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Global stability and sharp decay estimates for 3D MHD equations with only vertical dissipation near a background magnetic field

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ABSTRACT

This paper is concerned with the stability and large-time behavior of 3D incompressible MHD equations with only vertical dissipation near a background magnetic field. By making full use of the dissipation generated by the background magnetic field, we first establish the global stability of the solutions in H^3-norm. Then, the optimal decay rates of the solutions are obtained, which are consistent with the 2D classical heat equation. Moreover, some enhanced decay rates of (u_1, b_1) are also achieved. In other words, the decay estimates of the second or third component of velocity/magnetic field coincide with those of 2D heat kernel, while the first component behaves like the 3D heat kernel.

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Partial dissipation
Stability
Decay estimates

This is mainly due to the divergence-free condition and the anisotropic structure.

1. Introduction

Magnetohydrodynamics (MHD) is one of the most important branches of continuum mechanics, which deals with the interplay between electromagnetic fields and conducting fluids. The MHD system is coupled by the Navier-Stokes equations for fluid dynamics and the Maxwell equations for electromagnetism. Since the pioneering work of Alfvén [4], MHD is widely used in many fields, such as geomagnetism and planetary magnetism, astrophysics, nuclear fusion (plasma) physics, and liquid metal technology (see, e.g., [6,10,31]). In recent years, the stability of MHD equations has attracted a lot of attention. A significant nonlinear phenomena of MHD observed by physical experiments and numerical simulations is that a background magnetic field can smooth and stabilize turbulent electrically conducting fluids (see, e.g., [2,3,14,16,27]).

In this paper, we consider the following 3D MHD equations with only vertical velocity dissipation:

$$\begin{cases} U_t - \mu\partial_3^2 U = B\cdot\nabla B - U\cdot\nabla U - \nabla P, \quad x\in\mathbb{R}^3,\, t>0, \\ B_t - \eta\partial_2^2 B - \eta\partial_3^2 B = B\cdot\nabla U - U\cdot\nabla B, \\ \nabla\cdot U = \nabla\cdot B = 0, \\ (U,B)(x,t)|_{t=0} = (U_0,B_0)(x), \end{cases} \tag{1.1}$$

where $U=(U_1,U_2,U_3)^\top$, $B=(B_1,B_2,B_3)^\top$ and P are the velocity field, the magnetic field, and the pressure, respectively. The positive constants $\mu>0$ and $\eta>0$ are the viscosity coefficient and the magnetic diffusivity coefficient, respectively.

Our goal is to understand the stability of 3D anisotropic MHD system (1.1) near a equilibrium state $(U^{(0)},B^{(0)})$,

$$U^{(0)}\equiv 0, \quad B^{(0)}\equiv e_2 := (0,1,0).$$

Thus, the perturbation (u,b) around the steady state $(U^{(0)},B^{(0)})$ with

$$u := U - U^{(0)}, \qquad b := B - B^{(0)},$$

is governed by

$$\begin{cases} u_t - \mu\partial_3^2 u - \partial_2 b = b\cdot\nabla b - u\cdot\nabla u - \nabla p, \quad x\in\mathbb{R}^3,\, t>0,\\ b_t - \eta\partial_2^2 b - \eta\partial_3^2 b - \partial_2 u = b\cdot\nabla u - u\cdot\nabla b,\\ \nabla\cdot u = \nabla\cdot b = 0,\\ u(x,0) = u_0(x),\ \ b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

In fact, if $B = 0$ in (1.1), then it reduces to the 3D Navier-Stokes equations with only vertical dissipation,

$$\begin{cases} U_t - \mu\partial_3^2 U = B\cdot\nabla B - U\cdot\nabla U - \nabla P, \quad x\in\mathbb{R}^3,\, t>0,\\ \nabla\cdot U = 0,\\ U(x,t)|_{t=0} = U_0(x). \end{cases} \tag{1.3}$$

The global stability and large-time behavior of (1.3) is a challenging problem. The major difficulty lies in the fact that the dissipation in only one direction is too weak to control all the nonlinear terms in the whole space $\mathbb{R}^3$. In a recent work [28], Paicu-Zhang established the global well-posedness of smooth solutions with small data for (1.3) on the strip domain $\Omega = \mathbb{R}^2\times[0,1]$. The analysis of [28] strongly relies on the geometry of the domain and a Poincaré type inequality, which no longer holds true for the case of whole space. Motivated by this, the main purpose of this paper is to study the global stability of MHD and to reveal the influence of magnetic field on fluid stability from a mathematical point of view.

In the following, we briefly recall some noteworthy works on the stability of MHD equations near a background magnetic field. In the pioneering work [5], Bardos-Sulem-Sulem first demonstrated the stability effect of magnetic fields in ideal MHD. Cai-Lei [8] and He-Xu-Yu [15], via different approaches, successfully solved the stability problem of the ideal MHD system and its fully dissipative counterpart (with identical viscosity and resistivity) near a background magnetic field. Wei-Zhang [39] allowed the viscosity and resistivity coefficients to be slightly different. The stability of 2D viscous and non-resistive MHD equations was first considered by Lin-Xu-Zhang [21], and then was extended and improved by [9,17,32,33,42,44]. We also refer to [1,11,30,36] for the stability theory of 3D viscous and non-resistive MHD equations. In [45], Zhou-Zhu proved the stability of 2D inviscid and resistive MHD equations on the periodic domain by using the oddity conditions proposed in [30]. Wu-Wu-Xu [41] (also cf. [12]) studied the stability of 2D MHD system with only velocity damping.

Recently, the stability of the incompressible MHD equations with partial dissipation has attracted more and more attention, since it mathematically reveals the stability mechanism of a background magnetic field, compared with the Navier-Stokes equations. For 2D MHD equations with partial dissipation, one can refer to [7,13,18–20,22,29] and the references therein for interest. Next, we would like to introduce some works on the stability of 3D MHD equations with partial dissipation, which is certainly more complicated and difficult. The first work is due to Wu-Zhu [43], where the authors considered

the 3D MHD equations with horizontal dissipation $\Delta_h u$ and vertical magnetic diffusion $\partial_3^2 b$. Then, Lin-Wu-Zhu [24] proved the stability of 3D MHD equations with only one-directional dissipation $\partial_1^2 u$ and horizontal magnetic diffusion $\Delta_h b$ in Sobolev space $H^4(\mathbb{R}^3)$. Lin-Wu-Zhu [25] improved the stability result under the assumption that initial data (u_0, b_0) satisfy

$$\|(u_0, b_0)\|_{H^3} + \|\|\partial_1^k (u_0, b_0)\|_{L^2_{x_1} L^1_{x_2 x_3}} \leq \varepsilon, \quad \text{with} \quad k = 0, 1, 2.$$

Recently, the stability of an initial-boundary value problem of 3D MHD equations with only vertical dissipation $\partial_3^2 u$ and $\partial_3^2 b$ was studied by Lin-Wu-Suo [23] in the case of strip domain $\Omega = \mathbb{R}^2 \times [0, 1]$ with Dirichlet boundary conditions by using the Poincaré type inequality.

The main purpose of this paper is to study the stability theory and large-time behavior of the Cauchy problem (1.2) on the whole domain $\mathbb{R}^3$. Our first result of this paper, concerning the global solvability and stability in the H^3-framework, is formulated in the following theorem.

Theorem 1.1. *Assume that $(u_0, b_0) \in H^3$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. There exists an absolutely positive constant $\varepsilon > 0$, depending only on μ and η, such that if*

$$\|(u_0, b_0)\|_{H^3} \leq \varepsilon,$$

then the problem (1.2) has a unique global solution (u, b) on $\mathbb{R}^3 \times [0, \infty)$, satisfying

$$\|(u, b)(t)\|_{H^3}^2 + \int_0^T \left(\|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2 \right) dt \leq \varepsilon^2. \tag{1.4}$$

The issue of large-time behavior and decay rates of the global solutions is more subtle. We will derive the sharp decay rates of the solutions by the method of spectral analysis. For the notational convenience, we denote by

$$\begin{cases} \|f\|_{L^p} \triangleq \|f\|_{L^p(\mathbb{R}^3)}, \quad \|f\|_{W^{k,p}} \triangleq \|f\|_{W^{k,p}(\mathbb{R}^3)}, \\ \|f\|_{L^2} \triangleq \|f\|_{L^2(\mathbb{R}^3)}, \quad \|f\|_{W^{k,2}} \triangleq \|f\|_{H^k} = \|f\|_{H^k(\mathbb{R}^3)}, \\ \|f\|_{L^r L^q L^p} \triangleq \left\| \left\| \|f\|_{L^p(\mathbb{R})} \right\|_{L^q(\mathbb{R})} \right\|_{L^r(\mathbb{R})}, \quad \|f\|_{L^q_{x_1} L^p_{x_2, x_3}} \triangleq \left\| \|f\|_{L^p_{x_2, x_3}(\mathbb{R}^2)} \right\|_{L^q_{x_1}(\mathbb{R})}. \end{cases}$$

Moreover, the same letter C will be used to denote the generic positive constants, which may be different from line to line.

