3^2 u_0, \partial_3^2 b_0) \in L^2_{x_3} L^1_{x_1x_2}(\mathbb{R}^3).$$

Then there exists a sufficiently small constant $\delta > 0$ *such that, if*

$$\begin{aligned} &\|(u_0, b_0)\|_{H^3(\mathbb{R}^3)} + \|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1x_2}(\mathbb{R}^3)} + \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{x_3} L^1_{x_1x_2}(\mathbb{R}^3)} \\ &\quad + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2_{x_3} L^1_{x_1x_2}(\mathbb{R}^3)} \le \delta, \end{aligned} \tag{1.5}$$

then (1.4) *admits a unique global solution* $(u, b) \in C\big([0, \infty); H^3(\mathbb{R}^3)\big)$. *In addition,* (u, b) *is stable in the sense that, for an absolute constant* $C > 0$,

$$\|(u, b)(t)\|^2_{H^3(\mathbb{R}^3)} + \int_0^t \big(\|\partial_1 u(\tau)\|^2_{H^3(\mathbb{R}^3)} + \|\partial_2 u(\tau)\|^2_{H^2(\mathbb{R}^3)} + \|\nabla_h b(\tau)\|^2_{H^3(\mathbb{R}^3)}\big)\, \mathrm{d}\tau \le C\delta^2$$

for any $t > 0$.

Furthermore, (u, b) *obeys the following time decay estimates, for* $0 < \varepsilon \le \frac{1}{36}$:

$$\begin{aligned} &\|(u, b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{2}}, \qquad \|(\nabla_h u, \nabla_h b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-1}, \\ &\|(\partial_3 u, \partial_3 b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{2}+\varepsilon}, \\ &\|(\partial_1 \partial_j u, \partial_1 \partial_j b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{5}{4}+\varepsilon},\ j = 1, 2, \\ &\|(\partial_1 \partial_3 u, \partial_1 \partial_3 b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-1+\varepsilon}, \\ &\|(\partial_2 \partial_j u, \partial_2 \partial_j b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{2}{3}+\varepsilon},\ j = 2, 3, \\ &\|(\partial_3^2 u, \partial_3^2 b)\|_{L^2(\mathbb{R}^3)} \le C(1+t)^{-\frac{1}{4}}. \end{aligned}$$

Theorem 1.1 rigorously confirms the smoothing and stabilizing effect of the magnetic field on the electrically conducting fluids. Without the magnetic field, the fluid motion is governed by the 3D anisotropic Navier–Stokes equation (1.2) alone and whether or not the velocity is stable in Sobolev spaces remains an outstanding open problem. When coupled with magnetic field, Theorem 1.1 ensures that any perturbation near a background magnetic is stable and decays to zero at explicit rates as $t \to \infty$.

We clarify the differences between Theorem 1.1 and some of the closely related results. Wu and Zhu [94] solved the stability problem for the MHD system with horizontal dissipation $\Delta_h u$ and vertical magnetic diffusion $\partial_3^2 b$. It appears that the situation considered here is more difficult. This is due to the handling of the velocity nonlinearity $(u \cdot \nabla)u$. When the velocity dissipation is only in one direction, the triple-product term $((u \cdot \nabla)u, u)_{H^3}$ is much more difficult to control than any triple product terms generated for the MHD system considered in [94]. In fact, this term is exactly the reason why the well-posedness problem on the 3D anisotropic Navier–Stokes (1.2) is open. One main contribution of this paper is the handling of the Navier–Stokes nonlinearity when the dissipation of the velocity is only in a single direction. The smoothing and stabilizing effect of the magnetic field on the fluids, and the elaborate construction of time-weighted energy functional are the key ingredients of this successful story. We remark that there is a very large mathematical literature on the incompressible MHD equations. In particular, there have been substantial recent developments on the well-posedness and stability problems, and significant progress has been made (see, example, [1,8–10,14–16,21,29,33,34,36,37,40–42,45,47–49,51,53,55,56, 58–64,71,73,74,78–80,85–87,90–92,95,97–99,101,102,105–107]).

We explain the proof of Theorem 1.1. Due to the lack of velocity dissipation in two directions, we take the functional setting to be the Sobolev space H^3 in order to guarantee the uniqueness. The local existence follows from a standard procedure (see, example, [67]), so we focus on the global *a priori* bounds of (u, b). This is accomplished via the bootstrapping argument (see, example, [83]). A crucial step is to construct a suitable energy functional. Naturally it should include the H^3-norm together with the time integral pieces from the dissipative terms

$$E_0^{(1)}(t) = \sup_{0 \le \tau \le t} \|(u(\tau), b(\tau))\|_{H^3}^2 + \int_0^t \Big(\|\partial_1 u(\tau)\|_{H^3}^2 + \|\nabla_h b(\tau)\|_{H^3}^2 \Big) \mathrm{d}\tau. \tag{1.6}$$

However, due to the lack of velocity dissipation in two directions, the triple product generated by the nonlinearity, namely $((u \cdot \nabla)u, u)_{H^3}$ can not be bounded in terms of $E_0^{(1)}(t)$. The most difficult piece is the following triple product:

$$\int \partial_3^3 (u \cdot \nabla u) \cdot \partial_3^3 u \, \mathrm{d}x.$$

Here we have used $\int$ to denote the integral in x over $\mathbb{R}^3$. To distinguish the derivatives in different directions, we further write that as

$$\begin{aligned} &\int \partial_3^3 (u \cdot \nabla u) \cdot \partial_3^3 u \, \mathrm{d}x \\ &= 3 \int \partial_3 u_h \cdot \nabla_h \partial_3^2 u \cdot \partial_3^3 u \, \mathrm{d}x + 3 \int \partial_3^2 u_h \cdot \nabla_h \partial_3 u \cdot \partial_3^3 u \, \mathrm{d}x \\ &\quad + \int \partial_3^3 u_h \cdot \nabla_h u \cdot \partial_3^3 u \, \mathrm{d}x \\ &\quad + 3 \int \partial_3 u_3 \, \partial_3^3 u \cdot \partial_3^3 u \, \mathrm{d}x + 3 \int \partial_3^2 u_3 \, \partial_3^2 u \cdot \partial_3^3 u \, \mathrm{d}x + \int \partial_3^3 u_3 \, \partial_3 u \cdot \partial_3^3 u \, \mathrm{d}x. \end{aligned} \tag{1.7}$$

Clearly we need to seek enhanced dissipation in the x_2 or the x_3 direction to complement the existing dissipation in the x_1-direction. The background magnetic field is along the x_2 direction and it is in this direction that the extra regularization is generated. Mathematically this is reflected in the wave structure. We explain this. To avoid unnecessary complications, we look at the linearized system of (1.4), namely,

$$\begin{cases} \partial_t u = \mu\partial_1^2 u + \partial_2 b, \\ \partial_t b = \eta\Delta_h b + \partial_2 u, \\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{1.8}$$

By differentiating the first equation of (1.8) in t and making several substitutions, we obtain

$$\begin{aligned} \partial_{tt} u &= \mu\partial_1^2\partial_t u + \partial_2\partial_t b = \mu\partial_1^2\partial_t u + \partial_2(\eta\Delta_h b + \partial_2 u) \\ &= \mu\partial_1^2\partial_t u + \eta\Delta_h(\partial_t u - \mu\partial_1^2 u) + \partial_2^2 u \\ &= (\mu\partial_1^2 + \eta\Delta_h)\partial_t u - \mu\eta\partial_1^2\Delta_h u + \partial_2^2 u. \end{aligned}$$

Similarly, we have, for b, that

$$\partial_{tt} b = (\mu\partial_1^2 + \eta\Delta_h)\partial_t b - \mu\eta\partial_1^2\Delta_h b + \partial_2^2 b.$$

Therefore, (1.8) is converted into the following system of wave equations:

$$\begin{cases} \partial_{tt} u - (\mu\partial_1^2 + \eta\Delta_h)\partial_t u + \mu\eta\partial_1^2\Delta_h u - \partial_2^2 u = 0, \\ \partial_{tt} b - (\mu\partial_1^2 + \eta\Delta_h)\partial_t b + \mu\eta\partial_1^2\Delta_h b - \partial_2^2 b = 0, \\ \nabla\cdot u = \nabla\cdot b = 0. \end{cases} \tag{1.9}$$

(1.9) is a system of anisotropic and degenerate wave equations. In comparison with (1.8), (1.9) exhibits much more smoothing and stabilizing properties. In particular, the two terms $\partial_2^2 u$ and $\partial_2^2 b$ in (1.9), emerged from the interaction of the velocity and the magnetic field, generates the dissipation in the x_2-direction. This confirms the stabilizing effect of the background magnetic field. In fact, by energy estimates, we can show that, any solution (u, b) of the wave equations (1.9) satisfies

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(2\|\partial_t u\|_{L^2}^2 - 2(\mu\partial_1^2 u + \eta\Delta_h u, \partial_t u) + \|\mu\partial_1^2 u + \eta\Delta_h u\|_{L^2}^2 \\ &\quad +\mu\eta\|\partial_1\nabla_h u\|_{L^2}^2 + \|\partial_2 u\|_{L^2}^2\Big) \\ &\quad + \mu\|\partial_1\partial_t u\|_{L^2}^2 + \eta\|\nabla_h\partial_t u\|_{L^2}^2 + \mu^2\eta\|\partial_1^2\nabla_h u\|_{L^2}^2 + \mu\eta^2\|\partial_1\nabla_h^2 u\|_{L^2}^2 \\ &\quad + \mu\|\partial_1\partial_2 u\|_{L^2}^2 + \eta\|\nabla_h\partial_2 u\|_{L^2}^2 = 0. \end{aligned} \tag{1.10}$$

Clearly,

$$\begin{aligned} &2\|\partial_t u\|_{L^2}^2 - 2(\mu\partial_1^2 u + \eta\Delta_h u, \partial_t u) + \|\mu\partial_1^2 u \\ &\quad +\eta\Delta_h u\|_{L^2}^2 \geq \frac{1}{4}(\|\partial_t u\|_{L^2}^2 + \|\mu\partial_1^2 u + \eta\Delta_h u\|_{L^2}^2). \end{aligned}$$

Integrating (1.10) in time yields an upper bound on various norms of u. The upper bound for b is the same. These upper bounds reflect the smoothing and stabilizing effect of the wave structure. In particular, we gain a regularization in the

x_2-direction. As we shall see in the proof of Theorem 1.1, this regularization can be realized via a Lyapunov functional with a mixed scalar product. However, as we can see from the simple energy estimate above, the wave structure provides much more regularization. When we prove the decay estimates of Theorem 1.1, we need to take full advantage of all these smoothing properties. This is done through the integral representation in (5.5) and (5.6), which involves typical kernels for wave equations.

We remark that the stabilizing phenomenon and the wave structure appear to be universal for perturbations near steady states of many fluids. In particular, the Boussinesq system governing the perturbations near the hydrostatic equilibrium share many common features with the MHD systems such as the anisotropy and the wave structure. The stability problems on the Boussinesq systems near the hydrostatic equilibrium and/or the shear flow have recently attracted considerable interests and important progress has been made (see, example, [5–7,17,30, 32,39,57,68,81,82,85,93,100,108,109]). The Boussinesq wave structure reveals the smoothing and stabilizing effect of the buoyancy on the fluids near the hydrostatic equilibrium. The stabilizing effect in the Boussinesq systems is weaker than the corresponding one for the MHD systems.

To include this regularizing property in the energy functional, we define that

$$E_0^{(2)}(t) = \int_0^t \|\partial_2 u(\tau)\|_{H^2}^2 \mathrm{d}\tau. \tag{1.11}$$

We emphasize that the extra dissipative effect in the x_2-direction is one-derivative lower than what a standard dissipation term $\partial_2^2 u$ provides. This is due to the fact that such (partial) Laplace term is inside a wave-like equation (of second order in time). As a consequence, this energy functional only allows the time integrability of $\|\partial_2 u\|_{H^2}^2$, not $\|\partial_2 u\|_{H^3}^2$. Combining $E_0^{(1)}$ and $E_0^{(2)}$ gives

$$\begin{aligned} E_0(t) &= E_0^{(1)} + E_0^{(2)} \\ &= \sup_{0\le\tau\le t} \|(u(\tau), b(\tau))\|_{H^3}^2 + \int_0^t \\ &\quad \Big(\|\partial_1 u(\tau)\|_{H^3}^2 + \|\partial_2 u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^3}^2\Big)\mathrm{d}\tau. \end{aligned}$$

However, there are still two terms in (1.7) (the third term and the fourth term) that can not be bounded in terms of $E_0(t)$. After invoking the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, these terms are reduced to the difficult term

$$\int |\partial_2 u|\, |\partial_3^3 u|\, |\partial_3^3 u| \mathrm{d}x. \tag{1.12}$$

Due to the aforementioned weaker smoothing effect in the x_2-direction, (1.12) can not be bounded by $E_0(t)$. Extra maneuvers are necessary. Our idea is to include two extra time-weighted energy functionals

$$
\begin{aligned}
E_1(t) = & \sup_{0\le\tau\le t} (1+\tau)\|(\nabla_h u(\tau), \nabla_h b(\tau))\|_{H^1}^2 \\
& + \int_0^t (1+\tau)\Big(\|\partial_1\nabla_h u(\tau)\|_{H^1}^2 + \|\partial_2\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h^2 b(\tau)\|_{H^1}^2\Big)\mathrm{d}\tau, \\
E_2(t) = & \sup_{0\le\tau\le t} \Big((1+\tau)\|(u(\tau), b(\tau))\|_{L^2}^2 + (1+\tau)^2\|(\nabla_h u(\tau), \nabla_h b(\tau))\|_{L^2}^2 \\
& + (1+\tau)^{1-2\varepsilon}\|(\partial_3 u(\tau), \partial_3 b(\tau))\|_{L^2}^2 \\
& + \sum_{j=1}^{2}(1+\tau)^{\frac{5}{2}-2\varepsilon}\|(\partial_1\partial_j u(\tau), \partial_1\partial_j b(\tau))\|_{L^2}^2 \\
& + \sum_{j=2}^{3}(1+\tau)^{\frac{4}{3}-2\varepsilon}\|(\partial_2\partial_j u(\tau), \partial_2\partial_j b(\tau))\|_{L^2}^2 \\
& + (1+\tau)^{2-2\varepsilon}\|(\partial_1\partial_3 u(\tau), \partial_1\partial_3 b(\tau))\|_{L^2}^2 + (1+\tau)^{\frac{1}{2}}\|(\partial_3^2 u(\tau), \partial_3^2 b(\tau))\|_{L^2}^2.\Big).
\end{aligned}
$$

We shall show that the time-weighted terms involving the x_2-derivatives in $E_1(t)$ and $E_2(t)$ enables us to bound the term in (1.12) suitably and thus establish a closed energy inequality. The definition of $E_2(t)$ is certainly not simple. It takes into account of the precise time decay rate of each norm involved in $E_2(t)$. We will resort to the integral representation of (1.4) and spectral analysis to control the terms in $E_2(t)$. Having obtained the necessary components of the energy functional, we sum them up to form our total energy functional

$$E(t) = E_0(t) + E_1(t) + E_2(t).$$

Our main efforts are devoted to proving the estimate

$$E(t) \le C_1 F(u_0, b_0) + C_2\Big(E^{\frac{3}{2}}(t) + E^2(t)\Big), \tag{1.13}$$

where C_1 and C_2 are constants, and

$$
\begin{aligned}
F(u_0, b_0) = & \|(u_0, b_0)\|_{H^3}^2 + \|(u_0, b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2 \\
& + \|(\partial_3 u_0, \partial_3 b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2.
\end{aligned}
$$

Verifying (1.13) is a very lengthy process. For the sake of clarity, we divide the whole process into the proofs of the following inequalities:

$$E_0(t) \le CE(0) + CE^{\frac{3}{2}}(t), \tag{1.14}$$

$$E_1(t) \le CE(0) + CE_0(t) + CE^{\frac{3}{2}}(t), \tag{1.15}$$

$$
\begin{aligned}
E_2(t) \le & C\Big(E^{\frac{3}{2}}(t) + E^2(t)\Big) + C\Big(\|(u_0, b_0)\|_{H^2}^2 + \|(u_0, b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2 \\
& + \|(\partial_3 u_0, \partial_3 b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L_{x_3}^2 L_{x_1x_2}^1}^2\Big).
\end{aligned} \tag{1.16}
$$

To prove (1.14), we realize that $E_0(t)$ consists of two different types of terms $E_0^{(1)}(t)$ and $E_0^{(2)}(t)$, as aforementioned in (1.6) and (1.11). The boundedness of $E_0^{(2)}(t)$ relies on the enhanced dissipation from the wave structure. Naturally the proof of (1.14) is further split into two parts,

$$\begin{aligned}&(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2)\\&\quad + 2\int_0^t \Big(\mu\|\partial_1 u(\tau)\|_{H^3}^2 + \eta\|\nabla_h b(\tau)\|_{H^3}^2\Big)\mathrm{d}\tau \le CE(0) + CE^{\frac{3}{2}}(t)\end{aligned}$$

and

$$\begin{aligned}&-(\partial_2 u(t), b(t))_{H^2} + \frac{1}{2}\int_0^t \|\partial_2 u(\tau)\|_{H^2}^2\\&\quad - \int_0^t \Big(\|\partial_2 b(\tau)\|_{H^2}^2 + (\mu^2+\eta^2)\|\Delta_h b(\tau)\|_{H^2}^2\Big)\mathrm{d}\tau\\&\le CE(0) + CE^{\frac{3}{2}}(t).\end{aligned}$$

The detailed estimates are provided in Sect. 3. To prove (1.15), we also need to divide the terms in $E_1(t)$ into two parts,

$$\int_0^t (1+\tau)\|\partial_2\nabla_h u(\tau)\|_{L^2}^2\,\mathrm{d}\tau,$$

and the rest of the terms. The regularization from the wave structure in (1.9) is used to gain the time integrability of the vertical derivative. More technical details are left to Sect. 4.

The proof of (1.16) is extremely elaborate and relies on the precise decay rates of the norms involved in $E_2(t)$. Direct energy estimates are not sufficient for this purpose. Instead we solve the system of linear equations (1.8) and recast the nonlinear system (1.4) into an integral form. This form relies on three kernel functions. They are degenerate and anisotropic in the frequency space. We first perform a detailed spectral analysis in suitably divided subdomains of the frequency space to obtain sharp and precise upper bounds for the kernel functions. The terms in $E_2(t)$ are then estimated according to the orders and directions of their derivatives. After a lengthy process, we finally obtain (1.16).

Once (1.13) is at our disposal, a direct application of the bootstrapping argument yields the desired global bounds and Theorem 1.1 then follows.

The rest of this paper is divided into four sections. Section 2 applies the bootstrapping argument to the a priori inequality (1.13) to establish Theorem 1.1. In addition, several anisotropic inequalities for products and triple products are provided here as well. They will be used in the subsequent sections. Section 3 details the proof of (1.14). Section 4 proves (1.15) while Sect. 5 is devoted to (1.16).

2. Proof of Theorem 1.1 and Anisotropic Sobolev Inequalities

This section serves two purposes. The first is to prove Theorem 1.1 by applying the bootstrapping argument to the *a priori* inequality in (1.13). The second is to provide anisotropic inequalities for several products and triple products, which will be used in the proofs in subsequent sections.