Our second and main result of this paper is concerned with the sharp decay rates of the global solutions.

Theorem 1.2. *In addition to the conditions of Theorem 1.1, assume that* $(u_0, b_0) \in L^1$ *and* $\partial_1^k(u_0, b_0) \in L^2_{x_1} L^1_{x_2,x_3}$ *for* $k = 0, 1, 2$. *There exists an absolutely positive constant* $\varepsilon > 0$, *such that if*

$$\|(u_0, b_0)\|_{H^3} + \|(u_0, b_0)\|_{L^1} + \|\partial_1^k(u_0, b_0)\|_{L^2_{x_1} L^1_{x_2,x_3}} \le \varepsilon, \tag{1.5}$$

then the global solutions (u, b) *of the problem* (1.2) *satisfy*

$$\begin{aligned}
&\|\partial_1^k(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}}, && \|\partial_i(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1},\\
&\|\partial_i\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{2}}, && \|\partial_l\partial_2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{12}},\\
&\|\partial_1\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1}, && \|\partial_1^2(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{8}},\\
&\|\partial_l\partial_2\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{12}}, && \|\partial_1^2\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|\partial_1\partial_3^2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{5}{4}}, && \|\partial_2\partial_3^2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{6}},\\
&\|\partial_3^3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{23}{12}}, && \|(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{4}},\\
&\|\partial_i(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{5}{4}}, && \|\partial_i\partial_3(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{7}{4}},\\
&\|\partial_3^3(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{9}{4}}, &&
\end{aligned}$$

where $k = 0, 1,\ i = 2,\ 3,\ l = 1,\ 2$.

If we assume further that $(u_0, b_0) \in H^4$, then we can show that the decay rates of the partial derivatives involving the third direction x_3 are optimal, compared with the 2D heat equation.

Theorem 1.3. *Assume that for* $k = 0, 1, 2, 3$, $\partial_1^k(u_0, b_0) \in L^2_{x_1} L^1_{x_2,x_3}$, $(u_0, b_0) \in H^4$, $\nabla \cdot u_0 = 0$, *and* $\nabla \cdot b_0 = 0$. *Then, there exists sufficiently small* $\varepsilon > 0$, *such that if*

$$\|(u_0, b_0)\|_{H^4} + \|(u_0, b_0)\|_{L^1} + \|\partial_1^k(u_0, b_0)\|_{L^2_{x_1} L^1_{x_2,x_3}} \le \varepsilon, \tag{1.6}$$

then the following large-time decay estimates hold for the global solutions (u, b)*:*

$$\begin{aligned}
&\|\partial_1^k(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}}, && \|\partial_i(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1},\\
&\|\partial_1\partial_i(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1}, && \|\partial_i\partial_j(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{2}},\\
&\|\partial_1^2(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}}, && \|\partial_1\partial_i\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{2}},\\
&\|\partial_l\partial_2^2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1}, && \|\partial_1^2\partial_i(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1},\\
&\|\partial_i\partial_j\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-2}, && \|\partial_1^3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{8}},\\
&\|\partial_1^3\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}}, && \|\partial_1^2\partial_3^2(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{8}},\\
&\|\partial_l\partial_h\partial_2\partial_3(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-1}, && \|\partial_1\partial_3^3(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-2},\\
&\|\partial_l\partial_2\partial_3^2(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-2}, && \|\partial_i\partial_3^3(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{5}{2}},
\end{aligned}$$

$$\|(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{3}{4}},\qquad \|\partial_i(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{5}{4}},$$
$$\|\partial_i\partial_j(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{7}{4}},\qquad \|\partial_i\partial_3^2(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{9}{4}},$$
$$\|\partial_i\partial_3^3(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{11}{4}},$$

where $k=0,1,\ i,\ j=2,\ 3,\ l,h=1,\ 2.$

Remark 1.1. The method herein can be applied to the H^N-framework with $N\ge 4$. In particular, if

$$\|(u_0,b_0)\|_{H^N}+\|(u_0,b_0)\|_{L^1}+\|\partial_1^k(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}}\le\varepsilon$$

with $k=0,1,\cdots,N-1$, then it holds that

$$\|(\partial_3^N u,\partial_3^N b)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{N+1}{2}}.$$

Remark 1.2. Due to the resemblance of mathematical structure, the same results as the ones stated in Theorem 1.1–1.3 can be achieved for the 3D MHD equations with only vertical magnetic diffusion:

$$\begin{cases}\partial_t u+u\cdot\nabla u+\nabla p=\nu\partial_2^2u+\nu\partial_3^2u+b\cdot\nabla b+\partial_2 b,\\ \partial_t b+u\cdot\nabla b=\eta\partial_3^2 b+b\cdot\nabla u+\partial_2 u,\\ \nabla\cdot u=\nabla\cdot b=0,\\ (u,b)(x,t)|_{t=0}=(u_0,b_0)(x).\end{cases}$$

Next, we briefly sketch the main ideas of the proofs in Theorems 1.1–1.3. Since the local-in-time existence result can be shown by the standard method (see, e.g., [26]), to prove Theorem 1.1, our main task is to derive the global-in-time a priori estimates of the solutions. The main difficulties arise from the lack of velocity dissipations in both x_1 and x_2 directions, which will be supplemented by the additional dissipation generated by the background magnetic field. To do this, we adopt the method of "double energy", which have been successfully used in the previous works (see, for example, [18,19], [22–25], etc). The first energy functional is the natural H^3-energy $\mathcal{E}_1(t)$ induced by the partial dissipations of (1.2),

$$\mathcal{E}_1(t)\triangleq\sup_{0\le\tau\le t}\|(u,b)(\tau)\|_{H^3}^2+2\int_0^t\left(\mu\|\partial_3u(\tau)\|_{H^3}^2+\eta\|\left(\partial_2b,\partial_3b\right)(\tau)\|_{H^3}^2\right)d\tau,\tag{1.7}$$

and the second one $\mathcal{E}_2(t)$ is the additional H^2-energy of $\partial_2 u$,

$$\mathcal{E}_2(t)\triangleq\int_0^t\|\partial_2u(\tau)\|_{H^2}^2d\tau,\tag{1.8}$$

which is indeed generated by the background magnetic field. The estimates $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ will be built up, based on the standard L^2-method and the applications of anisotropic Sobolev inequalities (cf. Lemmas 2.1 and 2.2). However, some of the nonlinear terms cannot be bounded by $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ directly. The most difficult terms are

$$\mathrm{D}_1 \triangleq 2\int \partial_2 u_2 \left(\partial_1^3 u_2\right)^2 dx,\ \mathrm{D}_2 \triangleq \int \partial_1^3 u_2 \partial_2 u_3 \partial_1^3 u_3\ dx,\ \mathrm{D}_3 \triangleq 3\int \partial_2 u_2 \left(\partial_1^3 u_3\right)^2 dx.$$

To deal with these terms, we make use of (1.2) to replace $\partial_2 u_2$ and $\partial_2 u_3$ by (see (3.34))

$$\begin{aligned}
\partial_2 u_2 &= \partial_t(b_2 - \eta\partial_2 u_2) - \eta\partial_3^2 b_2 + \mu\eta\partial_2\partial_3^2 u_2 \\
&\quad - \eta\partial_2\left(u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p\right) + u\cdot\nabla b_2 - b\cdot\nabla u_2, \\
\partial_2 u_3 &= \partial_t(b_3 - \eta\partial_2 u_3) - \eta\partial_3^2 b_3 + \mu\eta\partial_2\partial_3^2 u_3 \\
&\quad - \eta\partial_2\left(u\cdot\nabla u_3 - b\cdot\nabla b_3 + \partial_3 p\right) + u\cdot\nabla b_3 - b\cdot\nabla u_3.
\end{aligned} \tag{1.9}$$

For example, by several substitutions and integration by parts, the difficulty of the term D_1 is shifted to the treatment of F_4 (see (3.51)),

$$F_4 \triangleq 4\int b\cdot\nabla\partial_1^3 b_2 \partial_1^3 u_2 (b_2 - \eta\partial_2 u_2)\ dx.$$

Also, this cannot be bounded directly. To circumvent this difficulty, we artificially retain the term E_1 (see (3.27)),

$$E_1 \triangleq 2\int \left(\partial_1^3 b_2\right)^2 \partial_2 u_2\ dx,$$

which arises from the treatment of the nonlinear term $b\cdot\nabla u$. Based on (1.9) and integration by parts, the term E_1 will produce a "good" term R_4 (see (3.59)),

$$R_4 \triangleq 4\int b\cdot\nabla\partial_1^3 u_2 \partial_1^3 b_2 (b_2 - \eta\partial_2 u_2) dx,$$

which, combined with F_4 and integrated by parts, leads to the desired bound of $D_1 + E_1$ (see (3.62)). The estimates of the other two terms D_2 and D_3 can be done in a similar manner, if we add the terms E_2 and E_3,

$$E_2 \triangleq \int \partial_1^3 b_2 \partial_2 u_3 \partial_1^3 b_3\ dx, \quad E_3 \triangleq 3\int \left(\partial_1^3 b_3\right)^2 \partial_2 u_2\ dx,$$

to D_2 and D_3, respectively. With all the global a priori estimates at hand, Theorem 1.1 then follows from the bootstrap arguments (see, e.g. [37]). We refer to Section 3 for more technical details.

The proofs of the decay rates stated in Theorems 1.2–1.3 are built upon the spectral analysis and the integral representations in (2.7) and (2.8). In particular, the precise upper bounds for kernel functions play a mathematically important role. This forces us to divide the frequency space into suitable sub-domains and to derive the sharp estimates of each kernel function in these sub-domains (see Proposition 2.1). By virtue of Proposition 2.1, we then carry out some reasonable ansatz of the decay rates and provide the decay estimates of the nonlinear terms. With these preparations at hand, we can adopt the bootstrap argument to establish the desired decay estimates by elaborate calculations. The entire analysis strongly relies on Corollaries 2.1 and 2.2, which coincide respectively with the estimates of 2D and 3D heat kernel. It is worth mentioning that Corollary 2.2 is a consequence of the divergence-free condition and will be used to prove the enhanced decay rates of (u_1, b_1). Compared to the previous work [25], the analysis of the present paper is more delicate and technical. The details can be found in Section 4 and 5.

The rest of the paper is organized as follows. In Section 2, we present some useful mathematical tools and establish the sharp upper bounds of kernel functions in different sub-domains. The global stability result, i.e. Theorem 1.1, will be shown in Section 3. The decay rates stated in Theorems 1.2 and 1.3 will be derived in Section 4 and 5, respectively.

2. Preliminaries

2.1. Some elementary inequalities

In this subsection, we will recall some elementary inequalities and results which will be used frequently later. We begin with the anisotropic inequalities for triple products (see [24,25,43]).

Lemma 2.1. *Assume f, $\partial_1 f$, g, $\partial_2 g$, h and $\partial_3 h$ are all in $L^2(\mathbb{R}^3)$. Then,*

$$\int_{\mathbb{R}^3} |fgh| dx \le C\|f\|_{L^2}^{\frac{1}{2}} \|\partial_1 f\|_{L^2}^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh| dx &\le C\|f\|_{L^2} \|g\|_{L^2}^{\frac{1}{4}} \|\partial_i g\|_{L^2}^{\frac{1}{4}} \|\partial_j g\|_{L^2}^{\frac{1}{4}} \|\partial_i \partial_j g\|_{L^2}^{\frac{1}{4}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_k h\|_{L^2}^{\frac{1}{2}} \\ &\le C\|f\|_{L^2} \|g\|_{H^1}^{\frac{1}{2}} \|\partial_i g\|_{H^1}^{\frac{1}{2}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_k h\|_{L^2}^{\frac{1}{2}},\end{aligned}$$

where $i, j, k \in 1, 2, 3$ and $i \neq j \neq k$.

The following lemma provides the anisotropic upper bounds for quadruple product, which is very mathematically helpful in the treatments of nonlinear terms.

Lemma 2.2. *The following estimates hold when the right-hand sides are all bounded.*

$$\int_{\mathbb{R}^3} |efgh|dx \le C\|e\|_{L^2}^{\frac{1}{2}}\|\partial_i e\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}\|\partial_i f\|_{L^2}^{\frac{1}{2}}\|g\|_{H^1}^{\frac{1}{2}}\|\partial_k g\|_{H^1}^{\frac{1}{2}}\|h\|_{H^1}^{\frac{1}{2}}\|\partial_k h\|_{H^1}^{\frac{1}{2}}$$
$$\le C(\|(e,f)\|_{L^2}^2 + \|(g,h)\|_{H^1}^2)(\|(\partial_i e, \partial_i f)\|_{L^2}^2 + \|(\partial_k g, \partial_k h)\|_{H^1}^2),$$

where $i,k \in 1,2,3$ *and* $i \neq k$.

Proof. The proof relies on the basic one-dimensional Sobolev inequality:

$$\|g\|_{L^\infty(\mathbb{R})} \le \sqrt{2}\|g\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\|g'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}, \tag{2.1}$$

and the Minkowski inequality

$$\|\|f\|_{L_y^q(\mathbb{R}^n)}\|_{L_x^p(\mathbb{R}^m)} \le \|\|f\|_{L_x^p(\mathbb{R}^m)}\|_{L_y^q(\mathbb{R}^n)}, \quad \forall\ 1 \le q \le p \le \infty, \tag{2.2}$$

where $f = f(x,y)$ is a measurable function on $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$. With the help of (2.1)–(2.2), we have

$$\int_{\mathbb{R}^3} |efgh|dx \le C\|e\|_{L^2_{x_j,x_k}L^\infty_{x_i}}\|f\|_{L^2_{x_j,x_k}L^\infty_{x_i}}\|g\|_{L^\infty_{x_j,x_k}L^2_{x_i}}\|h\|_{L^\infty_{x_j,x_k}L^2_{x_i}}$$
$$\le C\left\|\|e\|_{L^2_{x_i}}^{\frac{1}{2}}\|\partial_i e\|_{L^2_{x_i}}^{\frac{1}{2}}\right\|_{L^2_{x_j,x_k}}\left\|\|f\|_{L^2_{x_i}}^{\frac{1}{2}}\|\partial_i f\|_{L^2_{x_i}}^{\frac{1}{2}}\right\|_{L^2_{x_j,x_k}}\|g\|_{L^2_{x_i}L^\infty_{x_j,x_k}}\|h\|_{L^2_{x_i}L^\infty_{x_j,x_k}}$$
$$\le C\|e\|_{L^2}^{\frac{1}{2}}\|\partial_i e\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}\|\partial_i f\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2_{x_i}L^\infty_{x_j,x_k}}\|h\|_{L^2_{x_i}L^\infty_{x_j,x_k}}. \tag{2.3}$$

By virtue (2.1) and Hölder's inequality, we obtain

$$\|g\|_{L^2_{x_i}L^\infty_{x_j,x_k}} \le C\left\|\|g\|_{L^2_{x_k}}^{\frac{1}{2}}\|\partial_k g\|_{L^2_{x_k}}^{\frac{1}{2}}\right\|_{L^2_{x_i}L^\infty_{x_j}} \le C\left\|\|g\|_{L^\infty_{x_j}}\right\|_{L^2_{x_i,x_k}}^{\frac{1}{2}}\left\|\|\partial_k g\|_{L^\infty_{x_j}}\right\|_{L^2_{x_i,x_k}}^{\frac{1}{2}}$$
$$\le C\|g\|_{L^2}^{\frac{1}{4}}\|\partial_j g\|_{L^2}^{\frac{1}{4}}\|\partial_k g\|_{L^2}^{\frac{1}{4}}\|\partial_{jk} g\|_{L^2}^{\frac{1}{4}} \le C\|g\|_{H^1}^{\frac{1}{2}}\|\partial_k g\|_{H^1}^{\frac{1}{2}}, \tag{2.4}$$

and analogously,

$$\|h\|_{L^2_{x_i}L^\infty_{x_j}L^\infty_{x_k}} \le C\|h\|_{H^1}^{\frac{1}{2}}\|\partial_k h\|_{H^1}^{\frac{1}{2}}. \tag{2.5}$$