Proof of Theorem 1.1. The local (in time) well-posedness of (1.4) in H^3 can be shown via standard procedures (see, example, [67]). It suffices to establish the global bounds stated in Theorem 1.1 in order to obtain the global existence. This is accomplished by applying the bootstrapping argument to (1.13), namely,

$$E(t) \le C_1 F(u_0, b_0) + C_2\Big(E^{\frac{3}{2}}(t) + E^2(t)\Big), \tag{2.1}$$

where

$$F(u_0, b_0) = \|(u_0, b_0)\|_{H^3}^2 + \|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1x_2}}^2 + \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{x_3} L^1_{x_1x_2}}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2_{x_3} L^1_{x_1x_2}}^2.$$

A useful description of the bootstrapping argument can be found in [83, p.21]. In order to apply the bootstrapping argument, we make the ansatz that

$$E(t) \le M := \min\left\{1, \frac{1}{(4C_2)^2}\right\}. \tag{2.2}$$

We then verify that $E(t)$ actually admits a smaller bound,

$$E(t) \le \frac{M}{2}.$$

Inserting (2.2) in (2.1) and recalling the initial assumption (1.5), we have

$$\begin{aligned} E(t) &\le C_1 F(u_0, b_0) + C_2\Big(M^{\frac{1}{2}} + M\Big)E(t) \\ &\le C_1\delta^2 + 2C_2 M^{\frac{1}{2}} E(t) \\ &\le C_1\delta^2 + \frac{1}{2}E(t), \end{aligned}$$

or

$$E(t) \le 2C_1\delta^2.$$

If the initial data is sufficiently small, say

$$\delta^2 \le \frac{M}{4C_1},$$

then we derive that

$$E(t) \le 2C_1\delta^2 \le \frac{M}{2}.$$

The bootstrapping argument then implies $T = \infty$ and asserts that, for any time $t > 0$,

$$E(t) \le C\delta^2,$$

which, in particular, implies the desired global bound on the solution (u, b). As a consequence, we obtain the global existence of solutions. The uniqueness is obvious due to the high regularity of the solution. The global bound on $E_2(t)$ yields the desired decay rates stated in Theorem 1.1. This completes the proof of Theorem 1.1. □

In the second part of this section, we provide several anisotropic upper bounds for products and triple products. The bounds stated in the following lemma are powerful tools in controlling the nonlinearity in terms of the anisotropic dissipation:

Lemma 2.1. *For some constants $C > 0$, $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, we have*

$$\begin{aligned}
\int |fgh|\,\mathrm{d}x &\leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_1 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \\
&\qquad \|\partial_2 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_3 h\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, \qquad (2.3)\\
\int |fgh|\,\mathrm{d}x &\leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_i f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_i\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \\
&\qquad \times \|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_k g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|h\|_{L^2(\mathbb{R}^3)}, \qquad (2.4)\\
\|fg\|_{L^2(\mathbb{R}^3)} &\leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_i f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \\
&\qquad \times \|\partial_i\partial_j f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_k g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}, \qquad (2.5)\\
\|f\,g\|_{L^2_{x_3}L^1_{x_1x_2}} &\leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_3 f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^3)}. \qquad (2.6)
\end{aligned}$$

Proof. The first two estimates have been stated and proven in [94]. Here we give the proof of (2.5) and (2.6). Without loss of generality, we assume $i = 2$, $j = 3$, $k = 1$ in (2.5). Now we prove (2.5). By Hölder's inequality, for $l = 1, 2, 3$, we have the simple fact

$$\|f\|_{L^\infty_{x_l}(\mathbb{R})} \leq \sqrt{2}\|f\|_{L^2_{x_l}(\mathbb{R})}^{\frac{1}{2}}\|\partial_l f\|_{L^2_{x_l}(\mathbb{R})}^{\frac{1}{2}}. \qquad (2.7)$$

By (2.7),

$$\begin{aligned}
\|fg\|_{L^2(\mathbb{R}^3)} &\leq \Big\| \|f\|_{L^2_{x_1}}\|g\|_{L^\infty_{x_1}} \Big\|_{L^2_{x_2x_3}} \\
&\leq C\Big\| \|f\|_{L^2_{x_1}}\|g\|_{L^2_{x_1}}^{\frac{1}{2}}\|\partial_1 g\|_{L^2_{x_1}}^{\frac{1}{2}} \Big\|_{L^2_{x_2x_3}} \\
&\leq C\|f\|_{L^\infty_{x_2x_3}L^2_{x_1}}\|g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}\|\partial_1 g\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}.
\end{aligned}$$

By Minkowski's inequality, (2.7) and Hölder's inequality,

$$\begin{aligned}\|f\|_{L^\infty_{x_2x_3}L^2_{x_1}} &\leq \big\| \|f\|_{L^\infty_{x_2}} \big\|_{L^2_{x_1}L^\infty_{x_3}} \leq C\big\| \|f\|^{\frac12}_{L^2_{x_2}} \|\partial_2 f\|^{\frac12}_{L^2_{x_2}} \big\|_{L^2_{x_1}L^\infty_{x_3}}\\ &\leq C\Big\| \big\| \|f\|_{L^\infty_{x_3}} \big\|^{\frac12}_{L^2_{x_2}} \big\| \|\partial_2 f\|_{L^\infty_{x_3}} \big\|^{\frac12}_{L^2_{x_2}} \Big\|_{L^2_{x_1}}\\ &\leq C\big\| \|f\|_{L^\infty_{x_3}} \big\|^{\frac12}_{L^2_{x_1x_2}} \big\| \|\partial_2 f\|_{L^\infty_{x_3}} \big\|^{\frac12}_{L^2_{x_1x_2}}\\ &\leq C\|f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_3 f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_2 f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_2\partial_3 f\|^{\frac14}_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|fg\|_{L^2(\mathbb{R}^3)} \leq \quad & C\|f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_2 f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_3 f\|^{\frac14}_{L^2(\mathbb{R}^3)}\|\partial_2\partial_3 f\|^{\frac14}_{L^2(\mathbb{R}^3)}\\ & \|g\|^{\frac12}_{L^2(\mathbb{R}^3)}\|\partial_1 g\|^{\frac12}_{L^2(\mathbb{R}^3)}.\end{aligned}$$

To prove (2.6), we apply Hölder's inequality, Minkowski's inequality and (2.7) to obtain

$$\begin{aligned}\|f\,g\|_{L^2_{x_3}L^1_{x_1x_2}} &\leq C\Big\| \|f\|_{L^2_{x_1x_2}} \|g\|_{L^2_{x_1x_2}} \Big\|_{L^2_{x_3}} \leq C\Big\| \|f\|_{L^\infty_{x_3}} \Big\|_{L^2_{x_1x_2}} \|g\|_{L^2(\mathbb{R}^3)}\\ &\leq C\|f\|^{\frac12}_{L^2(\mathbb{R}^3)}\|\partial_3 f\|^{\frac12}_{L^2(\mathbb{R}^3)}\|g\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

This completes the proof of Lemma 2.1. □

3. Estimate for $E_0(t)$

This section is devoted to proving the *a priori* estimate (1.14) for $E_0(t)$. More precisely, we prove the next proposition. We exploit the extra smoothing reflected in the wave structure (1.9) to make up for the lack of vertical dissipation in the velocity equation. The idea is to consider a Lyapunov functional involving an inner product besides the standard H^3-norm.

Proposition 3.1. *Let* (u, b) *be a solution of the system* (1.4)*. Then, for some constant* $C > 0$*, we have*

$$E_0(t) \leq CE(0) + CE^{\frac32}(t). \tag{3.1}$$

To prove (3.1), we work with the Lyapunov functional defined by

$$L(u, b)(t) = \|(u(t), b(t))\|^2_{H^3} + \lambda\big(\partial_2 u(t), b(t)\big)_{H^2},$$

where $0 < \lambda < 2$ is a small parameter. Next we show the bound of $L(u, b)$. We evaluate the time evolution of each part in this Lyapunov functional. For the sake of clarity, we divide this process into two lemmas. The first focuses on bounding $\|(u(t), b(t))\|^2_{H^3}$ while the second handles the inner product $\big(\partial_2 u(t), b(t)\big)_{H^2}$.

Lemma 3.2. *Assume that* (u, b) *is a solution to* (1.4). *Then we have*

$$(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2) + 2\int_0^t \Big(\mu\|\partial_1 u(\tau)\|_{H^3}^2 + \eta\|\nabla_h b(\tau)\|_{H^3}^2\Big)\mathrm{d}\tau \le CE(0) + CE^{\frac{3}{2}}(t).$$

Proof of Lemma 3.2. First we take the L^2-inner product of (1.4) with (u, b) to obtain

$$\frac{1}{2}\frac{d}{dt}(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2) + \big(\mu\|\partial_1 u\|_{L^2}^2 + \eta\|\nabla_h b\|_{L^2}^2\big) = 0. \tag{3.2}$$

Due to the equivalence of the norm $\|(u(t), b(t))\|_{H^3}$ with $\|(u(t), b(t))\|_{L^2} + \|(u(t), b(t))\|_{\dot{H}^3}$, it suffices to bound $\|(u(t), b(t))\|_{\dot{H}^3}$. Applying ∂_i^3 $(i = 1, 2, 3)$ to the equations (1.4) and taking the L^2-inner product of the resulting equations with $(\partial_i^3 u, \partial_i^3 b)$, we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^3 \frac{d}{dt}\Big(\|\partial_i^3 u(t)\|_{L^2}^2 + \|\partial_i^3 b(t)\|_{L^2}^2\Big) + \sum_{i=1}^3\Big(\mu\|\partial_i^3\partial_1 u\|_{L^2}^2 + \eta\|\partial_i^3\nabla_h b\|_{L^2}^2\Big)\\
&= -\sum_{i=1}^3\int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,\mathrm{d}x + \sum_{i=1}^3\int \partial_i^3(b\cdot\nabla b)\cdot\partial_i^3 u\,\mathrm{d}x\\
&\quad -\sum_{i=1}^3\int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\,\mathrm{d}x + \sum_{i=1}^3\int \partial_i^3(b\cdot\nabla u)\cdot\partial_i^3 b\,\mathrm{d}x\\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.3}$$

By the Leibniz formula, integration by parts and $\nabla\cdot u = 0$, we have

$$\begin{aligned}
I_1 &= -\sum_{i=1}^2\sum_{k=1}^3 C_3^k\int \partial_i^k u\cdot\nabla\partial_i^{3-k}u\cdot\partial_i^3 u\,\mathrm{d}x - \sum_{k=1}^3 C_3^k\int \partial_3^k u\cdot\nabla\partial_3^{3-k}u\cdot\partial_3^3 u\,\mathrm{d}x\\
&:= I_{11} + I_{12},
\end{aligned}$$

where C_3^k is the standard binomial coefficient. By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}
I_{11} &= -\sum_{i=1}^2\Big(3\int \partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,\mathrm{d}x + 3\int \partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,\mathrm{d}x\\
&\qquad + \int \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,\mathrm{d}x\Big)\\
&\le C(\|\nabla u\|_{L^\infty}\|\nabla\nabla_h^2 u\|_{L^2} + \|\nabla_h^2 u\|_{L^4}\|\nabla_h\nabla u\|_{L^4})\|\nabla_h^3 u\|_{L^2}\\
&\le C(\|\nabla u\|_{H^2}\|\nabla\nabla_h^2 u\|_{L^2} + \|\nabla_h^2 u\|_{H^1}\|\nabla_h\nabla u\|_{H^1})\|\nabla_h^3 u\|_{L^2}\\
&\le C\|\nabla u\|_{H^2}\|\nabla_h^2 u\|_{H^1}^2.
\end{aligned} \tag{3.4}$$

Rewriting the terms I_{12} in components, we have

$$\begin{aligned}
I_{12} &\le 4\int |\partial_3 u|\,|\partial_3^2\nabla_h u|\,|\partial_3^3 u|\,\mathrm{d}x + 6\int |\nabla_h\partial_3 u|\,|\partial_3^2 u|\,|\partial_3^3 u|\,\mathrm{d}x\\
&\quad - 3\int \partial_3 u_3\,\partial_3^3 u\cdot\partial_3^3 u\,\mathrm{d}x - \int \partial_3^3 u_h\cdot\nabla_h u\cdot\partial_3^3 u\,\mathrm{d}x\\
&\le 4\int |\partial_3 u|\,|\partial_3^2\nabla_h u|\,|\partial_3^3 u|\,\mathrm{d}x + 6\int |\nabla_h\partial_3 u|\,|\partial_3^2 u|\,|\partial_3^3 u|\,\mathrm{d}x\\
&\quad + 8\int |u|\,|\partial_3^3 u|\,|\partial_1\partial_3^3 u|\,\mathrm{d}x + 4\int |\partial_2 u|\,|\partial_3^3 u|^2\,\mathrm{d}x\\
&:= I_{121} + I_{122} + I_{123} + I_{124},
\end{aligned}$$

where we have used the divergence-free condition, $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$. By the anisotropic inequalities (2.3) and (2.4),

$$\begin{aligned}
I_{121} + I_{122} &\le C\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_3^2 u\|_{L^2}^{\frac14}\|\partial_3^2\partial_2 u\|_{L^2}^{\frac14}\|\partial_3^2\nabla_h u\|_{L^2}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_3^3\partial_1 u\|_{L^2}^{\frac12}\\
&\quad + C\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 u\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12}\\
&\le C\|\nabla u\|_{H^2}(\|\nabla_h\nabla u\|_{H^1}^2 + \|\partial_1\nabla^3 u\|_{L^2}^2).
\end{aligned}$$

Applying (2.4) again, I_{123} can be bounded by

$$\begin{aligned}
I_{123} &\le C\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac32}\\
&\le C\|u\|_{H^3}(\|\partial_2 u\|_{H^1}^2 + \|\partial_1\nabla^3 u\|_{L^2}^2).
\end{aligned}$$

Therefore,

$$I_{12} \le C\|u\|_{H^3}(\|\nabla_h u\|_{H^2}^2 + \|\partial_1\nabla^3 u\|_{L^2}^2) + I_{124}, \tag{3.5}$$

where I_{124} will be estimated at the end of the proof. Consequently, (3.4), together with (3.5), leads to

$$I_1 \le C\|u\|_{H^3}(\|\nabla_h u\|_{H^2}^2 + \|\partial_1\nabla^3 u\|_{L^2}^2) + I_{124}. \tag{3.6}$$

Since b has better dissipation than u, it is simpler to bound I_2. By Leibniz's formula,

$$\begin{aligned}
I_2 &= \sum_{i=1}^{2}\sum_{k=1}^{3} C_3^k\int \partial_i^k b\cdot\nabla\partial_i^{3-k} b\cdot\partial_i^3 u\,\mathrm{d}x + \sum_{k=1}^{3} C_3^k\int \partial_3^k b\cdot\nabla\partial_3^{3-k} b\cdot\partial_3^3 u\,\mathrm{d}x\\
&\quad + \int b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,\mathrm{d}x\\
&:= I_{21} + I_{22} + \int b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,\mathrm{d}x.
\end{aligned}$$

As in I_{11}, we first have

$$\begin{aligned} I_{21} &\le C(\|\nabla b\|_{L^\infty}\|\nabla\nabla_h^2 b\|_{L^2} + \|\nabla_h^2 b\|_{L^4}\|\nabla_h\nabla b\|_{L^4})\|\nabla_h^3 u\|_{L^2} \\ &\le C(\|\nabla b\|_{H^2}\|\nabla\nabla_h^2 b\|_{L^2} + \|\nabla_h^2 b\|_{H^1}\|\nabla_h\nabla b\|_{H^1})\|\nabla_h^3 u\|_{L^2} \\ &\le C\|\nabla b\|_{H^2}(\|\nabla_h^2 b\|_{H^1}^2 + \|\nabla_h^3 u\|_{L^2}^2). \end{aligned} \tag{3.7}$$

For I_{22}, we further split it into two parts and then apply (2.3) to get

$$\begin{aligned} I_{22} &= \sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_3^k b_h \cdot \nabla_h \partial_3^{3-k} b \cdot \partial_3^3 u \,\mathrm{d}x + \sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_3^k b_3\, \partial_3^{4-k} b \cdot \partial_3^3 u \,\mathrm{d}x \\ &\le C\sum_{k=1}^{3} \|\partial_3^k b_h\|_{L^2}^{\frac12}\|\partial_2\partial_3^k b_h\|_{L^2}^{\frac12}\|\nabla_h\partial_3^{3-k} b\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_3^{3-k} b\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12} \\ &\quad + C\sum_{k=1}^{3} \|\partial_3^k b_3\|_{L^2}^{\frac12}\|\partial_3^{k+1} b_3\|_{L^2}^{\frac12}\|\partial_3^{4-k} b\|_{L^2}^{\frac12}\|\partial_2\partial_3^{4-k} b\|_{L^2}^{\frac12}\|\partial_3^3 u\|_{L^2}^{\frac12}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12} \\ &\le C(\|\nabla b\|_{H^2} + \|\nabla^3 u\|_{L^2})(\|\nabla_h b\|_{H^3}^2 + \|\partial_1\partial_3^3 u\|_{L^2}^2). \end{aligned} \tag{3.8}$$

Therefore, (3.7) and (3.8) yield

$$\begin{aligned} I_2 \le{}& C(\|\nabla b\|_{H^2} + \|\nabla^3 u\|_{L^2})(\|\nabla_h b\|_{H^3}^2 + \|\nabla_h^3 u\|_{L^2}^2 + \|\partial_1\nabla^3 u\|_{L^2}^2) \\ &+ \int b\cdot\nabla\partial_i^3 b\cdot\partial_i^3 u\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

We proceed to deal with I_3. I_3 is firstly divided into three parts,

$$\begin{aligned} I_3 &= -\sum_{i=1}^{2}\sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_i^k u\cdot\nabla\partial_i^{3-k} b\cdot\partial_i^3 b\,\mathrm{d}x - \sum_{k=1}^{3}\mathcal{C}_3^k\int \partial_3^k u_h\cdot\nabla_h\partial_3^{3-k} b\cdot\partial_3^3 b\,\mathrm{d}x \\ &\quad - \sum_{k=1}^{3}\mathcal{C}_3^k\int \partial_3^k u_3\,\partial_3^{4-k} b\cdot\partial_3^3 b\,\mathrm{d}x \\ &:= I_{31} + I_{32} + I_{33}. \end{aligned}$$

By (2.3),

$$\begin{aligned} &I_{31} + I_{32} \\ &\le C\sum_{i=1}^{2}\sum_{k=1}^{3} \|\partial_i^k u\|_{L^2}^{\frac12}\|\partial_1\partial_i^k u\|_{L^2}^{\frac12}\|\nabla\partial_i^{3-k} b\|_{L^2}^{\frac12}\|\partial_2\nabla\partial_i^{3-k} b\|_{L^2}^{\frac12}\|\partial_i^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_i^3 b\|_{L^2}^{\frac12} \\ &\quad + C\sum_{k=1}^{3} \|\partial_3^k u_h\|_{L^2}^{\frac12}\|\partial_1\partial_3^k u_h\|_{L^2}^{\frac12}\|\nabla_h\partial_3^{3-k} b\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_i^{3-k} b\|_{L^2}^{\frac12}\|\partial_3^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 b\|_{L^2}^{\frac12} \\ &\le C(\|\nabla u\|_{H^2} + \|\nabla b\|_{H^2})(\|\partial_1\nabla u\|_{H^2}^2 + \|\nabla_h b\|_{H^3}^2). \end{aligned} \tag{3.10}$$