Now, substituting (2.4) and (2.5) into (2.3), we infer from Cauchy-Schwar's inequality that

$$\int_{\mathbb{R}^3} |efgh|dx \le C\|e\|_{L^2}^{\frac{1}{2}}\|\partial_i e\|_{L^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}\|\partial_i f\|_{L^2}^{\frac{1}{2}}\|g\|_{H^1}^{\frac{1}{2}}\|\partial_k g\|_{H^1}^{\frac{1}{2}}\|h\|_{H^1}^{\frac{1}{2}}\|\partial_k h\|_{H^1}^{\frac{1}{2}}$$
$$\le C\|e\|_{L^2}^2\|\partial_i e\|_{L^2}^2 + \|f\|_{L^2}^2\|\partial_i f\|_{L^2}^2 + \|g\|_{H^1}^2\|\partial_k g\|_{H^1}^2 + \|h\|_{H^1}^2\|\partial_k h\|_{H^1}^2$$

$$\leq C\left(\|(e,f)\|_{L^2}^2+\|(g,h)\|_{H^1}^2\right)\left(\|(\partial_i e,\partial_i f)\|_{L^2}^2+\|(\partial_k g,\partial_k h)\|_{H^1}^2\right).$$

The proof of Lemma 2.2 is thus complete. □

2.2. Integral representation and the kernels

In this subsection, we aim to derive integral representation of (1.2) and study the sharp bounds for the kernel functions, which will be used to prove the decay rates of the solutions. To begin, we first operate the Fourier transform of (1.2) to get that ($\widehat{u}=(\widehat{u_1},\widehat{u_2},\widehat{u_3})$ and $\widehat{b}=(\widehat{b_1},\widehat{b_2},\widehat{b_3})$)

$$\partial_t\begin{pmatrix}\widehat{u_i}\\ \widehat{b_i}\end{pmatrix}=\mathbb{A}\begin{pmatrix}\widehat{u_i}\\ \widehat{b_i}\end{pmatrix}+\begin{pmatrix}\widehat{N_{1i}}\\ \widehat{N_{2i}}\end{pmatrix},\quad \forall\, i=1,2,3,$$

where $\mathbb{A}$ comes from the linear operators, and N_1, N_2 are the nonlinear terms,

$$\mathbb{A}\triangleq\begin{pmatrix}-\mu\xi_3^2 & i\xi_2\\ i\xi_2 & -\eta\left(\xi_2^2+\xi_3^2\right)\end{pmatrix}$$

and

$$\begin{cases}N_1=(N_{11},N_{12},N_{13})\triangleq\mathbb{P}\left(b\cdot\nabla b-u\cdot\nabla u\right),\\ N_2=(N_{21},N_{22},N_{23})\triangleq b\cdot\nabla u-u\cdot\nabla b.\end{cases}$$

The characteristic polynomial of A is determined by

$$\lambda^2+\left[\mu\xi_3^2+\eta\left(\xi_2^2+\xi_3^2\right)\right]\lambda+\mu\eta\xi_3^2\left(\xi_2^2+\xi_3^2\right)+\xi_2^2=0,$$

from which we obtain the eigenvalues of $\mathbb{A}$:

$$\lambda_1=\frac{-\left[\mu\xi_3^2+\eta\left(\xi_2^2+\xi_3^2\right)\right]-\sqrt{\Gamma}}{2},\quad \lambda_2=\frac{-\left[\mu\xi_3^2+\eta\left(\xi_2^2+\xi_3^2\right)\right]+\sqrt{\Gamma}}{2}\tag{2.6}$$

with

$$\Gamma\triangleq\left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2-4\left(\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2\right),\quad \xi_\nu^2\triangleq\xi_2^2+\xi_3^2.$$

By computing the corresponding eigenvectors and diagonalizing $\mathbb{A}$, we find

$$\widehat{u}(\xi,t)=\widehat{K_1}(t)\widehat{u_0}+\widehat{K_2}(t)\widehat{b_0}+\int_0^t\left(\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)+\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right)d\tau\tag{2.7}$$

and

$$\widehat{b}(\xi,t)=\widehat{K_2}(t)\widehat{u_0}+\widehat{K_3}(t)\widehat{b_0}+\int_0^t\left(\widehat{K_2}(t-\tau)\widehat{N_1}(\tau)+\widehat{K_3}(t-\tau)\widehat{N_2}(\tau)\right)d\tau, \tag{2.8}$$

where the kernel functions $\widehat{K_1},\widehat{K_2}$ and $\widehat{K_3}$ are given by

$$\widehat{K_1}=-\mu\xi_3^2G_1+G_2,\quad \widehat{K_2}=i\xi_2G_1,\quad \widehat{K_3}=\mu\xi_3^2G_1+G_3, \tag{2.9}$$

with

$$G_1=\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1},\quad G_2=\frac{\lambda_2e^{\lambda_1 t}-\lambda_1e^{\lambda_2 t}}{\lambda_2-\lambda_1}=e^{\lambda_1 t}-\lambda_1G_1=e^{\lambda_2 t}-\lambda_2G_1,$$
$$G_3=\frac{\lambda_2e^{\lambda_2 t}-\lambda_1e^{\lambda_1 t}}{\lambda_2-\lambda_1}=e^{\lambda_1 t}+\lambda_2G_1=e^{\lambda_2 t}+\lambda_1G_1.$$

It is worth pointing out that when $\lambda_1=\lambda_2$, the representation in (2.7) and (2.8) remains valid, provided we replace G_1 by its limiting form:

$$G_1=\lim_{\lambda_2\to\lambda_1}\frac{e^{\lambda_2 t}-e^{\lambda_1 t}}{\lambda_2-\lambda_1}=te^{\lambda_1 t}.$$

Obviously, the behavior of $\widehat{K_1}(\xi,t)$, $\widehat{K_2}(\xi,t)$ and $\widehat{K_3}(\xi,t)$ depends strongly on the Fourier frequencies ξ. So, we need to study the upper bounds for the kernel functions in the different sub-domains of frequency space.

Proposition 2.1. *Let Ω_1 and Ω_2 be the sub-domains of $\mathbb{R}^3$ defined by*

$$\Omega_1\triangleq\left\{\xi\in\mathbb{R}^3:\ \Gamma\le\frac{\left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2}{4}\quad or\quad \left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2\le\frac{16}{3}\left(\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2\right)\right\},$$
$$\Omega_2\triangleq\left\{\xi\in\mathbb{R}^3:\ \Gamma>\frac{\left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2}{4}\quad or\quad \left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2>\frac{16}{3}\left(\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2\right)\right\}.$$

Then G_1, G_2 and G_3, and $\widehat{K_1}$, $\widehat{K_2}$ and $\widehat{K_3}$ admit the following upper bounds:

(I) *There exists some $c=c(\nu,\eta)>0$ such that for any $\xi\in\Omega_1$,*

$$Re\lambda_1\le-\frac{1}{2}\left(\mu\xi_3^2+\eta\xi_\nu^2\right),\quad Re\lambda_2\le-\frac{1}{4}\left(\mu\xi_3^2+\eta\xi_\nu^2\right),$$
$$|G_1|\le te^{-\frac{1}{4}(\mu\xi_3^2+\eta\xi_\nu^2)t},\quad |\widehat{K_1}|,|\widehat{K_2}|,|\widehat{K_3}|\le Ce^{-c\xi_\nu^2 t}. \tag{2.10}$$