For I_{33}, we further decompose it, integrate by parts and use (2.4) to get

$$
\begin{aligned}
I_{33} &= -\sum_{k=2}^{3} \mathcal{C}_3^k \int \partial_3^k u_3 \, \partial_3^{4-k} b \cdot \partial_3^3 b \,\mathrm{d}x + 6 \int u_h \cdot \partial_3^2 \nabla_h b \cdot \partial_3^3 b \,\mathrm{d}x \\
&\le C \sum_{k=2}^{3} \|\partial_3^k u_3\|_{L^2} \|\partial_3^{4-k} b\|_{L^2}^{\frac14} \|\partial_1 \partial_3^{4-k} b\|_{L^2}^{\frac14} \\
&\quad \|\partial_3 \partial_3^{4-k} b\|_{L^2}^{\frac14} \|\partial_1 \partial_3 \partial_3^{4-k} b\|_{L^2}^{\frac14} \|\partial_3^3 b\|_{L^2}^{\frac12} \|\partial_2 \partial_3^3 b\|_{L^2}^{\frac12} \\
&\quad + C \|u\|_{L^2}^{\frac14} \|\partial_1 u\|_{L^2}^{\frac14} \|\partial_3 u\|_{L^2}^{\frac14} \|\partial_1 \partial_3 u\|_{L^2}^{\frac14} \|\partial_3^2 \nabla_h b\|_{L^2} \|\partial_3^3 b\|_{L^2}^{\frac12} \|\partial_2 \partial_3^3 b\|_{L^2}^{\frac12} \\
&\le C(\|u\|_{H^1} + \|\nabla b\|_{H^2})(\|\partial_1 u\|_{H^1}^2 + \|\nabla \nabla_h u\|_{H^1}^2 + \|\nabla \nabla_h b\|_{H^2}^2),
\end{aligned} \tag{3.11}
$$

where we have used $\nabla \cdot u = 0$. Combining (3.10) and (3.11) yields

$$
I_3 \le C(\|u\|_{H^3} + \|\nabla b\|_{H^2})(\|\nabla_h \nabla u\|_{H^1}^2 + \|\partial_1 u\|_{H^3}^2 + \|\nabla_h b\|_{H^3}^2). \tag{3.12}
$$

We now bound I_4. As in I_2, we decompose I_4 into three parts,

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{2} \sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_i^k b \cdot \nabla \partial_i^{3-k} u \cdot \partial_i^3 b \,\mathrm{d}x + \sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_3^k b \cdot \nabla \partial_3^{3-k} u \cdot \partial_3^3 b \,\mathrm{d}x \\
&\quad + \int b \cdot \nabla \partial_i^3 u \cdot \partial_i^3 b \,\mathrm{d}x \\
&:= I_{41} + I_{42} + \int b \cdot \nabla \partial_i^3 u \cdot \partial_i^3 b \,\mathrm{d}x.
\end{aligned}
$$

By Hölder's inequality and Sobolev's inequality,

$$
I_{41} \le C \sum_{k=1}^{3} \|\nabla_h^k b\|_{L^4} \|\nabla \nabla_h^{3-k} u\|_{L^2} \|\nabla_h^3 b\|_{L^4} \le C \|\nabla u\|_{H^2} \|\nabla_h b\|_{H^3}^2.
$$

The estimate for I_{42} is more subtle. We first further split it into three terms,

$$
\begin{aligned}
I_{42} &= 3 \int \partial_3 b_h \cdot \nabla_h \partial_3^2 u \cdot \partial_3^3 b \,\mathrm{d}x + \sum_{k=2}^{3} \mathcal{C}_3^k \int \partial_3^k b_h \cdot \nabla_h \partial_3^{3-k} u \cdot \partial_3^3 b \,\mathrm{d}x \\
&\quad + \sum_{k=1}^{3} \mathcal{C}_3^k \int \partial_3^k b_3 \, \partial_3^{4-k} u \cdot \partial_3^3 b \,\mathrm{d}x \\
&:= I_{421} + I_{422} + I_{423}.
\end{aligned}
$$

Applying (2.4) to I_{421}, and (2.3) to I_{422} and I_{423}, respectively, we obtain

$$
\begin{aligned}
I_{421} &\le C \|\partial_3 b_h\|_{L^2}^{\frac14} \|\partial_2 \partial_3 b_h\|_{L^2}^{\frac14} \|\partial_3^2 b_h\|_{L^2}^{\frac14} \|\partial_2 \partial_3^2 b_h\|_{L^2}^{\frac14} \|\nabla_h \partial_3^2 u\|_{L^2} \|\partial_3^3 b\|_{L^2}^{\frac12} \|\partial_1 \partial_3^3 b\|_{L^2}^{\frac12} \\
&\le C \|\nabla b\|_{H^2} (\|\nabla^2 \nabla_h u\|_{L^2}^2 + \|\nabla \nabla_h b\|_{H^2}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&I_{422}+I_{423}\\
&\quad\le C\sum_{k=2}^{3}\|\partial_3^k b_h\|_{L^2}^{\frac12}\|\partial_1\partial_3^k b_h\|_{L^2}^{\frac12}\|\nabla_h\partial_3^{3-k}u\|_{L^2}^{\frac12}\|\partial_3\nabla_h\partial_3^{3-k}u\|_{L^2}^{\frac12}\|\partial_3^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 b\|_{L^2}^{\frac12}\\
&\qquad+C\sum_{k=1}^{3}\|\partial_3^k b_3\|_{L^2}^{\frac12}\|\partial_3^{k+1}b_3\|_{L^2}^{\frac12}\|\partial_3^{4-k}u\|_{L^2}^{\frac12}\|\partial_1\partial_3^{4-k}u\|_{L^2}^{\frac12}\|\partial_3^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_3^3 b\|_{L^2}^{\frac12}\\
&\quad\le C(\|\nabla u\|_{H^2}+\|\nabla^2 b\|_{H^1})(\|\partial_1\nabla u\|_{H^2}^2+\|\nabla_h u\|_{H^2}^2+\|\nabla_h b\|_{H^3}^2).
\end{aligned}
$$

Thus,

$$
I_4\le C(\|\nabla u\|_{H^2}+\|\nabla b\|_{H^2})(\|\nabla_h u\|_{H^2}^2+\|\partial_1\nabla u\|_{H^2}^2+\|\nabla_h b\|_{H^3}^2). \tag{3.13}
$$

Inserting (3.6), (3.9), (3.12) and (3.13) in (3.3) and combining with (3.2), we conclude

$$
\begin{aligned}
&\frac12\frac{d}{dt}\Big[\|(u(t),b(t))\|_{L^2}^2+\sum_{i=1}^{3}\|(\partial_i^3 u(t),\partial_i^3 b(t))\|_{L^2}^2\Big]+\Big[\mu\|\partial_1 u\|_{L^2}^2+\eta\|\nabla_h u\|_{L^2}^2\\
&\quad+\sum_{i=1}^{3}(\mu\|\partial_i^3\partial_1 u\|_{L^2}^2+\eta\|\partial_i^3\nabla_h b\|_{L^2}^2)\Big]\\
&\le C(\|u\|_{H^3}+\|b\|_{H^3})(\|\partial_2 u\|_{H^2}^2+\|\partial_1 u\|_{H^3}^2+\|\nabla_h b\|_{H^3}^2)+I_{124}.
\end{aligned} \tag{3.14}
$$

Integrating (3.14) over $[0,t]$ yields

$$
\begin{aligned}
&\|(u(t),b(t))\|_{H^3}^2+2\int_0^t\big(\mu\|\partial_1 u(\tau)\|_{H^3}^2+\eta\|\nabla_h b(\tau)\|_{H^3}^2\big)\,\mathrm{d}\tau\\
&\le C\int_0^t(\|u(\tau)\|_{H^3}+\|b(\tau)\|_{H^3})(\|\partial_2 u(\tau)\|_{H^2}^2+\|\partial_1 u(\tau)\|_{H^3}^2+\|\nabla_h b(\tau)\|_{H^3}^2)\mathrm{d}\tau\\
&\quad+C(\|u_0\|_{H^3}^2+\|b_0\|_{H^3}^2)+C\int_0^t I_{124}(\tau)\,\mathrm{d}\tau\\
&\le CE_0^{\frac32}(t)+CE(0)+C\int_0^t I_{124}(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

It remains to bound the integral of I_{124}. By means of (2.4), we have

$$
I_{124}\le C\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_2^2 u\|_{L^2}^{\frac14}\|\partial_3\partial_2 u\|_{L^2}^{\frac14}\|\partial_2^2\partial_3 u\|_{L^2}^{\frac14}\|\partial_3^3 u\|_{L^2}^{\frac32}\|\partial_1\partial_3^3 u\|_{L^2}^{\frac12}.
$$

Then, applying Hölder's inequality leads to

$$
\begin{aligned}
&\int_0^t I_{124}(\tau)\mathrm{d}\tau \\
&\quad \le C \sup_{0\le\tau\le t} (1+\tau)^{\frac14}\|\partial_2 u(\tau)\|_{L^2}^{\frac14}(1+\tau)^{\frac14(\frac23-\varepsilon)}\|\partial_2\partial_3 u(\tau)\|_{L^2}^{\frac14}\|\partial_3^3 u(\tau)\|_{L^2}^{\frac32} \\
&\qquad \times \int_0^t (1+\tau)^{\frac18}\|\partial_2^2 u(\tau)\|_{L^2}^{\frac14}\|\partial_2^2\partial_3 u(\tau)\|_{L^2}^{\frac14}\|\partial_1\partial_3^3 u(\tau)\|_{L^2}^{\frac12}(1+\tau)^{-\frac{13}{24}+\frac{\varepsilon}{4}}\mathrm{d}\tau \\
&\quad \le C E_2^{\frac14}(t)E_0^{\frac34}(t)\Big(\int_0^t (1+\tau)\|\partial_2^2 u(\tau)\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac18}\Big(\int_0^t \|\partial_2^2\partial_3 u(\tau)\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac18} \\
&\qquad \times \Big(\int_0^t \|\partial_1\partial_3^3 u(\tau)\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac14}\Big(\int_0^t (1+\tau)^{-\frac{13}{12}+\frac{\varepsilon}{2}}\mathrm{d}\tau\Big)^{\frac12} \\
&\quad \le C E_2^{\frac14}(t)E_1^{\frac18}(t)E_0^{\frac98}(t) \le C E^{\frac32}(t).
\end{aligned}
$$

Therefore,

$$
\|(u(t),b(t))\|_{H^3}^2 + 2\int_0^t (\mu\|\partial_1 u(\tau)\|_{H^3}^2 + \eta\|\nabla_h b(\tau)\|_{H^3}^2)\,\mathrm{d}\tau \le C E^{\frac32}(t) + C E(0).
$$

This completes the proof of Lemma 3.2. □

Next we evaluate the inner product $(\partial_2 u(t), b(t))_{H^2}$ and prove the following lemma:

Lemma 3.3. *Assume* (u,b) *is a solution to* (1.4)*. Then*

$$
\begin{aligned}
&-(\partial_2 u(t), b(t))_{H^2} + \frac12\int_0^t \|\partial_2 u(\tau)\|_{H^2}^2 \\
&\quad -\int_0^t \Big(\|\partial_2 b(\tau)\|_{H^2}^2 + (\mu^2+\eta^2)\|\Delta_h b(\tau)\|_{H^2}^2\Big)\mathrm{d}\tau \\
&\le C E(0) + C E^{\frac32}(t).
\end{aligned}
\tag{3.15}
$$

Proof of Lemma 3.3. Invoking the equations of u and b in (1.4), we have

$$
\begin{aligned}
&-\frac{d}{dt}(\partial_2 u(t), b(t))_{H^2} + \|\partial_2 u\|_{H^2}^2 - \|\partial_2 b\|_{H^2}^2 \\
&= (\partial_2(u\cdot\nabla u), b)_{H^2} - (\partial_2(b\cdot\nabla b), b)_{H^2} + (\partial_2 u, u\cdot\nabla b)_{H^2} - (\partial_2 u, b\cdot\nabla u)_{H^2} \\
&\quad - \mu(\partial_2\partial_1^2 u, b)_{H^2} - \eta(\partial_2 u, \Delta_h b)_{H^2} \\
&:= I_5 + \cdots + I_{10}.
\end{aligned}
\tag{3.16}
$$

By integration by parts, I_5 can be rewritten as

$$
\begin{aligned}
I_5 &= -\int u\cdot\nabla u\cdot(\partial_2 b - \partial_2\Delta b)\,\mathrm{d}x + \int \nabla(u\cdot\nabla u)\cdot\partial_2\nabla^3 b\,\mathrm{d}x \\
&= -\int u\cdot\nabla u\cdot(\partial_2 b - \partial_2\Delta b)\,\mathrm{d}x + \int (\nabla u\cdot\nabla)u\cdot\partial_2\nabla^3 b\,\mathrm{d}x \\
&\quad + \int (u\cdot\nabla)\nabla u\cdot\partial_2\nabla^3 b\,\mathrm{d}x.
\end{aligned}
$$

Applying (2.3) and (2.4) leads to

$$
\begin{aligned}
I_5 &\leq C\|u\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2 b+\partial_2\Delta b\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_2 b+\partial_3\partial_2\Delta b\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_3\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^3 b\|_{L^2}\\
&\quad + C\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla^3 b\|_{L^2}\\
&\leq C\|u\|_{H^2}(\|\nabla_h u\|_{H^2}^2+\|\nabla_h b\|_{H^3}^2).
\end{aligned}
$$

Similarly,

$$
I_6 \leq C\|b\|_{H^2}\|\nabla_h b\|_{H^3}^2.
$$

For I_7, we split it into two parts

$$
I_7 = \int u\cdot\nabla b\cdot(\partial_2 u-\partial_2\Delta u)\,\mathrm{d}x + \int \Delta(u\cdot\nabla b)\cdot\partial_2\Delta u\,\mathrm{d}x := I_{71}+I_{72}.
$$

By (2.4),

$$
\begin{aligned}
I_{71} &\leq C\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{2}}\|\partial_2 u+\partial_2\Delta u\|_{L^2}\\
&\leq C(\|u\|_{H^1}+\|\nabla b\|_{L^2})(\|\nabla_h u\|_{H^2}^2+\|\partial_2\nabla b\|_{L^2}^2).
\end{aligned}
$$

Similarly, making use of the inequality (2.4) again, we get

$$
\begin{aligned}
I_{72} &= \int(\Delta u\cdot\nabla b+2\nabla u\cdot\nabla^2 b+u\cdot\nabla\Delta b)\cdot\partial_2\Delta u\,\mathrm{d}x\\
&\leq C\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\partial_1\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla b\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla b\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3\nabla b\|_{L^2}^{\frac{1}{4}}\|\partial_2\Delta u\|_{L^2}\\
&\quad + C\|\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_3\nabla u\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3\nabla u\|_{L^2}^{\frac{1}{4}}\|\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u\|_{L^2}\\
&\quad + C\|u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\nabla\Delta b\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\Delta b\|_{L^2}^{\frac{1}{2}}\|\partial_2\Delta u\|_{L^2}\\
&\leq C(\|u\|_{H^2}+\|\nabla b\|_{H^2})(\|\nabla_h u\|_{H^2}^2+\|\nabla_h\nabla b\|_{H^2}^2),
\end{aligned}
$$

which, together with the estimate of I_{71}, gives

$$
I_7 \leq C(\|u\|_{H^2}+\|\nabla b\|_{H^2})(\|\nabla_h u\|_{H^2}^2+\|\nabla_h\nabla b\|_{H^2}^2).
$$

I_8 can be estimated with the same process as I_7. Firstly,

$$
I_8 = -\int b\cdot\nabla u\cdot(\partial_2 u-\partial_2\Delta u)\mathrm{d}x - \int\Delta(b\cdot\nabla u)\cdot\partial_2\Delta u\,\mathrm{d}x = I_{81}+I_{82}.
$$

Then we can derive

$$
I_{81} \leq C(\|\nabla u\|_{L^2}+\|b\|_{H^1})(\|\partial_2 u\|_{H^2}^2+\|\partial_1 b\|_{H^1}^2),
$$

and

$$
\begin{aligned}
I_{82} = & -\int (\Delta b \cdot \nabla u + 2\nabla b \cdot \nabla^2 u + b \cdot \nabla \Delta u) \cdot \partial_2 \Delta u \, \mathrm{d}x \\
\le & \; C\|\Delta b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \Delta b\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \partial_3 \nabla u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \Delta u\|_{L^2} \\
& + C\|\nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_2 \nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_3 \nabla b\|_{L^2}^{\frac{1}{4}} \|\partial_2 \partial_3 \nabla b\|_{L^2}^{\frac{1}{4}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta u\|_{L^2} \\
& + C\|b\|_{L^2}^{\frac{1}{4}} \|\partial_2 b\|_{L^2}^{\frac{1}{4}} \|\partial_3 b\|_{L^2}^{\frac{1}{4}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{4}} \|\nabla \Delta u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla \Delta u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \Delta u\|_{L^2} \\
\le & \; C(\|\nabla u\|_{H^2} + \|b\|_{H^2})(\|\partial_2 \nabla u\|_{H^1}^2 + \|\partial_1 \nabla^2 u\|_{H^1}^2 + \|\nabla_h b\|_{H^2}^2).
\end{aligned}
$$

Thus,

$$
I_8 \le C(\|\nabla u\|_{H^2} + \|b\|_{H^2})(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 \nabla^2 u\|_{H^1}^2 + \|\nabla_h b\|_{H^2}^2).
$$

By Hölder's inequality and Young's inequality,

$$
\begin{aligned}
I_9 + I_{10} = & -\mu(\partial_2 u, \partial_1^2 b)_{H^2} - \eta(\partial_2 u, \Delta_h b)_{H^2} \\
\le & \; \frac{1}{2}\|\partial_2 u\|_{H^2}^2 + \mu^2 \|\partial_1^2 b\|_{H^2} + \eta^2 \|\Delta_h b\|_{H^2}^2.
\end{aligned}
$$

In summary, we have obtained

$$
\begin{aligned}
& -\frac{d}{dt}(\partial_2 u(t), b(t))_{H^2} + \frac{1}{2}\|\partial_2 u\|_{H^2}^2 - \Big(\|\partial_2 b\|_{H^2}^2 + (\mu^2 + \eta^2)\|\Delta_h b\|_{H^2}^2\Big) \\
& \le C(\|u\|_{H^3} + \|b\|_{H^3})(\|\partial_2 u\|_{H^2}^2 + \|\partial_1 u\|_{H^3}^2 + \|\nabla_h b\|_{H^3}^2).
\end{aligned} \tag{3.17}
$$

Then integrating (3.17) leads to the desired estimate (3.15). This completes the proof of Lemma 3.3. □

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. According to Lemma 3.2 and 3.3, we have

$$
\begin{aligned}
& \Big(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 - \lambda(\partial_2 u(t), b(t))_{H^2}\Big) + \int_0^t \Big[2\mu\|\partial_1 u(\tau)\|_{H^3}^2 \\
& \quad + \Big(2\eta - \lambda(1 + \mu^2 + \eta^2)\Big)\|\nabla_h b(\tau)\|_{H^3}^2 + \frac{\lambda}{2}\|\partial_2 u(\tau)\|_{H^2}^2\Big]\mathrm{d}\tau \\
& \le CE(0) + CE^{\frac{3}{2}}(t),
\end{aligned}
$$

where λ is a parameter. Now we select λ to be sufficiently small to obtain

$$
\begin{aligned}
& \|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \Big(\|\partial_1 u(\tau)\|_{H^3}^2 + \|\partial_2 u(\tau)\|_{H^2}^2 + \|\nabla_h b(\tau)\|_{H^3}^2\Big)\mathrm{d}\tau \\
& \le CE(0) + CE^{\frac{3}{2}}(t).
\end{aligned}
$$

This completes the proof of Proposition 3.1. □

4. Estimate for $E_1(t)$

The section proves the *a priori* inequality (1.15) for $E_1(t)$. That is, we establish the following proposition. Since the velocity equation does not have the vertical dissipation, we need to make use of the extra smoothing and stabilization revealed by the wave structure in (1.9). Our idea is to use the inner product $(1+t)(\partial_2\nabla_h u, \nabla_h b)$ to decode this regularizing property. As a consequence, we obtain the time integrability of $(1+t)\,\|\partial_2\nabla_h u\|_{L^2}^2$. More details are given in Lemma 4.3 and its proof.

Proposition 4.1. *For some constants $C>0$, it holds that*

$$E_1(t) \le CE(0) + CE_0(t) + CE^{\frac{3}{2}}(t). \tag{4.1}$$

We shall divide the proof of (4.1) into two main parts. The first one bounds the time-weighted energy $(1+t)\|(\nabla_h u, \nabla_h b)\|_{H^1}^2$ while the second handles the inner product $(1+t)(\partial_2\nabla_h u, \nabla_h b)$ to generate the time-weighted dissipation $(1+t)\,\|\partial_2\nabla_h u\|_{L^2}^2$.