(II) *There exists some $c = c(\nu, \eta) > 0$ such that for any $\xi \in \Omega_2$,*

$$\begin{aligned}
&\lambda_1 \le -\frac{3}{4}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right), \quad \lambda_2 \le -\frac{\mu\eta\xi_3^2\xi_\nu^2 + \xi_2^2}{\mu\xi_3^2 + \eta\xi_\nu^2},\\
&|G_1| \le 2\left(\mu\xi_3^2 + \eta\xi_\nu^2\right)^{-1}\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right),\\
&|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}| \le Ce^{-c(\mu\xi_3^2+\eta\xi_\nu^2)t} + Ce^{-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2}t}.
\end{aligned} \tag{2.11}$$

(III) *Let Ω_{2i} with $i = 1, 2, 3$ be the subsets of Ω_2 as follows,*

$$\begin{aligned}
&\Omega_{21} \triangleq \left\{\xi \in \Omega_2, \quad \mu\xi_3^2 > \eta\xi_\nu^2\right\},\\
&\Omega_{22} \triangleq \left\{\xi \in \Omega_2, \quad \mu\xi_3^2 \le \eta\xi_\nu^2 \quad \textit{and} \quad |\xi_2| \le |\xi_3|\right\},\\
&\Omega_{23} \triangleq \left\{\xi \in \Omega_2, \quad \mu\xi_3^2 \le \eta\xi_\nu^2 \quad \textit{and} \quad |\xi_2| > |\xi_3|\right\}.
\end{aligned}$$

Then the following bounds hold for K_1, K_2 and K_3,

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}| \le Ce^{-c\xi_\nu^2 t}, \quad \forall\ \xi \in \Omega_{21}, \tag{2.12}$$

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}| \le Ce^{-c\xi_\nu^2 t}, \quad \forall\ \xi \in \Omega_{22}, \tag{2.13}$$

$$|\widehat{K_1}|, |\widehat{K_2}|, |\widehat{K_3}| \le C\left(e^{-c(1+\xi_3^2)t} + e^{-c\xi_\nu^2 t}\right), \quad \forall\ \xi \in \Omega_{23}. \tag{2.14}$$

Proof. (I) For $\xi \in \Omega_1$, the eigenvalues λ_1 and λ_2 given by (2.6) obviously satisfy

$$Re\lambda_1 \le -\frac{1}{2}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right), \quad Re\lambda_2 \le -\frac{1}{4}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right).$$

We divide the analysis into two cases,

$$\Gamma < 0 \quad \text{and} \quad 0 \le \sqrt{\Gamma} \le \frac{\mu\xi_3^2 + \eta\left(\xi_2^2 + \xi_3^2\right)}{2}.$$

Case 1. $\Gamma < 0$**.** In this case, the eigenvalues λ_1 and λ_2 are a pair of complex conjugates, satisfying

$$|\lambda_1|, \ |\lambda_2| = \sqrt{\mu\eta\xi_3^2\left(\xi_2^2 + \xi_3^2\right) + \xi_2^2}.$$

In addition,

$$G_1 = \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} = e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t}\,\frac{2\sin(\frac{1}{2}Qt)}{Q}, \tag{2.15}$$

where

$$Q := \sqrt{4\left(\mu\eta\xi_3^2\left(\xi_2^2+\xi_3^2\right)+\xi_2^2\right)-\left[\mu\xi_3^2+\eta\left(\xi_2^2+\xi_3^2\right)\right]^2}.$$

By the simple fact that $|\sin\rho| \le |\rho|$ for any $\rho \in \mathbb{R}$, we infer from (2.15) that

$$|G_1| \le t\, e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t}.$$

It follows from the definitions of $\widehat{K}_1$ and $\widehat{K}_3$ in (2.9) that

$$|\widehat{K_1}| = |\mu\xi_3^2 G_1 + G_2| = |\mu\xi_3^2 G_1 + e^{\lambda_1 t} - \lambda_1 G_1|, \tag{2.16}$$

$$|\widehat{K_3}| = |\mu\xi_3^2 G_1 + G_3| = |\mu\xi_3^2 G_1 + e^{\lambda_2 t} + \lambda_1 G_1|. \tag{2.17}$$

Next, we first estimate the term $|\lambda_1 G_1|$. On one hand, if $|\lambda_1| \le \sqrt{-\Gamma}$, then

$$|\lambda_1 G_1| = |\lambda_1| \frac{\left|e^{\lambda_2 t} - e^{\lambda_1 t}\right|}{\left|\sqrt{-\Gamma}\right|} \le |e^{\lambda_1 t}| + |e^{\lambda_2 t}| \le C e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t}.$$

On the other hand, if $|\lambda_1| > \sqrt{-\Gamma}$, then

$$3\left(\mu\eta\xi_3^2\left(\xi_2^2+\xi_3^2\right)+\xi_2^2\right) < \left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2 \quad \text{or} \quad |\lambda_1| < \frac{1}{\sqrt{3}}\left(\mu\xi_3^2+\eta\xi_\nu^2\right),$$

which, together with (2.15), yields

$$\begin{aligned} |\lambda_1 G_1| &= |\lambda_1| e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t} \left|\frac{2\sin(\frac{1}{2}Qt)}{Q}\right| \\ &\le \frac{1}{\sqrt{3}}\left(\mu\xi_3^2+\eta\xi_\nu^2\right) t\, e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t} \le C e^{-c\xi_\nu^2 t}. \end{aligned}$$

So, in the case that $\Gamma < 0$ we have $|\lambda_1 G_1| \le C e^{-c\xi_\nu^2 t}$. As a result,

$$|\widehat{K_1}| = |\mu\xi_3^2 G_1 + e^{\lambda_1 t} - \lambda_1 G_1| \le C e^{-c\xi_\nu^2 t},$$

and analogously,

$$|\widehat{K_3}| = |\mu\xi_3^2 G_1 + e^{\lambda_2 t} + \lambda_1 G_1| \le C e^{-c\xi_\nu^2 t}.$$

To deal with $\widehat{K_2}$, we first observe that if $|\xi_2| \le \sqrt{-\Gamma}$, then

$$|\widehat{K_2}| = |\xi_2| \frac{\left|e^{\lambda_2 t} - e^{\lambda_1 t}\right|}{\sqrt{-\Gamma}} \le |e^{\lambda_1 t}| + |e^{\lambda_2 t}| \le C e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t}.$$

Moreover, if $|\xi_2| > \sqrt{-\Gamma}$, then

$$\xi_2^2 > 4\left(\mu\eta\xi_3^2\left(\xi_2^2+\xi_3^2\right)+\xi_2^2\right) - \left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2,$$

which implies

$$3\xi_2^2 < \left(\mu\xi_3^2 + \eta\xi_\nu^2\right)^2 \quad \text{i.e.} \quad |\xi_2| < \frac{1}{\sqrt{3}}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right), \tag{2.18}$$

so that, it follows from (2.9), (2.15) and (2.18) that

$$|\widehat{K_2}| = |\xi_2| e^{-\frac{1}{2}(\mu\xi_3^2+\eta\xi_\nu^2)t}\left|\frac{2\sin(\frac{1}{2}Qt)}{Q}\right| \le Ce^{-c\xi_\nu^2 t}.$$

This finishes the proof of (2.10) in the case $\Gamma < 0$.

Case 2. $0 \le \sqrt{\Gamma} \le \frac{\mu\xi_3^2+\eta\xi_\nu^2}{2}$**.** In this case, both λ_1 and λ_2 are real and satisfy

$$-\frac{3}{4}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right) \le \lambda_1 \le -\frac{1}{2}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right),$$
$$-\frac{1}{2}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right) \le \lambda_2 \le -\frac{1}{4}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right).$$

It follows from the mean-value theorem that there is a $\zeta \in (\lambda_1, \lambda_2)$ such that

$$G_1 = te^{\zeta} \le te^{-\frac{1}{4}(\mu\xi_3^2+\eta\xi_\nu^2)t},$$

and thus,

$$|\lambda_1 G_1| \le \frac{3}{4}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right) te^{-\frac{1}{4}(\mu\xi_3^2+\eta\xi_\nu^2)t} \le Ce^{-c\xi_\nu^2 t}. \tag{2.19}$$

With (2.19) at hand, we can derive the desired bounds of $|\widehat{K_1}|$ and $|\widehat{K_3}|$ in a similar manner as that in Case 1.

To bound $|\widehat{K_2}|$, we first use the fact that $\sqrt{\Gamma} \ge 0$ to get

$$\left(\mu\xi_3^2 + \eta\xi_\nu^2\right)^2 \ge 4\left(\mu\eta\xi_3^2\xi_\nu^2 + \xi_2^2\right),$$

and hence,

$$4\xi_2^2 \le 4\left(\mu\eta\xi_3^2\xi_\nu^2 + \xi_2^2\right) \le \left(\mu\xi_3^2 + \eta\xi_\nu^2\right)^2 \quad \text{i.e.} \quad |\xi_2| \le \frac{1}{2}\left(\mu\xi_3^2 + \eta\xi_\nu^2\right),$$

from which it immediately follows that

$$|\widehat{K_2}| = |\xi_2 G_1| \le |\xi_2| te^{-\frac{1}{4}(\mu\xi_3^2+\eta\xi_\nu^2)t} \le Ce^{-c\xi_\nu^2 t}.$$

This, together with Case 1, shows that (2.10) holds for all $\xi \in \Omega_1$.

(II) For $\xi \in \Omega_2$, it is clear that λ_1 and λ_2 are real and

$$\begin{aligned} -\left(\mu\xi_3^2+\eta\xi_\nu^2\right) \le \lambda_1 &\le -\frac{3}{4}\left(\mu\xi_3^2+\eta\xi_\nu^2\right), \\ \lambda_2 &= \frac{-\left[\mu\xi_3^2+\eta\left(\xi_2^2+\xi_3^2\right)\right]+\sqrt{\Gamma}}{2} \\ &= -\frac{2\left(\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2\right)}{\sqrt{\Gamma}+\mu\xi_3^2+\eta\xi_\nu^2} \le -\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2}, \end{aligned} \tag{2.20}$$

from which it is easily seen that

$$\lambda_2-\lambda_1=\sqrt{\Gamma} \ge \frac{1}{2}\left(\mu\xi_3^2+\eta\xi_\nu^2\right),$$

and consequently, the term G_1 admits the following upper bound:

$$|G_1| \le 2\left(\mu\xi_3^2+\eta\xi_\nu^2\right)^{-1}\left(e^{\lambda_1 t}+e^{\lambda_2 t}\right). \tag{2.21}$$

As an immediate result of (2.20) and (2.21), one has

$$|\lambda_1 G_1| \le 2\left(e^{\lambda_1 t}+e^{\lambda_2 t}\right),$$

so that, it is easily derived from (2.16)–(2.17) and

$$|\widehat{K_1}|,\ |\widehat{K_3}| \le C\left(e^{\lambda_1 t}+e^{\lambda_2 t}\right) \le Ce^{-c(\mu\xi_3^2+\eta\xi_\nu^2)t}+Ce^{-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2}t}.$$

For $\widehat{K_2}$, since $\xi \in \Omega_2$, we have

$$\left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2 > \frac{16}{3}\left(\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2\right),$$

that is,

$$\frac{16}{3}\xi_2^2 < \left(\mu\xi_3^2+\eta\xi_\nu^2\right)^2 \quad \text{or} \quad |\xi_2| < \frac{\sqrt{3}}{4}\left(\mu\xi_3^2+\eta\xi_\nu^2\right),$$

from which and (2.21), we find

$$|\widehat{K_2}| = |\xi_2 G_1| \le C\left(e^{\lambda_1 t}+e^{\lambda_2 t}\right) \le Ce^{-c(\mu\xi_3^2+\eta\xi_\nu^2)t}+Ce^{-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2}t}.$$

This finishes the proof of the assertions stated in (2.11).

(III) For $\xi \in \Omega_{21}$,

$$-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2} \le -\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{2\mu\xi_3^2} \le -\frac{\eta}{2}\xi_\nu^2.$$

For $\xi \in \Omega_{22}$,

$$-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2} \le -\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{2\eta\xi_\nu^2} \le -\frac{\mu}{2}\xi_3^2 \le -\frac{\mu}{4}\xi_\nu^2.$$

For $\xi \in \Omega_{23}$,

$$-\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{\mu\xi_3^2+\eta\xi_\nu^2} \le -\frac{\mu\eta\xi_3^2\xi_\nu^2+\xi_2^2}{2\eta\xi_\nu^2} = -\frac{\mu}{2}\xi_3^2 - \frac{\xi_2^2}{2\eta\,(\xi_2^2+\xi_3^2)} \le -\frac{\mu}{2}\xi_3^2 - \frac{1}{4\eta}.$$

Thus, combining these bounds with $(2.11)_3$, we immediately arrive at (2.12)–(2.14). The proof of Proposition 2.1 is therefore complete. □

2.3. Mathematical tools for the decay estimates

In this subsection, we state some useful inequalities which play important roles in the derivations of the decay rates. We first recall the following lemma concerning the exact decay rate for the solution operator associated with a fractional Laplacian (cf. [34,40]).

Lemma 2.3. *Let $\alpha \ge 0$ and $\beta > 0$ be real numbers. Then for any $1 \le q \le p \le \infty$,*

$$\|(-\Delta)^\alpha e^{-(-\Delta)^\beta t} f\|_{L^p(\mathbb{R}^d)} \le C\, t^{-\frac{\alpha}{\beta}-\frac{d}{2\beta}(\frac{1}{q}-\frac{1}{p})}\, \|f\|_{L^q(\mathbb{R}^d)}, \quad \forall\ t > 0.$$

It is clear from Proposition 2.1 that there is lack of ξ_1-decay in kernel functions. To overcome this difficulty and to derive the decay rates, we have to assume that the initial data is equipped with $L^2_{x_1}L^1_{x_2,x_3}$-norm. Moreover, as an immediate consequence of Lemma 2.3, we have the following corollary which is technically used in the analysis.

Corollary 2.1. *Assume that $u \in L^2_{x_1}L^1_{x_2,x_3} \cap L^2$. Then for any $\alpha \ge 0$, there exists an absolutely positive constant $C > 0$, depending only on α, such that*

$$\||\xi_\nu|^\alpha e^{-\xi_\nu^2 t}\widehat{u}(\xi)\|_{L^2_\xi} \le C(1+t)^{-\frac{1+\alpha}{2}} \left\| \|u\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}}, \quad \forall\ t \ge 1, \tag{2.22}$$

where $\xi = (\xi_1, \xi_\nu)$ with $\xi_\nu = (\xi_2, \xi_3)$.

Proof. Let $\Delta_\nu \triangleq \partial_2^2 + \partial_3^2$. Then, it follows from Plancherel theorem, Fubini theorem and Lemma 2.3 that for any $t \ge 1$,

$$\begin{aligned}
&\||\xi_\nu|^\alpha e^{-\xi_\nu^2 t}\widehat{u}(\xi)\|_{L^2_\xi} \\
&\quad \le \|(-\Delta_\nu)^{\frac{\alpha}{2}} e^{\Delta_\nu t} u\|_{L^2} = \left\| \|(-\Delta_\nu)^{\frac{\alpha}{2}} e^{\Delta_\nu t} u\|_{L^2_{x_2,x_3}} \right\|_{L^2_{x_1}} \\
&\quad \le Ct^{-\frac{1+\alpha}{2}} \left\| \|u\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \le C(1+t)^{-\frac{1+\alpha}{2}} \left\| \|u\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}},
\end{aligned}$$

which ends the proof of (2.22). □

Furthermore, if u is a vector-valued function satisfying the divergence-free condition, then we have the following refined decay rate of u_1.