Lemma 4.2. *Assume that (u, b) solves (1.4). Then we have*

$$\begin{aligned}
&(1+t)\big(\|\nabla_h u(t)\|_{H^1}^2 + \|\nabla_h b(t)\|_{H^1}^2\big) \\
&\quad + 2\int_0^t (1+\tau)\big(\mu\|\partial_1\nabla_h u(\tau)\|_{H^1}^2 + \eta\|\Delta_h b(\tau)\|_{H^1}^2\big)\mathrm{d}\tau \\
&\quad \le E_0(t) + E(0) + CE^{\frac{3}{2}}(t).
\end{aligned} \tag{4.2}$$

Proof of Lemma 4.2. Taking the H^1-inner product of (1.4) with $(\Delta_h u, \Delta_h b)$, and multiplying by $(1+t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(1+t)(\|\nabla_h u(t)\|_{H^1}^2 + \|\nabla_h b(t)\|_{H^1}^2) + (1+t)(\mu\|\partial_1\nabla_h u\|_{H^1}^2 + \eta\|\Delta_h b\|_{H^1}^2) \\
&= \frac{1}{2}(\|\nabla_h u\|_{H^1}^2 + \|\nabla_h b\|_{H^1}^2) - (1+t)\big(\nabla_h(u\cdot\nabla u), \nabla_h u\big)_{H^1} \\
&\quad + (1+t)\big(\nabla_h(b\cdot\nabla b), \nabla_h u\big)_{H^1} \\
&\quad - (1+t)\big(\nabla_h(u\cdot\nabla b), \nabla_h b\big)_{H^1} + (1+t)\big(\nabla_h(b\cdot\nabla u), \nabla_h b\big)_{H^1} \\
&:= \frac{1}{2}(\|\nabla_h u\|_{H^1}^2 + \|\nabla_h b\|_{H^1}^2) + J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.3}$$

To bound J_1, we split J_1 into three parts:

$$\begin{aligned}
J_1 &= -(1+t)\Big(\int \nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,\mathrm{d}x + \int \nabla_h^2(u\cdot\nabla u)\cdot\nabla_h^2 u\,\mathrm{d}x \\
&\qquad + \int \nabla_h\partial_3(u\cdot\nabla u)\cdot\nabla_h\partial_3 u\,\mathrm{d}x\Big) \\
&:= -(1+t)(J_{11} + J_{12} + J_{13}).
\end{aligned}$$

By the anisotropic inequality (2.3),

$$
\begin{aligned}
J_{11} &= \int \nabla_h u_h \cdot \nabla_h u \cdot \nabla_h u \,\mathrm{d}x + \int \nabla_h u_3\, \partial_3 u \cdot \nabla_h u \,\mathrm{d}x \\
&\leq C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\nabla_h u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 \nabla_h u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\leq C\|\nabla_h u\|_{L^2}\|\nabla_h^2 u\|_{L^2}\|\nabla_h u\|_{H^1} + C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\nabla_h^2 u\|_{L^2}\|\nabla_h u\|_{H^1}.
\end{aligned} \tag{4.4}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t (1+\tau) J_{11}(\tau)\mathrm{d}\tau \\
&\leq C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}}\|\nabla_h u(\tau)\|_{L^2} \int_0^t (1+\tau)^{\frac{1}{2}}\|\nabla_h^2 u(\tau)\|_{L^2}\|\nabla_h u(\tau)\|_{H^1}\mathrm{d}\tau \\
&\quad + C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}}\|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\quad \int_0^t (1+\tau)^{\frac{1}{2}}\|\nabla_h^2 u(\tau)\|_{L^2}\|\nabla_h u(\tau)\|_{H^1}\mathrm{d}\tau \\
&\leq C E_1^{\frac{1}{2}}(t) E_1^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t) + E_2^{\frac{1}{4}}(t) E_0^{\frac{1}{4}}(t) E_1^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t) \\
&\leq C E^{\frac{3}{2}}(t).
\end{aligned} \tag{4.5}
$$

Applying (2.3) again and using Sobolev's inequality, J_{12} can be bounded as

$$
\begin{aligned}
J_{12} &= \int \nabla_h^2 u \cdot \nabla u \cdot \nabla_h^2 u \,\mathrm{d}x + 2\int \nabla_h u \cdot \nabla\nabla_h u \cdot \nabla_h^2 u \,\mathrm{d}x \\
&\leq \|\nabla u\|_{L^\infty}\|\nabla_h^2 u\|_{L^2}^2 + C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\quad \|\nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3 \nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1 \nabla_h^2 u\|_{L^2}^{\frac{1}{2}} \\
&\leq C\|\nabla u\|_{H^2}\|\nabla_h^2 u\|_{L^2}^2 + C\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h^2 u\|_{L^2}\|\nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla_h \nabla^2 u\|_{L^2}.
\end{aligned} \tag{4.6}
$$

Thus,

$$
\begin{aligned}
&\int_0^t (1+\tau)J_{12}(\tau)\mathrm{d}\tau \\
&\leq C \sup_{0\leq\tau\leq t} \|\nabla u(\tau)\|_{H^2} \int_0^t (1+\tau)\|\nabla_h^2 u(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\quad + C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\nabla\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\quad \int_0^t (1+\tau)^{\frac{1}{2}} \|\nabla_h^2 u(\tau)\|_{L^2} \|\nabla_h \nabla^2 u(\tau)\|_{L^2} \mathrm{d}\tau \\
&\leq C E_0^{\frac{1}{2}}(t) E_1(t) + C E_2^{\frac{1}{4}}(t) E_0^{\frac{1}{4}}(t) E_1^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t) \\
&\leq C E^{\frac{3}{2}}(t).
\end{aligned}
\tag{4.7}
$$

The bound for J_{13} is more complicated; we first decompose it as follows:

$$
\begin{aligned}
J_{13} &= \int \nabla_h \partial_3 u \cdot \nabla u \cdot \nabla_h \partial_3 u \,\mathrm{d}x + \int \nabla_h u \cdot \nabla \partial_3 u \cdot \nabla_h \partial_3 u \mathrm{d}x \\
&\quad + \int \partial_3 u \cdot \nabla\nabla_h u \cdot \nabla_h \partial_3 u \,\mathrm{d}x \\
&\leq 3 \int |\nabla_h u|\, |\partial_3 \nabla_h u|^2 \,\mathrm{d}x + 2 \int |\partial_3 u|\, |\nabla_h^2 u|\, |\partial_3 \nabla_h u| \,\mathrm{d}x \\
&\quad + \int |\nabla_h u_3|\, |\partial_3^2 u|\, |\partial_3 \nabla_h u| \,\mathrm{d}x \\
&:= J_{131} + J_{132} + J_{133}.
\end{aligned}
$$

By means of (2.3) and (2.4),

$$
\begin{aligned}
J_{131} &\leq C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{H^1}^{\frac{3}{2}},
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
J_{132} &\leq C \|\partial_3 u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \partial_3 u\|_{L^2}^{\frac{1}{4}} \|\partial_3^2 u\|_{L^2}^{\frac{1}{4}} \|\partial_2 \partial_3^2 u\|_{L^2}^{\frac{1}{4}} \|\nabla_h^2 u\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\partial_3 u\|_{H^1}^{\frac{1}{2}} \|\partial_2 \partial_3 u\|_{H^1}^{\frac{1}{2}} \|\nabla_h^2 u\|_{L^2} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}},
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
J_{133} &\leq C \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}} \\
&\leq C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u\|_{H^1} \|\partial_1 \partial_3 \nabla_h u\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
\tag{4.10}
$$

Thereby, applying Hölder's inequality gives

$$
\begin{aligned}
&\int_0^t (1+\tau) J_{131}(\tau) \mathrm{d}\tau \\
&\quad \leq C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} (1+\tau)^{\frac{1}{4}} \|\partial_3 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times \int_0^t (1+\tau)^{\frac{1}{4}} \|\partial_2 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u(\tau)\|_{H^1}^{\frac{3}{2}} \,\mathrm{d}\tau \\
&\quad \leq C E_2^{\frac{1}{4}}(t) E_1^{\frac{1}{2}}(t) E_0^{\frac{3}{4}}(t) \leq C E^{\frac{3}{2}}(t), \qquad (4.11)\\
&\int_0^t (1+\tau) J_{132}(\tau) \mathrm{d}\tau \\
&\quad \leq C \sup_{0\leq\tau\leq t} \|\partial_3 u(\tau)\|_{H^1}^{\frac{1}{2}} (1+\tau)^{\frac{1}{4}} \|\partial_3 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \int_0^t (1+\tau)^{\frac{1}{2}} \|\nabla_h^2 u(\tau)\|_{L^2} \\
&\qquad \times (1+\tau)^{\frac{1}{4}} \|\partial_3 \partial_1 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 u(\tau)\|_{H^1}^{\frac{1}{2}} \,\mathrm{d}\tau \qquad (4.12)\\
&\quad \leq C E_0^{\frac{1}{4}}(t) E_1(t) E_0^{\frac{1}{4}}(t) \leq C E^{\frac{3}{2}}(t),\\
&\int_0^t (1+\tau) J_{133}(\tau) \mathrm{d}\tau \\
&\quad \leq C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 u(\tau)\|_{L^2}^{\frac{1}{2}} \int_0^t (1+\tau)^{\frac{1}{4}} \|\nabla_h^2 u(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times (1+\tau)^{\frac{1}{4}} \|\partial_3 \partial_1 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u(\tau)\|_{H^1} \,\mathrm{d}\tau \\
&\quad \leq C E_2^{\frac{1}{4}}(t) E_0^{\frac{1}{4}}(t) E_1^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t) \leq C E^{\frac{3}{2}}(t). \qquad (4.13)
\end{aligned}
$$

Adding (4.11), (4.12) and (4.13) yields

$$\int_0^t (1+\tau) J_{13}(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t). \qquad (4.14)$$

Consequently, according to the estimates (4.5), (4.7) and (4.14), we derive

$$\int_0^t J_1(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t). \qquad (4.15)$$

In what following, we handle J_3. The terms J_2 and J_4 will be estimated together later. Firstly,

$$
\begin{aligned}
J_3 = -(1+t)\Big(&\int \nabla_h (u\cdot\nabla b)\cdot \nabla_h b \,\mathrm{d}x + \int \nabla_h^2 (u\cdot\nabla b)\cdot \nabla_h^2 b \,\mathrm{d}x \\
&+ \int \nabla_h \partial_3 (u\cdot\nabla b)\cdot \nabla_h \partial_3 b \,\mathrm{d}x \Big) \\
:= -(1+t)&(J_{31} + J_{32} + J_{33}).
\end{aligned}
$$

Invoking (4.4) and (4.6), we have

$$
\begin{aligned}
J_{31} &= \int \nabla_h u_h \cdot \nabla_h b \cdot \nabla_h b \, \mathrm{d}x + \int \nabla_h u_3 \, \partial_3 b \cdot \nabla_h b \, \mathrm{d}x \\
&\le C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla_h b\|_{L^2}^{\frac{1}{2}} \\
&\quad + C \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h u_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \|\nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla_h b\|_{L^2}^{\frac{1}{2}}, \\
J_{32} &= \int \nabla_h^2 u \cdot \nabla b \cdot \nabla_h^2 b \, \mathrm{d}x + 2 \int \nabla_h u \cdot \nabla \nabla_h b \cdot \nabla_h^2 b \, \mathrm{d}x \\
&\le C \|\nabla b\|_{H^2} \|\nabla_h^2 u\|_{L^2} \|\nabla_h^2 b\|_{L^2} \\
&\quad + C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\nabla_h^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla_h^2 b\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Then a similar argument to (4.5) and (4.7) gives

$$
\int_0^t (1+\tau)(J_{31} + J_{32})(\tau) \mathrm{d}\tau \le C E^{\frac{3}{2}}(t).
$$

For J_{33}, we still reformulate it into several integrals

$$
\begin{aligned}
J_{33} &\le 2 \int |\nabla_h u| \, |\partial_3 \nabla_h b|^2 \, \mathrm{d}x + \int |\nabla_h b| \, |\partial_3 \nabla_h u_h| |\partial_3 \nabla_h b| \, \mathrm{d}x \\
&\quad + \int |\partial_3 b| \, |\nabla_h^2 u| \, |\partial_3 \nabla_h b| \, \mathrm{d}x + \int |\partial_3 u| \, |\nabla_h^2 b| \, |\partial_3 \nabla_h b| \, \mathrm{d}x \\
&\quad + \int |\nabla_h u_3| \, |\partial_3^2 b| \, |\partial_3 \nabla_h b| \, \mathrm{d}x \\
&:= 2 J_{331} + \cdots + J_{335}.
\end{aligned}
$$

Going through a similar process as in J_{13}, we are able to establish the bound for J_{33}. Recalling (4.8), we have

$$
J_{331} \le C \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 \nabla_h b\|_{L^2}^{\frac{1}{2}}.
$$

Then

$$
\begin{aligned}
\int_0^t (1+\tau) J_{331}(\tau) \mathrm{d}\tau &\le C \sup_{0 \le \tau \le t} (1+\tau)^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} (1+\tau)^{\frac{1}{4}} \|\partial_3 \nabla_h b(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \int_0^t (1+\tau)^{\frac{1}{4}} \|\partial_2 \nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b(\tau)\|_{H^1}^{\frac{3}{2}} \, \mathrm{d}\tau \le C E^{\frac{3}{2}}(t).
\end{aligned}
$$

As in (4.9) and (4.12), J_{333} can be bounded by

$$
\begin{aligned}
&\int_0^t (1+\tau) J_{333}(\tau) \mathrm{d}\tau \\
&\quad \le C \sup_{0 \le \tau \le t} \|\partial_3 b(\tau)\|_{H^1}^{\frac{1}{2}} (1+\tau)^{\frac{1}{4}} \|\partial_3 \nabla_h b(\tau)\|_{L^2}^{\frac{1}{2}} \int_0^t (1+\tau)^{\frac{1}{2}} \|\nabla_h^2 u(\tau)\|_{L^2} \\
&\qquad \times (1+\tau)^{\frac{1}{4}} \|\partial_3 \partial_1 \nabla_h b(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_2 b(\tau)\|_{H^1}^{\frac{1}{2}} \, \mathrm{d}\tau \le C E^{\frac{3}{2}}(t).
\end{aligned}
$$

Also, from (4.10) and (4.13), we get

$$
\begin{aligned}
&\int_0^t (1+\tau) J_{335}(\tau)\mathrm{d}\tau \\
&\quad \leq C \sup_{0\leq\tau\leq t} (1+\tau)^{\frac{1}{2}} \|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3^2 b(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\quad \int_0^t (1+\tau)^{\frac{1}{4}} \|\nabla_h \partial_3 u_3(\tau)\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times (1+\tau)^{\frac{1}{4}} \|\partial_3\partial_1 \nabla_h b(\tau)\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h b(\tau)\|_{H^1} \,\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
\end{aligned}
$$

The rest terms J_{332} and J_{334} can be handled as J_{331} and J_{333}, respectively. Thus, we have

$$
\int_0^t (1+\tau)(J_{332}(\tau) + J_{334}(\tau)) \,\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

Consequently, we derive

$$
\int_0^t (1+\tau) J_{33}(\tau)\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

Combining all estimates above for J_{31} through J_{33}, we conclude that

$$
\int_0^t J_3(\tau)\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t). \tag{4.16}
$$

Finally we bound J_2 and J_4. J_2 and J_4 can be estimated with a nearly same argument as J_1 and J_3, respectively. We shall just sketch the proof. By integration by parts and the divergence-free condition, we split J_2 and J_4 into three parts as

$$
J_2 + J_4 := (J_{21} + J_{22} + J_{23})(1+t),
$$

where

$$
\begin{aligned}
J_{21} &= \int (\nabla_h b \cdot \nabla b \cdot \nabla_h u + \nabla_h b \cdot \nabla u \cdot \nabla_h b) \,\mathrm{d}x, \\
J_{22} &= \int \Big[(\nabla\nabla_h b \cdot \nabla) b \cdot \nabla\nabla_h u + (\nabla b \cdot \nabla)\nabla_h b \cdot \nabla\nabla_h u + (\nabla_h b \cdot \nabla)\nabla b \cdot \nabla\nabla_h u \Big] \mathrm{d}x, \\
J_{23} &= \int \Big[(\nabla\nabla_h b \cdot \nabla) u \cdot \nabla\nabla_h b + (\nabla b \cdot \nabla)\nabla_h u \cdot \nabla\nabla_h b + (\nabla_h b \cdot \nabla)\nabla u \cdot \nabla\nabla_h b \Big] \mathrm{d}x.
\end{aligned}
$$

It is easy to verify that

$$
\int_0^t (J_2(\tau) + J_4(\tau))\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t). \tag{4.17}
$$

According to (4.4) and (4.5),

$$
\int_0^t (1+\tau) J_{21}(\tau)\mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

For J_{22}, we further divide it into two parts:

$$
\begin{aligned}
J_{22} = & \int \left(\nabla_h^2 b \cdot \nabla b \cdot \nabla_h^2 u + 2\nabla_h b \cdot \nabla\nabla_h b \cdot \nabla_h^2 u\right) \mathrm{d}x \\
& + \int \left(\nabla_h \partial_3 b \cdot \nabla b \cdot \nabla_h \partial_3 u + \partial_3 b \cdot \nabla\nabla_h b \cdot \nabla_h \partial_3 u + \nabla_h b \cdot \nabla\partial_3 b \cdot \nabla_h \partial_3 u\right) \mathrm{d}x \\
:= & J_{221} + J_{222}.
\end{aligned}
$$

As in (4.6) and (4.7) for J_{12},

$$
\int_0^t (1+\tau) J_{221}(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

For J_{222}, we have

$$
\begin{aligned}
J_{222} \leq & \, 3 \int |\nabla_h b|\, |\partial_3 \nabla_h b|\, |\partial_3 \nabla_h u| \, \mathrm{d}x + 2 \int |\partial_3 b|\, |\nabla_h^2 b|\, |\partial_3 \nabla_h u| \, \mathrm{d}x \\
& + \int |\nabla_h b_3|\, |\partial_3^2 b|\, |\partial_3 \nabla_h u| \, \mathrm{d}x.
\end{aligned}
$$

Using the similarities between J_{222} and J_{131}, J_{132} and J_{133}, we can easily find that

$$
\int_0^t (1+\tau) J_{222}(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

Therefore,

$$
\int_0^t (1+\tau) J_{22}(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

To bound J_{23}, we decompose it into

$$
\begin{aligned}
J_{23} \leq & \int (\nabla_h^2 b \cdot \nabla u \cdot \nabla_h^2 b + 2\nabla_h b \cdot \nabla\nabla_h u \cdot \nabla_h^2 b) \, \mathrm{d}x \\
& + J_{331} + 2J_{332} + J_{333} + J_{334} + \int |\nabla_h b_3|\, |\partial_3^2 u|\, |\partial_3 \nabla_h b| \, \mathrm{d}x.
\end{aligned}
$$

The first term and the last term are similar to J_{32} and J_{335}, respectively. Thus,

$$
\int_0^t (1+\tau) J_{23}(\tau) \mathrm{d}\tau \leq C E^{\frac{3}{2}}(t).
$$

Integrating (4.3) over $[0, t]$ and invoking (4.15), (4.16) and (4.17), we derive the desired estimate (4.2). This competes the proof of Lemma 4.2. □

We now turn to the second lemma.

Lemma 4.3. *Assume* (u, b) *is a solution to* (1.4). *Then we have*

$$
\begin{aligned}
&-(1+t)(\partial_2\nabla_h u(t), \nabla_h b(t)) + \frac{1}{2}\int_0^t (1+\tau)\|\partial_2\nabla_h u(\tau)\|_{L^2}^2 \mathrm{d}\tau \\
&\quad - \frac{1}{2}\int_0^t (1+\tau)\Big(3\|\partial_2\nabla_h b(\tau)\|_{L^2}^2 + \mu^2\|\nabla_h\partial_1^2 u(\tau)\|_{L^2}^2 + \eta^2\|\nabla_h\Delta_h b(\tau)\|_{L^2}^2\Big)\mathrm{d}\tau \\
&\quad \le CE(0) + \frac{1}{2}E_0(t) + CE^{\frac{3}{2}}(t).
\end{aligned}
\tag{4.18}
$$

Proof of Lemma 4.3. As in (3.16), we have

$$
\begin{aligned}
&-\frac{d}{dt}(1+t)(\partial_2\nabla_h u, \nabla_h b) + (1+t)\|\partial_2\nabla_h u\|_{L^2}^2 - (1+t)\|\partial_2\nabla_h b\|_{L^2}^2 \\
&= -(\partial_2\nabla_h u, \nabla_h b) + (1+t)\big(\partial_2\nabla_h(u\cdot\nabla u), \nabla_h b\big) - (1+t)\big(\partial_2\nabla_h(b\cdot\nabla b), \nabla_h b\big) \\
&\quad + (1+t)\big(\partial_2\nabla_h u, \nabla_h(u\cdot\nabla b)\big) - (1+t)\big(\partial_2\nabla_h u, \nabla_h(b\cdot\nabla u)\big) \\
&\quad - \mu(1+t)(\partial_2\nabla_h\partial_1^2 u, \nabla_h b) - \eta(1+t)(\partial_2\nabla_h u, \nabla_h\Delta_h b) \\
&:= J_5 + \cdots + J_{11},
\end{aligned}
$$

where (F, G) denotes the L^2-inner product of F and G. It is clear that

$$
\begin{aligned}
&\int_0^t J_5(\tau)\mathrm{d}\tau \le \frac{1}{2}\int_0^t (\|\partial_2\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h b(\tau)\|_{L^2}^2)\mathrm{d}\tau \le \frac{1}{2}E_0(t), \\
&J_{10} = \mu(1+t)(\nabla_h\partial_1^2 u, \partial_2\nabla_h b) \le \frac{\mu^2}{2}(1+t)\|\partial_1^2\nabla_h u\|_{L^2}^2 + \frac{1}{2}(1+t)\|\partial_2\nabla_h b\|_{L^2}^2, \\
&J_{11} \le \frac{1}{2}(1+t)\|\partial_2\nabla_h u\|_{L^2}^2 + \frac{\eta^2}{2}(1+t)\|\Delta_h\nabla_h b\|_{L^2}^2.
\end{aligned}
$$

Next we bound the nonlinear integral terms. We mainly focus on J_6 and J_8. The estimates for J_7 and J_9 can be established similarly. By integration by parts and (2.3), we have

$$
\begin{aligned}
J_6 &= -(1+t)\int \nabla_h u\cdot\nabla u\cdot\partial_2\nabla_h b\,\mathrm{d}x - (1+t)\int u\cdot\nabla\nabla_h u\cdot\partial_2\nabla_h b\,\mathrm{d}x \\
&\le C(1+t)\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h b\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_2\nabla_h b\|_{L^2}^{\frac{1}{2}} \\
&\quad + C(1+t)\|u\|_{L^2}^{\frac{1}{2}}\|\partial_2 u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_1\nabla\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_2\nabla_h b\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_2\nabla_h b\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int_0^t J_6(\tau)\mathrm{d}\tau &\le C\sup_{0\le t\le\tau}(1+\tau)^{\frac{1}{2}}\|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}}\|\nabla u(\tau)\|_{L^2}^{\frac{1}{2}}\int_0^t \|\nabla_h\nabla u(\tau)\|_{L^2} \\
&\qquad \times (1+\tau)^{\frac{1}{2}}\|\partial_2\nabla_h b(\tau)\|_{H^1}\mathrm{d}\tau \\
&\quad + C\sup_{0\le t\le\tau}(1+\tau)^{\frac{1}{2}}\|\partial_2 u(\tau)\|_{L^2}^{\frac{1}{2}}\|u(\tau)\|_{L^2}^{\frac{1}{2}}\int_0^t \|\nabla_h\nabla u(\tau)\|_{H^1} \\
&\qquad \times (1+\tau)^{\frac{1}{2}}\|\partial_2\nabla_h b(\tau)\|_{H^1}\mathrm{d}\tau \\
&\le CE_2^{\frac{1}{4}}(t)E_0^{\frac{3}{4}}(t)E_1^{\frac{1}{2}}(t) \le CE^{\frac{3}{2}}(t).
\end{aligned}
$$

Similarly, we can bound J_7 as

$$\int_0^t J_7(\tau)\mathrm{d}\tau \le CE^{\frac{3}{2}}(t).$$

For J_8, applying the anisotropic inequality (2.4) yields

$$\begin{aligned}
J_8 &= (1+t)\int \nabla_h u\cdot\nabla b\cdot\partial_2\nabla_h u\,\mathrm{d}x + (1+t)\int u\cdot\nabla\nabla_h b\cdot\partial_2\nabla_h u\,\mathrm{d}x\\
&\le C(1+t)\|\nabla_h u\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac14}\|\partial_1\nabla b\|_{L^2}^{\frac14}\|\partial_3\nabla b\|_{L^2}^{\frac14}\|\partial_1\partial_3\nabla b\|_{L^2}^{\frac14}\|\partial_2\nabla_h u\|_{L^2}\\
&\quad + C(1+t)\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla\nabla_h b\|_{L^2}^{\frac12}\|\partial_1\nabla\nabla_h b\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^t J_8(\tau)\mathrm{d}\tau &\le C\sup_{0\le t\le\tau}(1+\tau)^{\frac14}\|\nabla_h u(\tau)\|_{L^2}^{\frac12}\|\nabla b(\tau)\|_{H^1}^{\frac12}\\
&\quad\int_0^t(1+\tau)^{\frac34}\|\partial_2\nabla_h u(\tau)\|_{L^2}^{\frac32}\|\partial_1\nabla b(\tau)\|_{H^1}^{\frac12}\mathrm{d}\tau\\
&\quad + C\sup_{0\le t\le\tau}(1+\tau)^{\frac14}\|\nabla\nabla_h b(\tau)\|_{L^2}^{\frac12}\|u(\tau)\|_{H^1}^{\frac12}\int_0^t(1+\tau)^{\frac14}\|\partial_1\nabla_h\nabla b(\tau)\|_{L^2}^{\frac12}\\
&\quad\times(1+\tau)^{\frac12}\|\partial_2\nabla_h u(\tau)\|_{L^2}\|\partial_2 u(\tau)\|_{H^1}^{\frac12}\mathrm{d}\tau\\
&\le CE_1(t)E_0^{\frac12}(t)\le CE^{\frac32}(t).
\end{aligned}$$

Also,

$$\int_0^t J_9(\tau)\mathrm{d}\tau \le CE^{\frac{3}{2}}(t).$$

Collecting all the estimates for J_5 through J_{11}, and integrating over $[0,t]$, we derive the desired bound (4.18). This completes the proof of lemma 4.3. □

We now put together the two lemmas above to obtain Proposition 4.1.

Proof of Proposition 4.1. According to Lemma 4.2 and 4.3, the combination (4.2)+ λ_1(4.18) yields

$$\begin{aligned}
&(1+t)\big(\|\nabla_h u(t)\|_{H^1}^2+\|\nabla_h b(t)\|_{H^1}^2-\lambda_1(\partial_2\nabla_h u,\nabla_h b)\big)\\
&\quad+\int_0^t(1+\tau)\Big[\big(2\mu-\frac{\mu^2}{2}\lambda_1\big)\|\partial_1\nabla_h u(\tau)\|_{H^1}^2\\
&\qquad+\frac{\lambda_1}{2}(1+\tau)\|\partial_2\nabla_h u(\tau)\|_{L^2}^2+\big(2\eta-\frac32\lambda_1-\frac{\eta^2}{2}\lambda_1\big)\|\Delta_h b(\tau)\|_{H^1}^2\Big]\mathrm{d}\tau\\
&\quad\le CE(0)+CE_0(t)+CE^{\frac32}(t),
\end{aligned}$$

where λ_1 is a parameter. If λ_1 is sufficiently small, then

$$
\begin{aligned}
&(1+t)\Big(\|\nabla_h u(t)\|_{H^1}^2 + \|\nabla_h b(t)\|_{H^1}^2\Big) \\
&+ \int_0^t (1+\tau)\Big(\|\partial_1 \nabla_h u(\tau)\|_{H^1}^2 + \|\partial_2 \nabla_h u(\tau)\|_{L^2}^2 + \|\nabla_h^2 b(\tau)\|_{H^1}^2\Big)\mathrm{d}\tau \\
&\le CE(0) + CE_0(t) + CE^{\frac{3}{2}}(t).
\end{aligned}
$$

This completes the proof of Proposition 4.1. □

5. Estimate for $E_2(t)$

This section establishes the *a priori* inequality (1.16) for $E_2(t)$. That is, we prove the following proposition:

Proposition 5.1. *Let (u, b) be a solution to the system (1.4). Then it holds*

$$
\begin{aligned}
E_2(t) \le C\Big(E^{\frac{3}{2}}(t) + E^2(t)\Big) + C\Big(&\|(u_0, b_0)\|_{H^2}^2 + \|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2 \\
&+ \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2 + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2\Big). \qquad (5.1)
\end{aligned}
$$

We remark that energy estimates are no longer sufficient for the proof of (5.1). We resort to the integral representation of (1.4). To convert (1.4) into an integral representation, we take the Fourier transform of (1.4), solve the linearized system and represent the nonlinear system into an integral form via Duhamel's principle. The integral representation involves three key kernel functions, which are degenerate and anisotropic. Due to the anisotropic nature, we divide the frequency space into subdomains to obtain sharp upper bounds on the kernel functions. This is done in Proposition 5.4. Once these bounds are at our disposal, we then estimate the L^2-norms of (u, b) and its derivatives via the integral representation. For the sake of clarity, we divide the rest of this section into two subsections.

5.1. Integral Representation and Bounds for the Kernels

The subsection derives the integral representation of (1.4) and establishes optimal upper bounds for the kernel functions. First we recall two basic tools. The first one specifies the decay rate of a general heat operator associated with a fractional Laplacian operator. Here the fractional Laplacian operator can be defined through the Fourier transform

$$
\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi).
$$

The decay rate is stated in the following lemma, whose proof can be found in many references (see, example, [89]).

Lemma 5.2. *Assume $\alpha \geq 0$ and $\beta > 0$ are real numbers. Let $1 \leq p \leq q \leq \infty$. Then there exists a constant $C > 0$ such that, for any $t > 0$,*

$$\|\Lambda^\alpha e^{-\Lambda^\beta t} f\|_{L^q(\mathbb{R}^d)} \leq C\, t^{-\frac{\alpha}{\beta}-\frac{d}{\beta}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^d)}.$$

The second tool is an elementary inequality providing upper bounds for a convolution type integral. Its proof is straightforward.

Lemma 5.3. *Assume $0 < s_1 \leq s_2$. Then, for some constant $C > 0$,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\, \mathrm{d}\tau \leq \begin{cases} C(1+t)^{-s_1}, & \text{if } s_2 > 1, \\ C(1+t)^{-s_1}\ln(1+t), & \text{if } s_2 = 1, \\ C(1+t)^{1-s_1-s_2}, & \text{if } s_2 < 1. \end{cases} \tag{5.2}$$

Now we derive an integral representation of (1.4). Applying the Leray-Hopf projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation in (1.4) and taking the Fourier transform of the resulting equations, we have

$$\partial_t \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = A \begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} + \begin{pmatrix} \widehat{N}_1 \\ \widehat{N}_2 \end{pmatrix}, \tag{5.3}$$

where

$$A = \begin{pmatrix} -\mu\xi_1^2 & i\xi_2 \\ i\xi_2 & -\eta|\xi_h|^2 \end{pmatrix}, \quad N_1 = \mathbb{P}(b\cdot\nabla b - u\cdot\nabla u), \quad N_2 = b\cdot\nabla u - u\cdot\nabla b$$

with $|\xi_h|^2 = \xi_1^2 + \xi_2^2$. To diagonalize A, we compute the eigenvalues of A,

$$\lambda_1 = \frac{-(\mu\xi_1^2 + \eta|\xi_h|^2) - \sqrt{\Gamma}}{2}, \qquad \lambda_2 = \frac{-(\mu\xi_1^2 + \eta|\xi_h|^2) + \sqrt{\Gamma}}{2},$$

where

$$\Gamma = (\mu\xi_1^2 + \eta|\xi_h|^2)^2 - 4(\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2).$$

The corresponding eigenvectors are

$$\rho_1 = \begin{pmatrix} \lambda_1 + \eta|\xi_h|^2 \\ i\xi_2 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \lambda_2 + \eta|\xi_h|^2 \\ i\xi_2 \end{pmatrix}.$$

Therefore, the matrix A can be diagonalized as

$$A = (\rho_1, \rho_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\rho_1, \rho_2)^{-1}. \tag{5.4}$$

We can now represent (5.3) as

$$\begin{pmatrix} \widehat{u} \\ \widehat{b} \end{pmatrix} = e^{At} \begin{pmatrix} \widehat{u_0} \\ \widehat{b_0} \end{pmatrix} + \int_0^t e^{A(t-\tau)} \begin{pmatrix} \widehat{N_1}(\tau) \\ \widehat{N_2}(\tau) \end{pmatrix} d\tau.$$

By making e^{At} more explicit via (5.4), we obtain the integral representation

$$\widehat{u}(\xi,t) = \widehat{Q}_1(t)\widehat{u}_0 + \widehat{Q}_2(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_1(t-\tau)\widehat{N}_1(\tau) + \widehat{Q}_2(t-\tau)\widehat{N}_2(\tau)\big)\, d\tau, \tag{5.5}$$

$$\widehat{b}(\xi,t) = \widehat{Q}_2(t)\widehat{u}_0 + \widehat{Q}_3(t)\widehat{b}_0 + \int_0^t \big(\widehat{Q}_2(t-\tau)\widehat{N}_1(\tau) + \widehat{Q}_3(t-\tau)\widehat{N}_2(\tau)\big)\, d\tau, \tag{5.6}$$

where

$$\begin{aligned} &\widehat{Q}_1(t) = \eta|\xi_h|^2 G_1(t) + G_2(t), \quad \widehat{Q}_2(t) = i\xi_2\, G_1(t), \\ &\widehat{Q}_3(t) = -\eta|\xi_h|^2 G_1(t) + G_3(t), \end{aligned} \tag{5.7}$$

with

$$\begin{aligned} &G_1(t) = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}, \qquad G_2(t) = \frac{\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{\lambda_2 t} + \lambda_1 G_1(t), \\ &G_3(t) = \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1} = e^{\lambda_1 t} - \lambda_1 G_1(t). \end{aligned}$$

We remark that when $\lambda_1 = \lambda_2$, the representation in (5.5) and (5.6) remains valid if we replace G_1 by its limiting form

$$G_1(t) = \lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = t\, e^{\lambda_1 t}.$$

Next we investigate the behaviors of the kernels $\widehat{Q}_i(\xi,t)$ $(i = 1, 2, 3)$, which play a crucial role in the estimate of $E_2(t)$. There kernels are anisotropic and degenerate. To obtain precise and sharp upper bounds, we divide the frequency space into subdomains and classify the behavior of the kernel functions in each subdomain.

Proposition 5.4. *The domain $\mathbb{R}^3$ is split into two subdomains, $\mathbb{R}^3 = A_1 \cup A_2$ with*

$$A_1 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{\Gamma} \le \frac{\mu\xi_1^2 + \eta|\xi_h|^2}{2} \ \ or\ \ 3(\mu\xi_1^2 + \eta|\xi_h|^2)^2 \le 16(\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2) \right\},$$

$$A_2 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{\Gamma} > \frac{\mu\xi_1^2 + \eta|\xi_h|^2}{2} \ \ or\ \ 3(\mu\xi_1^2 + \eta|\xi_h|^2)^2 > 16(\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2) \right\}.$$

Then we have

(1) There exist two constants $C > 0$ and $c_0 > 0$ such that, for any $\xi \in A_1$,

$$Re\lambda_1 \leq -\frac{\mu\xi_1^2 + \eta|\xi_h|^2}{2}, \ Re\lambda_2 \leq -\frac{\mu\xi_1^2 + \eta|\xi_h|^2}{4},$$
$$|G_1(t)| \leq te^{-\frac{\mu\xi_1^2+\eta|\xi_h|^2}{4}t}, \quad |\widehat{Q}_i(\xi,t)| \leq Ce^{-c_0|\xi_h|^2 t}, \ i = 1, 2, 3.$$

(2) There is a constant $C > 0$ such that, for any $\xi \in A_2$,

$$\lambda_1 < -\frac{3(\mu\xi_1^2 + \eta|\xi_h|^2)}{4}, \ \lambda_2 \leq -\frac{\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2}{\mu\xi_1^2 + \eta|\xi_h|^2},$$
$$|G_1(t)| < \frac{2}{\mu\xi_1^2 + \eta|\xi_h|^2}\left(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}\right),$$
$$|\widehat{Q}_i(t)| < C(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}), \ i = 1, 2, 3.$$

If we further divide A_2 into three subdomains A_{21}, A_{22}, A_{23},

$$A_{21} = \{\xi \in A_2, \quad |\xi_1| \sim |\xi_2|\},$$
$$A_{22} = \{\xi \in A_2, \quad |\xi_1| >> |\xi_2|\},$$
$$A_{23} = \{\xi \in A_2, \quad |\xi_1| << |\xi_2|\},$$

then, for some constants $C > 0, c_1 > 0, c_2 > 0, c_3 > 0$ and $i = 1, 2, 3$,

$$|\widehat{Q}_i(t)| \leq C\, e^{-c_1|\xi_h|^2 t}, \quad \text{if} \ \ \xi \in A_{21},$$
$$|\widehat{Q}_i(t)| \leq C\, e^{-c_1|\xi_h|^2 t}, \quad \text{if} \ \ \xi \in A_{22},$$
$$|\widehat{Q}_i(t)| \leq C\, (e^{-c_1|\xi_h|^2 t} + e^{-c_2\xi_1^2 t - c_3 t}), \quad \text{if} \ \ \xi \in A_{23}.$$

Proof of Proposition 5.4. (1) For $\xi \in A_1$, $\sqrt{\Gamma} \leq \frac{\mu\xi_1^2+\eta|\xi_h|^2}{2}$. Through the direct estimates and the mean-value theorem, we have

$$-\frac{3(\mu\xi_1^2 + \eta|\xi_h|^2)}{4} \leq Re\lambda_1 \leq -\frac{\mu\xi_1^2 + \eta|\xi_h|^2}{2},$$
$$Re\lambda_2 \leq -\frac{\mu\xi_1^2 + \eta|\xi_h|^2}{4}, \quad |G_1(t)| \leq te^{-\frac{\mu\xi_1^2+\eta|\xi_h|^2}{4}t}. \tag{5.8}$$

To bound the kernel functions $\widehat{Q}_1(t)$ and $\widehat{Q}_3(t)$, we consider two cases: λ_1 is a real number and λ_1 is an imaginary number. If λ_1 is a real number, for some pure constant c_0 dependent of μ and η, we have

$$\begin{aligned}
|\widehat{Q}_1(t)| &= \left|\eta|\xi_h|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right| \\
&\leq C(\mu\xi_1^2 + \eta|\xi_h|^2)te^{-\frac{\mu\xi_1^2+\eta|\xi_h|^2}{4}t} + e^{-\frac{\mu\xi_1^2+\eta|\xi_h|^2}{4}t} \\
&\leq Ce^{-c_0|\xi_h|^2 t},
\end{aligned}$$

where we have used the simple fact that $x\, e^{-x} \le C$ for $x \ge 0$. If λ_1 is an imaginary number, namely $\Gamma < 0$, then

$$|\lambda_1|^2 = \mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2, \quad \Gamma = 4|\lambda_1|^2 - (\mu\xi_1^2 + \eta|\xi_h|^2)^2.$$