Corollary 2.2. *Let $u = (u_1, u_2, u_3)$ be a smooth function, satisfying $u \in L^1 \cap L^2$ and $\nabla \cdot u = 0$. Then for any $\beta \geq 0$, there exists an absolutely positive constant $C > 0$, depending only on β, such that*

$$\||\xi_\nu|^\beta e^{-\xi_\nu^2 t}\widehat{u_1}(\xi)\|_{L^2_\xi} \leq C(1+t)^{-\frac{3+2\beta}{4}}\|u\|_{L^1}, \quad \forall\ t \geq 1. \tag{2.23}$$

Proof. Let $\nabla_\nu \triangleq (\partial_2, \partial_3)$. Then, it holds that $\partial_1 u_1 = -\partial_\nu u_\nu$ with $u_\nu \triangleq (u_2, u_3)$, due to the divergence-free condition $\nabla \cdot u = 0$. Thus, using Plancherel theorem and Fubini theorem, we deduce from Lemma 2.3 that for any $t \geq 1$,

$$\begin{aligned}
\||\xi_\nu|^\beta e^{-\xi_\nu^2 t}\widehat{u_1}(\xi)\|^2_{L^2_\xi} &= \|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t} u_1\|^2_{L^2} \\
&\leq C\left\|\|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t} u_1\|^{\frac12}_{L^1_{x_1}} \|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t} u_1\|^{\frac12}_{L^\infty_{x_1}}\right\|^2_{L^2_{x_2,x_3}} \\
&\leq C\left\|\|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t} u_1\|_{L^1_{x_1}}\right\|_{L^2_{x_2,x_3}} \left\|\|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t}\partial_1 u_1\|_{L^1_{x_1}}\right\|_{L^2_{x_2,x_3}} \\
&\leq C\left\|\|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t} u\|_{L^2_{x_2,x_3}}\right\|_{L^1_{x_1}} \left\|\|(-\Delta_\nu)^{\frac{\beta}{2}} e^{\Delta_\nu t}\nabla_\nu \cdot u_\nu\|_{L^2_{x_2,x_3}}\right\|_{L^1_{x_1}} \\
&\leq C\left(t^{-\frac{1+\beta}{2}}\left\|\|u\|_{L^1_{x_2,x_3}}\right\|_{L^1_{x_1}}\right)\left(t^{-\frac{2+\beta}{2}}\left\|\|u\|_{L^1_{x_2,x_3}}\right\|_{L^1_{x_1}}\right) \\
&\leq Ct^{-\frac{3+2\beta}{2}}\|u\|^2_{L^1} \leq (1+t)^{-\frac{3+2\beta}{2}}\|u\|^2_{L^1},
\end{aligned}$$

where we have also used Hölder and Minkowski inequalities. □

The following two lemmas are concerned with the upper bounds with sharp decay rates for two special integrals (see, e.g., [38]).

Lemma 2.4. *For any $0 < s_1 \leq s_2$, there exists a positive constant C, depending only on s_1 and s_2, such that for any $t \geq 0$,*

$$\int_0^t (1+t-\tau)^{-s_1}(1+\tau)^{-s_2}\,d\tau \leq \begin{cases} C(1+t)^{-s_1}, & \text{if} \quad s_2 > 1; \\ C(1+t)^{-s_1}\ln(1+t), & \text{if} \quad s_2 = 1; \\ C(1+t)^{1-s_1-s_2}, & \text{if} \quad s_2 < 1. \end{cases}$$

Lemma 2.5. *For any $c > 0$ and $s > 0$, it holds that*

$$\int_0^t e^{-c(t-\tau)}(1+\tau)^{-s}\,d\tau \leq C(1+t)^{-s}, \quad \forall\ t \geq 0.$$

3. Global stability and proof of Theorem 1.1

This section concerns the global well-posedness of smooth solutions stated in Theorem 1.1. Since the local existence result can be shown in a similar manner as that in [26], the main task of the present paper is to derive the global a priori estimates of the solutions. This will be done by using the method of dual-layer energy and applying the bootstrapping arguments.

3.1. A priori estimates (I)

The first step is to deal with the natural H^3-energy $\mathcal{E}_1(t)$, based on the anisotropic Sobolev inequalities and the divergence-free conditions.

Lemma 3.1. *Let $\mathcal{E}_1(t)$ and $\mathcal{E}_2(t)$ be the same ones defined in (1.7) and (1.8), respectively. Then for any $t \geq 0$,*

$$\mathcal{E}_1(t) \leq C\left(\mathcal{E}_1(0) + \mathcal{E}_1^{\frac{3}{2}}(0)\right) + C\left(\mathcal{E}_1^{\frac{3}{2}}(t) + \mathcal{E}_2^{\frac{3}{2}}(t)\right) + C\left(\mathcal{E}_1^2(t) + \mathcal{E}_2^2(t)\right). \tag{3.1}$$

Proof. Since $\|(u,b)\|_{H^3} \sim \|(u,b)\|_{L^2} + \|(\nabla^3 u, \nabla^3 b)\|_{L^2}$, it suffices to bound the L^2-norms and the homogeneous $\dot{H}^3$-norms of (u,b).

Step I. The estimate of L^2-norm

Based on the divergence-free conditions $\nabla \cdot u = \nabla \cdot b = 0$, it is easily derived from (1.2) that for any $0 \leq t \leq T$,

$$\|(u,b)(t)\|_{L^2}^2 + 2\int_0^t \left(\mu\|\partial_3 u\|_{L^2}^2 + \eta\|\partial_2 b\|_{L^2}^2 + \eta\|\partial_3 b\|_{L^2}^2\right) d\tau = \|(u_0,b_0)\|_{L^2}^2. \tag{3.2}$$

Step II. The preliminary bound of $\dot{H}^3$-norm

Applying ∂_i^3 with $i = 1,2,3$ to (1.2) and dotting them with $(\partial_i^3 u, \partial_i^3 b)$ in L^2, we obtain after integrating by parts that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 \|(\partial_i^3 u, \partial_i^3 b)\|_{L^2}^2 + \mu\sum_{i=1}^3 \|\partial_i^3\partial_3 u\|_{L^2}^2 + \eta\sum_{i=1}^3 \left(\|\partial_i^3\partial_2 b\|_{L^2}^2 + \|\partial_i^3\partial_3 b\|_{L^2}^2\right) \\ &\quad \triangleq H_1 + H_2 + H_3 + H_4 + H_5, \end{aligned} \tag{3.3}$$

where

$$H_1 \triangleq \sum_{i=1}^3 \int \left(\partial_i^3\partial_2 b \cdot \partial_i^3 u + \partial_i^3\partial_2 u \cdot \partial_i^3 b\right) dx, \quad H_2 \triangleq -\sum_{i=1}^3 \int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\, dx,$$

$$H_3 \triangleq \sum_{i=1}^{3} \int \left[\partial_i^3(b\cdot\nabla b) - b\cdot\nabla\partial_i^3 b\right]\cdot\partial_i^3 u\ dx, \quad H_4 \triangleq -\sum_{i=1}^{3}\int \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\ dx,$$

$$H_5 \triangleq \sum_{i=1}^{3} \int \left[\partial_i^3(b\cdot\nabla u) - b\cdot\nabla\partial_i^3 u\right]\cdot\partial_i^3 b\ dx.$$

We are now in a position of estimating each term on the right-hand side of (3.3). First, it is easily seen that

$$H_1 = 0. \tag{3.4}$$

Due to $\nabla\cdot u = 0$, by direct calculations we have

$$\begin{aligned} H_2 = &-\sum_{i=1}^{3}\int \partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\ dx - 3\sum_{i=1}^{3}\int \partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\ dx \\ &- 3\sum_{i=1}^{3}\int \partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\ dx \triangleq \sum_{j=1}^{3} H_{2j}. \end{aligned}$$

It follows from Hölder's and Sobolev's inequalities that

$$\begin{aligned} H_{21} &= -\int \left(\partial_1^3 u\cdot\nabla u\cdot\partial_1^3 u + \partial_2^3 u\cdot\nabla u\cdot\partial_2^3 u + \partial_3^3 u\cdot\nabla u\cdot\partial_3^3 u\right)\ dx \\ &\le C\|\nabla u\|_{L^\infty}\left(\|\partial_2^3 u\|_{L^2}^2 + \|\partial_3^2 u\|_{L^2}^2\right) + H_{211} \\ &\le C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^2}^2\right) + H_{211}, \end{aligned} \tag{3.5}$$

where

$$H_{211} \triangleq -\int \partial_1^3 u\cdot\nabla u\cdot\partial_1^3 u\ dx.$$

In view of the divergence-free condition $\nabla\cdot u = 0$, we get

$$\|\partial_1 u_1\|_{\dot H^k} \le \|\partial_2 u_2\|_{\dot H^k} + \|\partial_3 u_3\|_{\dot H^k} \le \|\partial_2 u\|_{\dot H^k} + \|\partial_3 u\|_{\dot H^k}, \quad k = 1, 2, \dots, \tag{3.6}$$