Clearly, (5.8) implies

$$\left|\eta|\xi_h|^2 G_1(t) + e^{\lambda_2 t}\right| \le Ce^{-c_0|\xi_h|^2 t}.$$

Now we bound $|\lambda_1 G_1(t)|$. we further divide the consideration into two subcases: $|\lambda_1| \le |\sqrt{\Gamma}\,|$ and $|\lambda_1| \ge |\sqrt{\Gamma}\,|$. In the case when $|\lambda_1| \le |\sqrt{\Gamma}|$, by the definition of G_1, we obtain

$$|\lambda_1 G_1(t)| = \frac{|\lambda_1|}{|\sqrt{\Gamma}|}\,|e^{\lambda_1 t} - e^{\lambda_2 t}| \le Ce^{-\frac{\mu\xi_1^2+\eta|\xi_h|^2}{4}t}.$$

In the case when $|\lambda_1| \ge |\sqrt{\Gamma}\,|$, we have

$$|\lambda_1|^2 \ge 4|\lambda_1|^2 - (\mu\xi_1^2 + \eta|\xi_h|^2)^2,$$

or

$$\sqrt{3}|\lambda_1| \le \mu\xi_1^2 + \eta|\xi_h|^2.$$

Thus,

$$|\lambda_1 G_1(t)| \le \frac{1}{\sqrt{3}}(\mu\xi_1^2 + \eta|\xi_h|^2)|G_1| \le C(\mu\xi_1^2+\eta|\xi_h|^2)te^{-\frac{(\mu\xi_1^2+\eta|\xi_h|^2)}{4}t} \le Ce^{-c_0|\xi_h|^2 t}.$$

Consequently, if λ_1 is an imaginary number, we derive

$$|\widehat{Q}_1(t)| \le Ce^{-c_0|\xi_h|^2 t}.$$

In summary, for $\xi \in A_1$,

$$|\widehat{Q}_1(t)| \le Ce^{-c_0|\xi_h|^2 t}.$$

Similarly, we have

$$|\widehat{Q}_3(t)| = \left|-\eta|\xi_h|^2 G_1(t) - \lambda_1 G_1(t) + e^{\lambda_1 t}\right| \le Ce^{-c_0|\xi_h|^2 t}.$$

Now we bound $\widehat{Q}_2(t)$. As in the estimate of $\widehat{Q}_1(t)$, we consider the following two cases: $|\xi_2| \le |\sqrt{\Gamma}|$ and $|\xi_2| \ge |\sqrt{\Gamma}|$. In the first case $|\xi_2| \le |\sqrt{\Gamma}\,|$, by the definition of $\widehat{Q}_2(t)$ in (5.7),

$$|\widehat{Q}_2(t)| = \left|\frac{\xi_2}{\sqrt{\Gamma}}\right||e^{\lambda_1 t} - e^{\lambda_2 t}| \le C\,e^{-c_0|\xi_h|^2 t}.$$

In the second case, $|\xi_2| \ge |\sqrt{\Gamma}\,|$,

$$\left|(\mu\xi_1^2 + \eta|\xi_h|^2)^2 - 4(\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2)\right| \le \xi_2^2,$$

or

$$-\xi_2^2 \le (\mu\xi_1^2 + \eta|\xi_h|^2)^2 - 4(\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2) \le \xi_2^2,$$

which implies

$$3\xi_2^2 + 4\mu\eta\xi_1^2|\xi_h|^2 \le (\mu\xi_1^2 + \eta|\xi_h|^2)^2.$$

In particular,

$$\sqrt{3}|\xi_2| \le \mu\xi_1^2 + \eta|\xi_h|^2.$$

Therefore,

$$|\widehat{Q}_2(t)| \le \frac{1}{\sqrt{3}}(\mu\xi_1^2 + \eta|\xi_h|^2)\ |G_1(t)| \le C\, e^{-c_0|\xi_h|^2 t}.$$

(2) For $\xi \in A_2$, we have $\frac{\mu\xi_1^2+\eta|\xi_h|^2}{2} < \sqrt{\Gamma} \le \mu\xi_1^2 + \eta|\xi_h|^2$. It then follows that

$$\begin{aligned} &-(\mu\xi_1^2 + \eta|\xi_h|^2) \le \lambda_1 < -\frac{3}{4}(\mu\xi_1^2 + \eta|\xi_h|^2),\\ &\lambda_2 = \frac{\Gamma - (\mu\xi_1^2 + \eta|\xi_h|^2)^2}{2(\mu\xi_1^2 + \eta|\xi_h|^2 + \sqrt{\Gamma})} \le -\frac{\mu\eta\xi_1^2|\xi_h|^2 + \xi_2^2}{\mu\xi_1^2 + \eta|\xi_h|^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |G_1(t)| &\le \frac{1}{\lambda_2 - \lambda_1}(e^{\lambda_1 t} + e^{\lambda_2 t})\\ &< \frac{2}{\mu\xi_1^2 + \eta|\xi_h|^2}\left(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}\right). \end{aligned}$$

As a consequence,

$$\begin{aligned} |\widehat{Q}_1(t)| &= \left|\eta|\xi_h|^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_2 t}\right|\\ &\le 2(\mu\xi_1^2 + \eta|\xi_h|^2)|G_1(t)| + e^{\lambda_2 t}\\ &< C\Big(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}\Big). \end{aligned}$$

Similarly,

$$\begin{aligned} |\widehat{Q}_3(t)| &= \left|-\eta|\xi_h|^2 G_1(t) - \lambda_1 G_1(t) + e^{\lambda_1 t}\right|\\ &< C\left(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}\right). \end{aligned}$$

Due to $\sqrt{\Gamma} > \frac{\mu\xi_1^2+\eta|\xi_h|^2}{2}$, we find

$$\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)^2 > 4(\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2) \geq \xi_2^2.$$

Therefore,

$$|\widehat{Q}_2(t)| < C(\mu\xi_1^2+\eta|\xi_h|^2)\;|G_1(t)| < C\left(e^{-\frac{3}{4}(\mu\xi_1^2+\eta|\xi_h|^2)t} + e^{-\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2}t}\right).$$

Finally, according to the upper bound for $|\widehat{Q}_i(t)|$ $(i = 1, 2, 3)$, by further division of A_2 into A_{21}, A_{22} and A_{23}, we can establish more definite upper bound. For $\xi \in A_{21}$, $\xi_1^2 \sim \xi_2^2$, we have

$$\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2} \quad\sim\quad |\xi_h|^2+1.$$

For $\xi \in A_{22}$, $\xi_1^2 >> \xi_2^2$, there exists a $c_1 > 0$ small sufficiently such that

$$\frac{\mu\eta\xi_1^2|\xi_h|^2+\xi_2^2}{\mu\xi_1^2+\eta|\xi_h|^2} \geq c_1|\xi_h|^2.$$

The behavior $\xi \in A_{23}$ can be similarly identified. This completes the proof of Proposition 5.4. □

5.2. *Proof of Proposition 5.1*

With these preparations at our disposal, we are now ready to prove Proposition 5.1. Since the process is complicated and long, the proof is divided into three lemmas. To do so, we make the following decomposition for $E_2(t)$:

$$E_2(t) = E_{21}(t) + E_{22}(t) + E_{23}(t).$$

Here

$$\begin{aligned}
E_{21}(t) &= \sup_{0\leq\tau\leq t}(1+\tau)\|(u(\tau), b(\tau))\|_{L^2}^2,\\
E_{22}(t) &= \sup_{0\leq\tau\leq t}(1+\tau)^2\|(\nabla_h u(\tau), \nabla_h b(\tau))\|_{L^2}^2\\
&\quad+\sup_{0\leq\tau\leq t}(1+\tau)^{1-2\varepsilon}\|(\partial_3 u(\tau), \partial_3 b(\tau))\|_{L^2}^2,\\
E_{23}(t) &= \sup_{0\leq\tau\leq t}\sum_{k=1}^{2}(1+\tau)^{\frac{5}{2}-2\varepsilon}\|(\partial_1\partial_k u(\tau), \partial_1\partial_k b(\tau))\|_{L^2}^2\\
&\quad+\sup_{0\leq\tau\leq t}(1+\tau)^{2-2\varepsilon}\|(\partial_1\partial_3 u(\tau), \partial_1\partial_3 b(\tau))\|_{L^2}^2\\
&\quad+\sup_{0\leq\tau\leq t}\sum_{k=2}^{3}(1+\tau)^{\frac{4}{3}-2\varepsilon}\|(\partial_2\partial_k u(\tau), \partial_2\partial_k b(\tau))\|_{L^2}^2\\
&\quad+\sup_{0\leq\tau\leq t}(1+\tau)^{\frac{1}{2}}\|(\partial_3^2 u(\tau), \partial_3^2 b(\tau))\|_{L^2}^2.
\end{aligned}$$

Without loss of generality, we assume $t > 1$. In fact, if $0 \le t \le 1$, Proposition 3.1 implies

$$E_2(t) \le C \sup_{0\le\tau\le t} \|(u(\tau), b(\tau))\|_{H^2}^2 \le CE_0(t) \le CE(0) + CE^{\frac{3}{2}}(t). \tag{5.9}$$

Next we present the estimates for $E_{21}(t)$, $E_{22}(t)$ and $E_{23}(t)$, which will be shown in three lemmas. Proposition 5.1 then follows as an immediate consequence.

Lemma 5.5. *Assume that* (u, b) *is a solution to* (1.4)*. Then we have*

$$E_{21}(t) \le CE^2(t) + C(\|(u_0, b_0)\|^2_{L^2_{x_3}L^1_{x_1x_2}} + \|(u_0, b_0)\|^2_{L^2}). \tag{5.10}$$

Proof of Lemma 5.5. Recalling (5.5) and (5.6), and applying Plancherel' theorem, we have

$$\begin{aligned}\|u(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{u}(t)\|_{L^2(\mathbb{R}^3)} &\le \|\widehat{Q}_1(t)\widehat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_2(t)\widehat{b}_0\|_{L^2(\mathbb{R}^3)} \\ &\quad + \int_0^t \|\widehat{Q}_1(t-\tau)\widehat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau + \int_0^t \|\widehat{Q}_2(t-\tau)\widehat{N}_2(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\|b(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{b}(t)\|_{L^2(\mathbb{R}^3)} &\le \|\widehat{Q}_2(t)\widehat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_3(t)\widehat{b}_0\|_{L^2(\mathbb{R}^3)} \\ &\quad + \int_0^t \|\widehat{Q}_2(t-\tau)\widehat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau + \int_0^t \|\widehat{Q}_3(t-\tau)\widehat{N}_2(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.\end{aligned} \tag{5.12}$$

We shall only provide the estimates for $\|u\|_{L^2(\mathbb{R}^3)}$. $\|b\|_{L^2(\mathbb{R}^3)}$ can be estimated in a similar way and admits the same bound as u due to the similarity of (5.12) with (5.11). We focus on the first term and the third term on the right side in (5.11). The estimates for the rest can be established similarly. By Proposition 5.4 and Lemma 5.2,

$$\begin{aligned}\|\widehat{Q}_1(t)\widehat{u}_0\|_{L^2(\mathbb{R}^3)} &\le C\|e^{-\widetilde{c}_0|\xi_h|^2 t}\,\widehat{u}_0\,\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3 t}\,\widehat{u}_0\|_{L^2(\mathbb{R}^3)} \\ &= C\big\|\,\|e^{-\widetilde{c}_0(\Lambda_1^2+\Lambda_2^2)t}\,u_0\|_{L^2_{x_1x_2}}\,\big\|_{L^2_{x_3}} + C\|e^{-c_3 t}\,\widehat{u}_0\|_{L^2} \\ &\le C(1+t)^{-\frac{1}{2}}(\|u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|u_0\|_{L^2}),\end{aligned} \tag{5.13}$$

where we have used the fact $e^{-c_3 t}(1+t)^m \le C(c_3, m)$ for any $m \ge 0$. For the third term, according to the upper bound for $\widehat{Q}_1(t)$,

$$\begin{aligned}&\int_0^t \|\widehat{Q}_1(t-\tau)\widehat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \le \int_0^t \|\widehat{Q}_1(t-\tau)\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\le C\int_0^t \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau + C\int_0^t e^{-c_3(t-\tau)}\|\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &= C\int_0^{t-1} \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau + C\int_{t-1}^t \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\quad + C\int_0^t e^{-c_3(t-\tau)}\|M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau,\end{aligned} \tag{5.14}$$

where $M_1 = b\cdot\nabla b - u\cdot\nabla u$ and we have used the fact that the projection operator $\mathbb{P}$ is bounded in L^2. Observing the simple facts, for any positive number m,

$$(1+t-\tau)^{-m} \ge 2^{-m} \text{ for } \tau\in[t-1,t] \quad\text{and}\quad e^{-c_3 t}(1+t)^m \le C(c_3,m) \text{ for } t>0,$$

we have

$$\int_{t-1}^t \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \le 2^m \int_{t-1}^t (1+t-\tau)^{-m}\|\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.$$

Then (5.14) can be further bounded as

$$\begin{aligned}&\int_0^t \|\widehat{Q}_1(t-\tau)\widehat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\ &\quad\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau + C\int_0^t (1+t-\tau)^{-m}\|\widehat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.\end{aligned}\tag{5.15}$$

Next we bound the terms on the right side in (5.15). It suffices to estimate the integral involving $u\cdot\nabla u$. The integral of $b\cdot\nabla b$ admits the same bound. As in (5.13), we have

$$\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{u\cdot\nabla u}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \le C\int_0^t (1+t-\tau)^{-\frac12}\|u\cdot\nabla u(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau.$$

By (2.6),

$$\|u\cdot\nabla u\|_{L^2_{x_3}L^1_{x_1x_2}} \le C\|u_h\|_{L^2}^{\frac12}\|\partial_3 u_h\|_{L^2}^{\frac12}\|\nabla_h u\|_{L^2} + C\|u_3\|_{L^2}^{\frac12}\|\partial_3 u_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}.\tag{5.16}$$

By Lemma 5.3,

$$\begin{aligned}&\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{u\cdot\nabla u}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\ &\le C\sup_{0\le\tau\le t}(1+\tau)^{\frac14}\|u_h(\tau)\|_{L^2}^{\frac12}(1+\tau)^{\frac14-\frac12\varepsilon}\|\partial_3 u_h(\tau)\|_{L^2}^{\frac12}(1+\tau)\|\nabla_h u(\tau)\|_{L^2}\\ &\quad\times\int_0^t (1+t-\tau)^{-\frac12}(1+\tau)^{-\frac32+\frac12\varepsilon}\,\mathrm{d}\tau\\ &+C\sup_{0\le\tau\le t}(1+\tau)^{\frac14}\|u_3(\tau)\|_{L^2}^{\frac12}(1+\tau)^{\frac12}\|\partial_3 u_3(\tau)\|_{L^2}^{\frac12}(1+\tau)^{\frac12-\varepsilon}\|\partial_3 u(\tau)\|_{L^2}\\ &\quad\times\int_0^t (1+t-\tau)^{-\frac12}(1+\tau)^{-\frac54+\varepsilon}\,\mathrm{d}\tau\\ &\le CE_2(t)(1+t)^{-\frac12}.\end{aligned}\tag{5.17}$$

Applying Hölder's inequality and Sobolev's inequality, the second integral involving $u \cdot \nabla u$ in (5.15) can be bounded as

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-m} \|u \cdot \nabla u(\tau)\|_{L^2(\mathbb{R}^3)} \, \mathrm{d}\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-m} \|u(\tau)\|_{L^4} \|\nabla u(\tau)\|_{L^4} \, \mathrm{d}\tau \\
&\quad \le C \int_0^t (1+t-\tau)^{-m} \|u(\tau)\|_{L^2}^{\frac{1}{4}} \|\nabla u(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{L^2}^{\frac{3}{4}} \, \mathrm{d}\tau \\
&\quad \le C E_2(t) \int_0^t (1+t-\tau)^{-m} (1+\tau)^{-\frac{13}{16}+\varepsilon} \, \mathrm{d}\tau \le C E_2(t) (1+t)^{-\frac{1}{2}},
\end{aligned}$$

where $m > 1$. As a consequence, we have

$$\int_0^t \|\widehat{Q}_1(t-\tau) \widehat{u \cdot \nabla u}(\tau)\|_{L^2(\mathbb{R}^3)} \, \mathrm{d}\tau \le C(1+t)^{-\frac{1}{2}} E_2(t).$$

Thereby, we infer

$$\int_0^t \|\widehat{Q}_1(t-\tau) \widehat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)} \, \mathrm{d}\tau \le C(1+t)^{-\frac{1}{2}} E_2(t). \tag{5.18}$$

The second term and the fourth term admit the similar bound as (5.13) and (5.18), respectively. Therefore, we can conclude

$$(1+t)^{\frac{1}{2}} \|u(t)\|_{L^2} \le C\Big(E(t) + \|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}} + \|(u_0, b_0)\|_{L^2}\Big),$$

which means that

$$(1+t) \|u(t)\|_{L^2}^2 \le C\Big(E^2(t) + \|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2 + \|(u_0, b_0)\|_{L^2}^2\Big).$$

Also, $\|b\|_{L^2}$ obeys the same bound. This complete the proof of Lemma 5.5. □

Lemma 5.6. *Let* (u, b) *be a solution to* (1.4)*. Then we have*

$$\begin{aligned}
E_{22}(t) \le C E^2(t) + C\big(&\|(u_0, b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2 \\
&+ \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}}^2 + \|(\nabla u_0, \nabla b_0)\|_{L^2}^2\big).
\end{aligned} \tag{5.19}$$

Proof of Lemma 5.6. By differentiating (5.5) and (5.6), we have, for $i = 1, 2, 3$,

$$\begin{aligned}
\widehat{\partial_i u}(\xi, t) &= \widehat{Q}_1(t) \widehat{\partial_i u_0} + \widehat{Q}_2(t) \widehat{\partial_i b_0} \\
&\quad + \int_0^t \big(\widehat{Q}_1(t-\tau) \widehat{\partial_i N_1}(\tau) + \widehat{Q}_2(t-\tau) \widehat{\partial_i N_2}(\tau)\big) \, \mathrm{d}\tau, \\
\widehat{\partial_i b}(\xi, t) &= \widehat{Q}_2(t) \widehat{\partial_i u_0} + \widehat{Q}_3(t) \widehat{\partial_i b_0} \\
&\quad + \int_0^t \big(\widehat{Q}_2(t-\tau) \widehat{\partial_i N_1}(\tau) + \widehat{Q}_3(t-\tau) \widehat{\partial_i N_2}(\tau)\big) \, \mathrm{d}\tau.
\end{aligned}$$

As in the proof of Lemma 5.5, we focus on the $\|\partial_i u(t)\|_{L^2}$. Clearly,

$$
\begin{aligned}
\|\partial_i u(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{\partial_i u}(t)\|_{L^2(\mathbb{R}^3)} &\le \|\widehat{Q}_1(t)\widehat{\partial_i u_0}\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_2(t)\widehat{\partial_i b_0}\|_{L^2(\mathbb{R}^3)} \\
&\quad + \int_0^t \|\widehat{Q}_1(t-\tau)\widehat{\partial_i N_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad + \int_0^t \|\widehat{Q}_2(t-\tau)\widehat{\partial_i N_2}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&:= H_{i1} + H_{i2} + H_{i3} + H_{i4}.
\end{aligned} \tag{5.20}
$$

It suffices to bound H_{i1} and H_{i3} in (5.20). H_{i2} and H_{i4} share similar estimates as H_{i1} and H_{i3}, respectively.

(1) $\mathbf{i = 1}$ or $\mathbf{i = 2}$.

We focus on the case $i = 2$. The case $i = 1$ is similar. By Proposition 5.4, Lemma 5.2 and Minkowski's inequality,

$$
\begin{aligned}
H_{21} &\le C\|e^{-\widetilde{c}_0|\xi_h|^2 t}\,\widehat{\partial_2 u_0}\,\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3 t}\,\widehat{\partial_2 u_0}\|_{L^2(\mathbb{R}^3)} \\
&= C\Big\|\,\|e^{-\widetilde{c}_0\Lambda_2^2 t}\,e^{-\widetilde{c}_0\Lambda_1^2 t}\partial_2 u_0\|_{L^2_{x_2}}\,\Big\|_{L^2_{x_1x_3}} + Ce^{-c_3 t}\|\,\partial_2 u_0\|_{L^2} \\
&\le C(1+t)^{-\frac{3}{4}}\Big\|\,\|e^{-\widetilde{c}_0\Lambda_1^2 t}u_0\|_{L^1_{x_2}}\,\Big\|_{L^2_{x_1x_3}} + C(1+t)^{-1}\|\partial_2 u_0\|_{L^2} \\
&\le C(1+t)^{-\frac{3}{4}}\|\,\|e^{-\widetilde{c}_0\Lambda_1^2 t}u_0\|_{L^2_{x_1}}\,\|_{L^2_{x_3}L^1_{x_2}} + C(1+t)^{-1}\|\partial_2 u_0\|_{L^2} \\
&\le C(1+t)^{-1}(\|u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_2 u_0\|_{L^2}).
\end{aligned} \tag{5.21}
$$

Similarly,

$$
H_{22} \le C(1+t)^{-1}(\|b_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_2 b_0\|_{L^2}). \tag{5.22}
$$

For H_{23}, similarly to (5.15), we first bound it by

$$
\begin{aligned}
H_{23} &\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_2 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad + C\int_0^t (1+t-\tau)^{-m}\|\partial_2 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau,
\end{aligned} \tag{5.23}
$$

where $M_1 = b\cdot\nabla b - u\cdot\nabla u$. We consider the first term involving $u\cdot\nabla u$ in (5.23). Firstly, from the estimates (5.16) and (5.17), we obtain

$$
\begin{aligned}
&\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\partial_2(\widehat{u\cdot\nabla u})(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad\le \int_0^t (1+t-\tau)^{-1}\|u\cdot\nabla u(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau \\
&\quad\le CE_2(t)\int_0^t (1+t-\tau)^{-1}\Big[(1+\tau)^{-\frac{3}{2}+\frac{1}{2}\varepsilon} + (1+\tau)^{-\frac{5}{4}+\varepsilon}\Big]\mathrm{d}\tau \\
&\quad\le CE_2(t)(1+t)^{-1}.
\end{aligned}
$$

For the second term in (5.23), it follows from Hölder's inequality and Sobolev's inequality that

$$
\begin{aligned}
\|\partial_2(u\cdot\nabla u)\|_{L^2} &\le \|\partial_2 u\|_{L^2}\|\nabla u\|_{L^\infty} + \|u_h\|_{L^\infty}\|\partial_2\nabla_h u\|_{L^2} + \|u_3\|_{L^4}\|\partial_2\partial_3 u\|_{L^4}\\
&\le C(\|\partial_2 u\|_{L^2}\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12} + \|\nabla u_h\|_{L^2}^{\frac12}\|\nabla^2 u_h\|_{L^2}^{\frac12}\|\partial_2\nabla_h u\|_{L^2}\\
&\quad + \|u_3\|_{L^2}^{\frac14}\|\nabla u_3\|_{L^2}^{\frac34}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla\partial_2\partial_3 u\|_{L^2}^{\frac34}).
\end{aligned}
\tag{5.24}
$$

Therefore, for $m>2$, we derive

$$
\begin{aligned}
&\int_0^t (1+t-\tau)^{-m}\|\partial_2(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\
&\quad\le C\sup_{0\le t\le\tau}(1+\tau)\|\partial_2 u(\tau)\|_{L^2}(1+\tau)^{\frac18}\|\nabla^2 u(\tau)\|_{L^2}^{\frac12}\|\nabla^3 u(\tau)\|_{L^2}^{\frac12}\\
&\quad\int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac98}\,\mathrm{d}\tau\\
&\qquad + C\sup_{0\le t\le\tau}(1+\tau)^{\frac12(\frac12-\varepsilon)}\|\nabla u_h\|_{L^2}^{\frac12}(1+\tau)^{\frac18}\|\nabla^2 u_h\|_{L^2}^{\frac12}(1+\tau)^{\frac23-\varepsilon}\|\partial_2\nabla_h u(\tau)\|_{L^2}\\
&\qquad\times\int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac{25}{24}+\frac32\varepsilon}\,\mathrm{d}\tau\\
&\qquad + C\sup_{0\le t\le\tau}(1+\tau)^{\frac18}\|u_3(\tau)\|_{L^2}^{\frac14}(1+\tau)^{\frac34}\|\nabla u_3(\tau)\|_{L^2}^{\frac34}\\
&\qquad(1+\tau)^{\frac14(\frac23-\varepsilon)}\|\partial_2\partial_3 u(\tau)\|_{L^2}^{\frac14}\|\nabla^3 u(\tau)\|_{L^2}^{\frac34}\\
&\qquad\times\int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac{25}{24}+\frac14\varepsilon}\,\mathrm{d}\tau\\
&\quad\le CE_2^{\frac34}(t)E_0^{\frac14}(t)(1+t)^{-\frac98} + CE_2(t)(1+t)^{-\frac{25}{24}+\frac32\varepsilon} + CE_2^{\frac58}(t)E_0^{\frac38}(t)(1+t)^{-\frac{25}{24}+\frac14\varepsilon}\\
&\quad E(t)(1+t)^{-1}.
\end{aligned}
\tag{5.25}
$$

Consequently,

$$
\int_0^t \|\widehat{Q}_1(t-\tau)\partial_2\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \le CE(t)(1+t)^{-1}.
$$

Similarly,

$$
\int_0^t \|\widehat{Q}_1(t-\tau)\partial_2\widehat{(b\cdot\nabla b)}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \le CE(t)(1+t)^{-1}.
$$

Hence,

$$
H_{23}\le CE(t)(1+t)^{-1}, \tag{5.26}
$$

Similarly,

$$
H_{24}\le CE(t)(1+t)^{-1}. \tag{5.27}
$$

(5.21), (5.22), (5.26) and (5.27) yield

$$(1+t)\|\partial_2 u(t)\|_{L^2} \leq CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_2 u_0, \partial_2 b_0)\|_{L^2}).$$

Similarly,

$$(1+t)\|\partial_2 b(t)\|_{L^2} \leq CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_2 u_0, \partial_2 b_0)\|_{L^2}).$$

For $i = 1$, $\|(\partial_1 u, \partial_1 b)\|_{L^2}$ obeys a similar bound to $\|(\partial_2 u, \partial_2 b)\|_{L^2}$ with only a minor modification of (5.24) and (5.25),

$$(1+t)\|(\partial_1 u(t), \partial_1 b(t))\|_{L^2} \leq CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_1 u_0, \partial_1 b_0)\|_{L^2}).$$

(2) i = 3

Invoking the estimate (5.13), we have

$$\begin{aligned} H_{31} &\leq C\|e^{-\widetilde{c}_0|\xi_h|^2 t}\widehat{\partial_3 u_0}\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3 t}\,\widehat{\partial_3 u_0}\|_{L^2(\mathbb{R}^3)} \\ &\leq C(1+t)^{-\frac{1}{2}}(\|\partial_3 u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_3 u_0\|_{L^2}). \end{aligned} \tag{5.28}$$

H_{33} can be similarly estimated as H_{23},

$$\begin{aligned} H_{33} \leq C &\int_0^{t-1} \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_3 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &+ C\int_0^t (1+t-\tau)^{-m}\|\partial_3 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau. \end{aligned} \tag{5.29}$$

Firstly, we have

$$\begin{aligned} &int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\partial_3(\widehat{u\cdot\nabla u})(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &int_0^{t-1}(1+t-\tau)^{-\frac{1}{2}}\|\partial_3(u\cdot\nabla u)(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau \end{aligned}$$

Applying the estimate (2.6) yields

$$\begin{aligned} \|\partial_3(u\cdot\nabla u)\|_{L^2_{x_3}L^1_{x_1x_2}} \leq & C(\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u_h\|_{L^2}^{\frac{1}{2}}\|\nabla_h u\|_{L^2} + \|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2} \\ &+ \|u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2} + \|u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2}). \end{aligned} \tag{5.30}$$

As a consequence, we arrive at

$$\begin{aligned} &\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\partial_3(\widehat{u\cdot\nabla u})(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\leq CE_2(t)\int_0^t (1+t-\tau)^{-\frac{1}{2}}\left[(1+\tau)^{-\frac{11}{8}+\frac{1}{2}\varepsilon} + (1+\tau)^{-\frac{7}{6}+\frac{3}{2}\varepsilon} + (1+\tau)^{-1}\right]\mathrm{d}\tau \\ &\leq CE(t)(1+t)^{-\frac{1}{2}+\varepsilon}. \end{aligned} \tag{5.31}$$

To bound the second term in (5.29), we apply Hölder's and Sobolev's inequalities to obtain

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-m}\|\partial_3(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\
&\le C\int_0^t (1+t-\tau)^{-m}(\|\partial_3 u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^\infty}+\|u(\tau)\|_{L^\infty}\|\nabla\partial_3 u(\tau)\|_{L^2})\,\mathrm{d}\tau\\
&\le C\int_0^t (1+t-\tau)^{-m}(\|\partial_3 u(\tau)\|_{L^2}\|\nabla^2 u(\tau)\|_{L^2}^{\frac12}\|\nabla^3 u(\tau)\|_{L^2}^{\frac12}\\
&\qquad\qquad +\|\nabla u(\tau)\|_{L^2}^{\frac12}\|\nabla^2 u(\tau)\|_{L^2}^{\frac12}\|\nabla\partial_3 u(\tau)\|_{L^2})\,\mathrm{d}\tau\\
&\le C\Big(E_2^{\frac34}(t)E_0^{\frac14}(t)+E_2(t)\Big)\int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac58+\varepsilon}\,\mathrm{d}\tau\\
&\le CE(t)(1+t)^{-\frac12}.
\end{aligned}\tag{5.32}$$

The estimates (5.31) and (5.32) then lead to

$$\int_0^t \|Q_1(t-\tau)\partial_3\widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\le CE(t)(1+t)^{-\frac12+\varepsilon}.$$

Therefore,

$$H_{33}\le CE(t)(1+t)^{-\frac12+\varepsilon}.\tag{5.33}$$

Similarly,

$$H_{32}+H_{34}\le C\Big(\|\partial_3 b_0\|_{L^2_{x_3}L^1_{x_1x_2}}+\|\partial_3 b_0\|_{L^2}+E(t)\Big)(1+t)^{-\frac12+\varepsilon}.\tag{5.34}$$

Finally, by the estimates (5.28), (5.33) and (5.34), we conclude

$$(1+t)^{\frac12-\varepsilon}\|\partial_3 u(t)\|_{L^2}\le CE(t)+C(\|(\partial_3 u_0,\partial_3 b_0)\|_{L^2_{x_3}L^1_{x_1x_2}}+\|(\partial_3 u_0,\partial_3 b_0)\|_{L^2}).$$

This completes the proof of Lemma 5.6. □

Next we bound $E_{23}(t)$, which involves the second-order derivatives of (u,b).

Lemma 5.7. *Let* (u,b) *be a solution to* (1.4). *Then it holds*

$$\begin{aligned}
E_{23}(t)\le CE^2(t)+C\Big(&\|(u_0,b_0)\|^2_{L^2_{x_3}L^1_{x_1x_2}}+(\|(\partial_3 u_0,\partial_3 b_0)\|^2_{L^2_{x_3}L^1_{x_1x_2}}\\
&+\|(\partial_3^2 u_0,\partial_3^2 b_0)\|^2_{L^2_{x_3}L^1_{x_1x_2}}+\|(\Delta u_0,\Delta b_0)\|^2_{L^2}).
\end{aligned}\tag{5.35}$$

Proof of Theorem 5.7. First of all, we have, for $i,j=1,2,3$,

$$\begin{aligned}
\widehat{\partial_i\partial_j u}(\xi,t)&=\widehat{Q}_1(t)\widehat{\partial_i\partial_j u}_0+\widehat{Q}_2(t)\widehat{\partial_i\partial_j b}_0\\
&\quad+\int_0^t\Big(\widehat{Q}_1(t-\tau)\widehat{\partial_i\partial_j N}_1(\tau)+\widehat{Q}_2(t-\tau)\widehat{\partial_i\partial_j N}_2(\tau)\Big)\,\mathrm{d}\tau,\\
\widehat{\partial_i\partial_j b}(\xi,t)&=\widehat{Q}_2(t)\widehat{\partial_i\partial_j u}_0+\widehat{Q}_3(t)\widehat{\partial_i\partial_j b}_0\\
&\quad+\int_0^t\Big(\widehat{Q}_2(t-\tau)\widehat{\partial_i\partial_j N}_1(\tau)+\widehat{Q}_3(t-\tau)\widehat{\partial_i\partial_j N}_2(\tau)\Big)\,\mathrm{d}\tau.
\end{aligned}\tag{5.36}$$

Throughout the proof, we only show the bound of $\|\partial_i\partial_j u(t)\|_{L^2}$. The estimates for $\|\partial_i\partial_j b(t)\|_{L^2}$ can be obtained similarly. Taking the L^2 norm on both side of (5.36), we have

$$\begin{aligned}
\|\partial_i\partial_j u(t)\|_{L^2(\mathbb{R}^3)} = \|\widehat{\partial_i\partial_j u}(t)\|_{L^2(\mathbb{R}^3)} &\le \|\widehat{Q}_1(t)\widehat{\partial_i\partial_j u_0}\|_{L^2(\mathbb{R}^3)} + \|\widehat{Q}_2(t)\widehat{\partial_i\partial_j b_0}\|_{L^2(\mathbb{R}^3)} \\
&\quad + \int_0^t \|\widehat{Q}_1(t-\tau)\widehat{\partial_i\partial_j N_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad {}^{t}\,\|\widehat{Q}_2(t-\tau)\widehat{\partial_i\partial_j N_2}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau. \\
&= K_{ij1} + K_{ij2} + K_{ij3} + K_{ij4}.
\end{aligned}$$

We focus on K_{ij1} and K_{ij3}. The bound for the other terms can be established in a similar way. The proof will be split into four cases: $i=1,\ j=1,2; i=1,\ j=3$; $i=2,\ j=2,3$; $i=j=3$.

(1) $\mathbf{i=1,\ j=1,2}$.

It suffices to investigate the case $i=1,\ j=2$. The case $i=1,\ j=1$ can be dealt with similarly. By Lemma 5.2,

$$\begin{aligned}
K_{121} &\le C\|e^{-\widetilde{c}_0|\xi_h|^2 t}|\xi_h|^2\widehat{u}_0\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3 t}\ \widehat{\partial_1\partial_2 u_0}\|_{L^2(\mathbb{R}^3)} \\
&\le C(1+t)^{-\frac{3}{2}}(\|u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_1\partial_2 u_0\|_{L^2}). \qquad (5.37)
\end{aligned}$$

Similarly,

$$K_{122} \le C(1+t)^{-\frac{3}{2}}(\|b_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_1\partial_2 b_0\|_{L^2}). \qquad (5.38)$$

For K_{123}, we first give a different bound from the ones in Lemma 5.5 and Lemma 5.6.

$$\begin{aligned}
{}_{123} &\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&{}_{t-1}^{t}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad + C\int_0^t e^{-c_3(t-\tau)}\|e^{-c_2\xi_1^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.
\end{aligned}$$

For $\tau\in[t-1,t]$, we have $e^{-c_3(t-\tau)}\ge e^{-c_3}$ and thus

$$\begin{aligned}
&int_{t-1}^{t}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\le e^{c_3}\textstyle\int_{t-1}^{t} e^{-c_3(t-\tau)}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.
\end{aligned}$$

As a consequence, for a constant $c_4>0$,

$$\begin{aligned}
K_{123} &\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad + C\int_0^t e^{-c_3(t-\tau)}\|e^{-c_4\xi_1^2(t-\tau)}\widehat{\partial_1\partial_2 M_1}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&:= K_{1231} + K_{1232}.
\end{aligned}$$

Invoking (5.37), (5.16) and (5.17), we have

$$
\begin{aligned}
&\int_0^{t-1} \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)} \partial_1\partial_2 \widehat{(u\cdot\nabla u)}(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|u\cdot\nabla u(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau \\
&\le CE_2(t)\int_0^t (1+t-\tau)^{-\frac{3}{2}} \Big((1+\tau)^{-\frac{3}{2}+\frac{1}{2}\varepsilon} + (1+\tau)^{-\frac{5}{4}+\varepsilon}\Big)\,\mathrm{d}\tau \\
&\le CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}.
\end{aligned}
$$

Hence,

$$
K_{1231} \le CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}. \tag{5.39}
$$

For K_{1232}, according to Lemma 5.2, we have

$$
K_{1232} \le C\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_2(u\cdot\nabla u - b\cdot\nabla b)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau.
$$

By the anisotropic inequality (2.5),

$$
\begin{aligned}
\|\partial_2(u\cdot\nabla u)\|_{L^2(\mathbb{R}^3)} \le{}& C\|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_2^2 u\|_{L^2}^{\frac{1}{4}} \|\partial_2\partial_3 u\|_{L^2}^{\frac{1}{4}} \|\partial_2^2\partial_3 u\|_{L^2}^{\frac{1}{4}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_1\nabla u\|_{L^2}^{\frac{1}{2}} \\
&+ C\|u\|_{L^2}^{\frac{1}{4}} \|\partial_1 u\|_{L^2}^{\frac{1}{4}} \|\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_2 u\|_{L^2}^{\frac{1}{4}} \|\nabla\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_3\nabla u\|_{L^2}^{\frac{1}{2}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_2(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\le CE_2^{\frac{7}{8}}(t)E_0^{\frac{1}{8}}(t)\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-\frac{4}{3}+\frac{3}{2}\varepsilon}\,\mathrm{d}\tau \\
&\quad + CE_2^{\frac{3}{4}}(t)E_0^{\frac{1}{4}}(t)\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-\frac{61}{48}+\frac{3}{4}\varepsilon}\,\mathrm{d}\tau \\
&\le CE(t)\int_0^t e^{-\frac{c_3}{2}(t-\tau)}(t-\tau)^{-\frac{1}{2}}\,(1+t-\tau)^{-m}(1+\tau)^{-\frac{5}{4}+\varepsilon}\,\mathrm{d}\tau,
\end{aligned}
$$

where we have used the simple fact: $e^{-ct}(1+t)^m \le C(m)$ for any $t \ge 0, m \ge 0$. Furthermore, selecting $m > 2$, and then applying Hölder's inequality with $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, we infer that

$$
\begin{aligned}
&\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_2(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\le CE(t)\Big(\int_0^t e^{-\frac{c_3 p}{2}(t-\tau)}(t-\tau)^{-\frac{p}{2}}\mathrm{d}\tau\Big)^{\frac{1}{p}} \Big(\int_0^t (1+t-\tau)^{-mq}(1+\tau)^{(-\frac{5}{4}+\varepsilon)q}\mathrm{d}\tau\Big)^{\frac{1}{q}} \\
&\le CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}.
\end{aligned} \tag{5.40}
$$

where we have used fact that the integration $\int_0^\infty x^{s-1}e^{-x}\mathrm{d}x$ $(s>0)$ converges to $\Gamma(s)$. Consequently,

$$K_{1232} \leq CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}. \tag{5.41}$$

(5.39) and (5.41) lead to

$$K_{123} \leq CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}. \tag{5.42}$$

With a similar argument, we obtain

$$K_{124} \leq CE(t)(1+t)^{-\frac{5}{4}+\varepsilon}. \tag{5.43}$$

Combining the estimates (5.37), (5.38), (5.43) and (5.42), we derive

$$(1+t)^{\frac{5}{4}-\varepsilon}\|\partial_1\partial_2 u(t)\|_{L^2} \leq CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_1\partial_2 u_0, \partial_1\partial_2 b_0)\|_{L^2}).$$

Similarly, we can also obtain

$$(1+t)^{\frac{5}{4}-\varepsilon}\|\partial_1^2 u(t)\|_{L^2} \leq CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_1^2 u_0, \partial_1^2 b_0)\|_{L^2}).$$

(2) $\mathbf{i=1, j=3}$.
Firstly, from (5.21), we have

$$K_{131} \leq C(1+t)^{-1}(\|\partial_3 u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_1\partial_3 u_0\|_{L^2}). \tag{5.44}$$

For K_{133}, similarly to K_{123}, we first bound things as

$$\begin{aligned}
K_{133} \leq{}& C\int_0^{t-1} \|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\partial_1\widehat{\partial_3 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&+ C\int_0^t e^{-c_3(t-\tau)}\|e^{-c_4\xi_1^2(t-\tau)}\partial_1\widehat{\partial_3 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
\leq{}& C\int_0^t (1+t-\tau)^{-1}\|\partial_3 M_1(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau \\
&+ C\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_3 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
:={}& K_{1331} + K_{1332}.
\end{aligned}$$

Invoking (5.30) and (5.31), we get

$$\begin{aligned}
K_{1331} \leq{}& CE_2(t)\int_0^t (1+t-\tau)^{-1}\Big[(1+\tau)^{-\frac{11}{8}+\frac{1}{2}\varepsilon} + (1+\tau)^{-\frac{7}{6}+\frac{3}{2}\varepsilon} + (1+\tau)^{-1}\Big]\mathrm{d}\tau \\
\leq{}& CE(t)(1+t)^{-1+\varepsilon}.
\end{aligned} \tag{5.45}$$

For K_{1332}, by Hölder's inequality and Sobolev's inequality, we first have

$$
\begin{aligned}
&\|\partial_3(u\cdot\nabla u)\|_{L^2}\\
&\quad\le \|\partial_3 u_j\|_{L^4}\|\partial_j u\|_{L^4}+\|u_j\|_{L^\infty}\|\partial_j\partial_3 u\|_{L^2}\\
&\quad\le C\|\partial_3 u_h\|_{L^2}^{\frac14}\|\partial_3\nabla u_h\|_{L^2}^{\frac34}\|\nabla_h u\|_{L^2}^{\frac14}\|\nabla\nabla_h u\|_{L^2}^{\frac34}\\
&\qquad+C\|\partial_3 u_3\|_{L^2}^{\frac14}\|\partial_3\nabla u_3\|_{L^2}^{\frac34}\|\partial_3 u\|_{L^2}^{\frac14}\|\nabla\partial_3 u\|_{L^2}^{\frac34}\\
&\qquad+C\|\nabla u_h\|_{L^2}^{\frac12}\|\nabla^2 u_h\|_{L^2}^{\frac12}\|\nabla_h\partial_3 u\|_{L^2}+C\|\nabla u_3\|_{L^2}^{\frac12}\|\nabla^2 u_3\|_{L^2}^{\frac12}\|\partial_3^2 u\|_{L^2}.
\end{aligned}
$$

Then, for $m>1$,

$$
\begin{aligned}
&\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac12}\|\partial_3(u\cdot\nabla u)(\tau)\|_{L^2}\,\mathrm{d}\tau\\
&\quad\le CE_2(t)\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac12}\\
&\qquad\left[(1+\tau)^{-\frac{17}{16}+\varepsilon}+(1+\tau)^{-\frac{25}{24}+\frac32\varepsilon}+(1+\tau)^{-\frac{13}{12}+\frac12\varepsilon}\right]\mathrm{d}\tau\\
&\quad\le CE(t)\int_0^t e^{-c_3(t-\tau)}(t-\tau)^{-\frac12}(1+\tau)^{-1+\varepsilon}\mathrm{d}\tau\\
&\quad\le CE(t)\int_0^t e^{-\frac{c_3}{2}(t-\tau)}(t-\tau)^{-\frac12}\,(1+t-\tau)^{-m}(1+\tau)^{-1+\varepsilon}\mathrm{d}\tau\\
&\quad\le CE(t)(1+t)^{-1+\varepsilon},
\end{aligned}
$$

where we have used a similar derivation with (5.40) for the last inequality. Thus, we get

$$K_{1332}\le CE(t)(1+t)^{-1+\varepsilon}.$$

which, together with (5.45), gives

$$K_{133}\le CE(t)(1+t)^{-1+\varepsilon}. \tag{5.46}$$

Therefore, by (5.44) and (5.46), we conclude that

$$(1+t)^{1-\varepsilon}\|\partial_1\partial_3 u(t)\|_{L^2}\le CE(t)+C(\|(\partial_3 u_0,\partial_3 b_0)\|_{L^2_{x_3}L^1_{x_1x_2}}+\|(\partial_1\partial_3 u_0,\partial_1\partial_3 b_0)\|_{L^2}).$$

(3) $\mathbf{i=2,j=2,3}$.
It suffices to bound $\|\partial_2\partial_3 u\|_{L^2}$. Firstly, a similar argument with (5.21) yields

$$
\begin{aligned}
K_{231}&\le C\|e^{-\widetilde{c}_0|\xi_h|^2 t}\widehat{\partial_2\partial_3 u_0}\|_{L^2(\mathbb{R}^3)}+C\|e^{-c_3 t}\,\widehat{\partial_2\partial_3 u_0}\|_{L^2(\mathbb{R}^3)}\\
&\le C(1+t)^{-1}(\|\partial_3 u_0\|_{L^2_{x_3}L^1_{x_1x_2}}+C\|\partial_2\partial_3 u_0\|_{L^2}).
\end{aligned}\tag{5.47}
$$

As in H_{23}, K_{233} is firstly bounded by

$$\begin{aligned}K_{233} &\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\partial_2\widehat{\partial_3 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\&\quad + C\int_0^t(1+t-\tau)^{-m}\|\partial_2\partial_3 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\&\le C\int_0^t(1+t-\tau)^{-1}\|\partial_3 M_1(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau\\&\quad + C\int_0^t(1+t-\tau)^{-m}\|\partial_2\partial_3 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\&:= K_{2331}+K_{2332}.\end{aligned}$$

Now we estimate K_{2331}. Recalling the bound (5.45) gives

$$K_{2331}\le CE_2(t)(1+t)^{-1+\varepsilon}. \tag{5.48}$$

By Hölder's inequality and Sobolev's inequality,

$$\begin{aligned}&\|\partial_2\partial_3(u\cdot\nabla u)\|_{L^2}\\&\quad\le\|\partial_2\partial_3 u\cdot\nabla u\|_{L^2}+\|\partial_2 u\cdot\nabla\partial_3 u\|_{L^2}+\|\partial_3 u\cdot\nabla\partial_2 u\|_{L^2}+\|u\cdot\nabla\partial_2\partial_3 u\|_{L^2}\\&\quad\le\|\nabla u\|_\infty\|\nabla\partial_2 u\|_{L^2}+\|\partial_2 u\|_{L^4}\|\nabla\partial_3 u\|_{L^4}+\|u\cdot\nabla\partial_2\partial_3 u\|_{L^2}\\&\quad\le C\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla\partial_2 u\|_{L^2}+\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac34}\|\nabla\partial_3 u\|_{L^2}^{\frac14}\|\nabla^2\partial_3 u\|_{L^2}^{\frac34}\\&\qquad+\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla\partial_2\partial_3 u\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac12},\end{aligned}$$

where we have used the anisotropic inequality (2.5) for $\|u\cdot\nabla\partial_2\partial_3 u\|_{L^2}$. Thus,

$$\begin{aligned}&\int_0^t(1+t-\tau)^{-m}\|\partial_2\partial_3(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau\\&\quad\le C\int_0^t(1+t-\tau)^{-m}\\&\quad\Big(\|\nabla^2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\nabla\partial_2 u\|_{L^2}+\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac34}\|\nabla\partial_3 u\|_{L^2}^{\frac14}\|\nabla^2\partial_3 u\|_{L^2}^{\frac34}\Big)\,\mathrm{d}\tau\\&\qquad+C\int_0^t(1+t-\tau)^{-m}\|u\|_{L^2}^{\frac14}\|\partial_2 u\|_{L^2}^{\frac14}\|\partial_3 u\|_{L^2}^{\frac14}\\&\quad\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\nabla\partial_2\partial_3 u\|_{L^2}^{\frac12}\|\nabla\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac12}\,\mathrm{d}\tau\\&\quad:= L_1+L_2.\end{aligned}$$

By means of (5.2), for $m > 1$, we infer that

$$
\begin{aligned}
L_1 \le & C \sup_{0\le\tau\le t} (1+\tau)^{\frac{2}{3}-\varepsilon} \|\nabla\partial_2 u(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{H^1} \\
& \int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac{2}{3}+\varepsilon} \mathrm{d}\tau \\
& + \sup_{0\le\tau\le t} (1+\tau)^{\frac{1}{4}} \|\partial_2 u(\tau)\|_{L^2}^{\frac{1}{4}} (1+\tau)^{\frac{3}{4}(\frac{2}{3}-\varepsilon)} \|\partial_2\nabla u(\tau)\|_{L^2}^{\frac{3}{4}} \|\nabla^2 u(\tau)\|_{H^1} \\
& \times \int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac{3}{4}+\frac{3}{4}\varepsilon} \mathrm{d}\tau \\
\le & E_2^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t)(1+t)^{-\frac{2}{3}+\varepsilon} \le CE(t)(1+t)^{-\frac{2}{3}+\varepsilon}.
\end{aligned}
$$

For L_2, applying Hölder's inequality yields, for $m > 1$,

$$
\begin{aligned}
L_2 \le & C \sup_{0\le\tau\le t} (1+\tau)^{\frac{1}{8}} \|u(\tau)\|_{L^2}^{\frac{1}{4}} (1+\tau)^{\frac{1}{4}} \|\partial_2 u(\tau)\|_{L^2}^{\frac{1}{4}} (1+\tau)^{\frac{1}{8}-\frac{1}{4}\varepsilon} \|\partial_3 u(\tau)\|_{L^2}^{\frac{1}{4}} \\
& \times (1+\tau)^{\frac{1}{6}-\frac{1}{4}\varepsilon} \|\partial_2\partial_3 u(\tau)\|_{L^2}^{\frac{1}{4}} \\
& \times \|\nabla\partial_2\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}} \int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac{2}{3}+\frac{1}{2}\varepsilon} \|\nabla\partial_1\partial_2\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}} \mathrm{d}\tau \\
\le & CE_2^{\frac{1}{2}}(t) E_0^{\frac{1}{4}}(t) \Big(\int_0^t (1+t-\tau)^{-\frac{4}{3}m}(1+\tau)^{-\frac{8}{9}+\frac{2}{3}\varepsilon} \mathrm{d}\tau \Big)^{\frac{3}{4}} \\
& \Big(\int_0^t \|\nabla\partial_1\partial_2\partial_3 u(\tau)\|_{L^2}^2 \mathrm{d}\tau \Big)^{\frac{1}{4}} \\
\le & CE_2^{\frac{1}{2}}(t) E_0^{\frac{1}{2}}(t)(1+t)^{-\frac{2}{3}+\frac{1}{2}\varepsilon} \le CE(t)(1+t)^{-\frac{2}{3}+\varepsilon}.
\end{aligned}
$$

Therefore,

$$
\int_0^t (1+t-\tau)^{-m} \|\partial_2\partial_3(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} \,\mathrm{d}\tau \le CE(t)(1+t)^{-\frac{2}{3}+\varepsilon}.
$$

Thus,

$$
K_{2332} \le CE(t)(1+t)^{-\frac{2}{3}+\varepsilon},
$$

which, together with (5.48), gives

$$
K_{233} \le CE(t)(1+t)^{-\frac{2}{3}+\varepsilon}. \tag{5.49}
$$

K_{232} and K_{234} can be bounded with similar arguments as those for K_{231} and K_{233}, respectively. Therefore, by (5.47) and (5.49), we conclude that

$$
\begin{aligned}
(1+t)^{\frac{2}{3}-\varepsilon} \|\partial_2\partial_3 u(t)\|_{L^2} \le & CE(t) + C(\|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{x_3} L^1_{x_1 x_2}} \\
& + \|(\partial_2\partial_3 u_0, \partial_2\partial_3 b_0)\|_{L^2}).
\end{aligned}
$$

Similarly,

$$(1+t)^{\frac{2}{3}-\varepsilon}\|\partial_2^2 u(t)\|_{L^2} \le CE(t) + C(\|(u_0,b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_2^2 u_0, \partial_2^2 b_0)\|_{L^2}).$$

(**4**) $\mathbf{i}=\mathbf{j}=\mathbf{3}$.

Firstly, we have

$$\begin{aligned} K_{331} &\le C\|e^{-\widetilde{c}_0|\xi_h|^2 t}\widehat{\partial_3^2 u_0}\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3 t}\,\widehat{\partial_3^2 u_0}\|_{L^2(\mathbb{R}^3)} \\ &\le C(1+t)^{-\frac{1}{2}}(\|\partial_3^2 u_0\|_{L^2_{x_3}L^1_{x_1x_2}} + \|\partial_3^2 u_0\|_{L^2}). \end{aligned} \tag{5.50}$$

K_{233} can be bounded as

$$\begin{aligned} K_{333} &\le C\int_0^{t-1}\|e^{-\widetilde{c}_0|\xi_h|^2(t-\tau)}\widehat{\partial_3^2 M}_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\quad + C\int_0^t (1+t-\tau)^{-m}\|\partial_3^2 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &\le C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|\partial_3^2 M_1(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau \\ &\quad + C\int_0^t (1+t-\tau)^{-m}\|\partial_3^2 M_1(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\ &:= K_{3331} + K_{3332}. \end{aligned}$$

We consider the integral

$$\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|\partial_3^2(u\cdot\nabla u)(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau. \tag{5.51}$$

It follows from (2.6) that

$$\begin{aligned} &\|\partial_3^2(u\cdot\nabla u)\|_{L^2_{x_3}L^1_{x_1x_2}} = \|\partial_3^2 u_j\,\partial_j u + 2\partial_3 u_j\,\partial_j\partial_3 u + u_j\,\partial_j\partial_3^2 u\|_{L^2_{x_3}L^1_{x_1x_2}} \\ &\le C(\|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u_3\|_{L^2} + \|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u_h\|_{L^2} \\ &\quad + \|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2} + \|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2} \\ &\quad + \|u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^3 u\|_{L^2} + \|u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\nabla_h u\|_{L^2}) \\ &\le C(\|\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2} + \|\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u\|_{L^2} \\ &\quad + \|u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3^3 u\|_{L^2} + \|u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\nabla_h u\|_{L^2}). \end{aligned} \tag{5.52}$$

Inserting (5.52) in (5.51), and using Lemma 5.3, the first three terms can be bounded by

$$
\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{1}{2}}\big(\|\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u(\tau)\|_{L^2}\\
&+\|\partial_3 u(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3^2 u(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3\nabla_h u(\tau)\|_{L^2}\\
&\quad+\|u_3(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_3(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3^3 u(\tau)\|_{L^2}\big)\mathrm{d}\tau\\
&\le CE_2(t)\int_0^t (1+t-\tau)^{-\frac{1}{2}}\Big((1+\tau)^{-\frac{13}{12}+\frac{\varepsilon}{2}}\mathrm{d}\tau+(1+\tau)^{-\frac{25}{24}+\frac{3\varepsilon}{2}}\Big)\mathrm{d}\tau\\
&\quad+CE_2^{\frac{1}{2}}(t)E_0^{\frac{1}{2}}(t)\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\
&\le CE(t)\Big((1+t)^{-\frac{1}{2}}+(1+t)^{-\frac{1}{4}}\Big).
\end{aligned}
$$

The last term needs more subtle estimates. We resort to Hölder's inequality and the integrability of $\|\partial_3^2\nabla_h u\|_{L^2}$ to get that

$$
\begin{aligned}
&\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|u_h(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3 u_h(\tau)\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\nabla_h u(\tau)\|_{L^2}\mathrm{d}\tau\\
&\le CE_2^{\frac{1}{2}}(t)\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}+\frac{\varepsilon}{2}}\|\partial_3^2\nabla_h u(\tau)\|_{L^2}\mathrm{d}\tau\\
&\le CE_2^{\frac{1}{2}}(t)\Big(\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-1+\varepsilon}\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big(\int_0^t \|\partial_3^2\nabla_h u(\tau)\|_{L^2}^2\mathrm{d}\tau\Big)^{\frac{1}{2}}\\
&\le CE_2^{\frac{1}{2}}(t)E_0^{\frac{1}{2}}(t)(1+t)^{-\frac{1}{4}}.
\end{aligned}
$$

Combining all of the estimates above, we get that

$$
\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|\partial_3^2(u\cdot\nabla u)(\tau)\|_{L^2_{x_3}L^1_{x_1x_2}}\,\mathrm{d}\tau\le CE(t)(1+t)^{-\frac{1}{4}}.
$$

Thus,

$$
K_{3331}\le CE(t)(1+t)^{-\frac{1}{4}}. \tag{5.53}
$$

Finally, applying Hölder's inequality and Sobolev's inequality, for $m > 1$, we infer that

$$\begin{aligned}
&\int_0^t (1+t-\tau)^{-m} \|\partial_3^2(u\cdot\nabla u)(\tau)\|_{L^2(\mathbb{R}^3)}\,\mathrm{d}\tau \\
&\quad\le C\int_0^t (1+t-\tau)^{-m}(\|\nabla^2 u(\tau)\|_{L^4}\|\nabla u(\tau)\|_{L^4} + \|u(\tau)\|_{L^\infty}\|\nabla\partial_3^2 u(\tau)\|_{L^2})\,\mathrm{d}\tau \\
&\quad\le C\int_0^t (1+t-\tau)^{-m}(\|\nabla^2 u(\tau)\|_{L^2}\|\nabla^3 u(\tau)\|_{L^2}^{\frac34}\|\nabla u(\tau)\|_{L^2}^{\frac14} \\
&\qquad + \|\nabla u(\tau)\|_{L^2}^{\frac12}\|\nabla^2 u(\tau)\|_{L^2}^{\frac12}\|\nabla\partial_3^2 u(\tau)\|_{L^2})\,\mathrm{d}\tau \\
&\quad\le CE_2^{\frac12}(t)E_0^{\frac12}(t)\Big(\int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac14}\mathrm{d}\tau \\
&\qquad + \int_0^t (1+t-\tau)^{-m}(1+\tau)^{-\frac38+\frac{\varepsilon}{2}}\,\mathrm{d}\tau\Big) \\
&\quad\le CE(t)(1+t)^{-\frac14}.
\end{aligned}$$

Thus,

$$K_{3332} \le CE(t)(1+t)^{-\frac14}.$$

which, together with (5.53), yields

$$K_{333} \le CE(t)(1+t)^{-\frac14}. \tag{5.54}$$

As a consequence of (5.50) and (5.54),

$$(1+t)^{\frac14}\|\partial_3^2 u(t)\|_{L^2} \le CE(t) + C(\|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2_{x_3}L^1_{x_1x_2}} + \|(\partial_3^2 u_0, \partial_3^2 b_0)\|_{L^2}).$$

Combining all the estimates for the four cases above, we derive the desired estimate (5.35). This completes the proof of Lemma 5.7. □

Proposition 5.1 then follows from the estimates (5.9), (5.10), (5.19) and (5.35). This completes the proof of Proposition 5.1.

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