so that, by Lemma 2.1 we have from integration by parts that

$$\begin{aligned} H_{211} = &-\int \partial_1^3 u_1\partial_1 u\cdot\partial_1^3 u\ dx - \int \partial_1^3 u_2\partial_2 u_1\partial_1^3 u_1\ dx - \int \partial_1^3 u_2\partial_2 u_2\partial_1^3 u_2\ dx \\ &- \int \partial_1^3 u_2\partial_2 u_3\partial_1^3 u_3\ dx + \int \partial_3\partial_1^3 u_3\left(u\cdot\partial_1^3 u\right)dx + \int \partial_1^3 u_3\left(u\cdot\partial_3\partial_1^3 u\right)dx \\ \le\ & C\|\partial_1^3 u_1\|_{L^2}\|\partial_1 u\|_{L^2}^{\frac14}\|\partial_1^2 u\|_{L^2}^{\frac14}\|\partial_1\partial_2 u\|_{L^2}^{\frac14}\|\partial_1^2\partial_2 u\|_{L^2}^{\frac14}\|\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac12} \\ &+ C\|\partial_1^3 u_1\|_{L^2}\|\partial_2 u_1\|_{L^\infty}\|\partial_1^3 u_2\|_{L^2} + D_1^1 + D_2 \end{aligned}$$

$$
\begin{aligned}
&+ C\|\partial_3\partial_1^3 u\|_{L^2}\|u\|_{L^2}^{\frac{1}{4}}\|\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\le C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right) + D_1^1 + D_2, \qquad (3.7)
\end{aligned}
$$

where

$$
D_1^1 \triangleq -\int \partial_2 u_2 \left(\partial_1^3 u_2\right)^2 dx, \qquad D_2 \triangleq -\int \partial_1^3 u_2 \partial_2 u_3 \partial_1^3 u_3 \, dx.
$$

Thus, substituting (3.7) to (3.5), we find

$$
H_{21} \le C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^2}^2\right) + D_1^1 + D_2. \qquad (3.8)
$$

In terms of (3.6) and Lemma 2.1, we can bound H_{22} by

$$
\begin{aligned}
H_{22} &= -3\int \left(\partial_1^2 u_1 \partial_1^2 u \cdot \partial_1^3 u + \partial_1^2 u_2 \partial_2\partial_1 u \cdot \partial_1^3 u + \partial_1^2 u_3 \partial_3\partial_1 u \cdot \partial_1^3 u\right) dx\\
&\quad - 3\int \left(\partial_2^2 u \cdot \nabla\partial_2 u \cdot \partial_2^3 u + \partial_3^2 u \cdot \nabla\partial_3 u \cdot \partial_3^3 u\right) dx\\
&\le C\|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|\partial_1^2 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\left(\|\partial_2^2 u\|_{L^3}\|\nabla\partial_2 u\|_{L^6}\|\partial_2^3 u\|_{L^2} + \|\partial_3^2 u\|_{L^3}\|\nabla\partial_3 u\|_{L^6}\|\partial_3^3 u\|_{L^2}\right)\\
&\le C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right), \qquad (3.9)
\end{aligned}
$$

and analogously,

$$
\begin{aligned}
H_{23} &= -3\int \left(\partial_1 u_1 \partial_1^3 u \cdot \partial_1^3 u + \partial_1 u_2 \partial_2\partial_1^2 u \cdot \partial_1^3 u + \partial_1 u_3 \partial_3\partial_1^2 u \cdot \partial_1^3 u\right) dx\\
&\quad - 3\int \left(\partial_2 u \cdot \nabla\partial_2^2 u \cdot \partial_2^3 u + \partial_3 u \cdot \nabla\partial_3^2 u \cdot \partial_3^3 u\right) dx\\
&\le C\|\partial_2\partial_1^2 u\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{4}}\|\partial_1^2 u_2\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 u_2\|_{L^2}^{\frac{1}{4}}\|\partial_1^2\partial_2 u_2\|_{L^2}^{\frac{1}{4}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|\partial_3\partial_1^2 u\|_{L^2}\|\partial_1 u_3\|_{L^2}^{\frac{1}{4}}\|\partial_1^2 u_3\|_{L^2}^{\frac{1}{4}}\|\partial_1\partial_2 u_3\|_{L^2}^{\frac{1}{4}}\|\partial_1^2\partial_2 u_3\|_{L^2}^{\frac{1}{4}}\|\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\left(\|\partial_2 u\|_{L^\infty}\|\nabla\partial_2 u\|_{L^2}\|\partial_2^3 u\|_{L^2} + \|\partial_3 u\|_{L^\infty}\|\nabla\partial_3^2 u\|_{L^2}\|\partial_3^3 u\|_{L^2}\right) + H_{231}\\
&\le C\|u\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right) + H_{231}, \qquad (3.10)
\end{aligned}
$$

where

$$
H_{231} \triangleq -3\int \partial_1 u_1 \partial_1^3 u \cdot \partial_1^3 u \, dx.
$$

Note that $\|\partial_1 u_1\|_{L^\infty} \le C\|\partial_1 u_1\|_{H^2} \le C\|(\partial_2 u_2, \partial_3 u_3)\|_{H^2}$, due to the divergence-free condition $\nabla \cdot u = 0$ and the Sobolev embedding inequality. So, integrating by parts and using Lemma 2.1, we obtain

$$\begin{aligned}
H_{231} &= -3\int \partial_1 u_1 \partial_1^3 u_1 \partial_1^3 u_1 \, dx - 3\int \partial_1 u_1 \partial_1^3 u_2 \partial_1^3 u_2 \, dx - 3\int \partial_1 u_1 \partial_1^3 u_3 \partial_1^3 u_3 \, dx \\
&= -3\int \partial_1 u_1 \partial_1^3 u_1 \partial_1^3 u_1 \, dx + 3\int \partial_2 u_2 \partial_1^3 u_2 \partial_1^3 u_2 \, dx - 6\int u_3 \partial_1^3 u_2 \partial_3 \partial_1^3 u_2 \, dx \\
&\quad + 3\int \partial_2 u_2 \partial_1^3 u_3 \partial_1^3 u_3 \, dx - 6\int u_3 \partial_1^3 u_3 \partial_3 \partial_1^3 u_3 \, dx \\
&\le C\|\partial_1 u_1\|_{L^\infty} \|\partial_1^3 u_1\|_{L^2}^2 + D_1^2 + D_3 \\
&\quad + C\|\partial_3 \partial_1^3 u\|_{L^2} \|u_3\|_{L^2}^{\frac{1}{4}} \|\partial_1 u_3\|_{L^2}^{\frac{1}{4}} \|\partial_2 u_3\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_2 u_3\|_{L^2}^{\frac{1}{4}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\le C\|u\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right) + D_1^2 + D_3,
\end{aligned} \tag{3.11}$$

where

$$D_1^2 \triangleq 3\int \partial_2 u_2 \left(\partial_1^3 u_2\right)^2 dx, \quad D_3 \triangleq 3\int \partial_2 u_2 \left(\partial_1^3 u_3\right)^2 dx.$$

Now, plugging (3.11) into (3.10), we arrive at

$$H_{23} \le C\|u\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right) + D_1^2 + D_3,$$

which, together with (3.8) and (3.9), gives

$$H_2 \le C\|u\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2\right) + D_1 + D_2 + D_3. \tag{3.12}$$

Here, the quantities D_2, D_3 are the same ones as above and

$$D_1 \triangleq D_1^1 + D_1^2 = 2\int \partial_2 u_2 \left(\partial_1^3 u_2\right)^2 dx.$$

The treatments of D_1, D_2 and D_3 are postponed to the next step. We proceed to estimate H_3, H_4 and H_5 term by term. For H_3, we have

$$\begin{aligned}
H_3 &= 3\sum_{i=1}^3 \int \partial_i b \cdot \nabla \partial_i^2 b \cdot \partial_i^3 u \, dx + 3\sum_{i=1}^3 \int \partial_i^2 b \cdot \nabla \partial_i b \cdot \partial_i^3 u \, dx \\
&\quad + \sum_{i=1}^3 \int \partial_i^3 b \cdot \nabla b \cdot \partial_i^3 u \, dx \triangleq \sum_{j=1}^3 H_{3j}.
\end{aligned} \tag{3.13}$$

To estimate the terms on the right-hand side, we first observe that

$$\|\partial_1 b_1\|_{\dot{H}^k} \le \|\partial_2 b_2\|_{\dot{H}^k} + \|\partial_3 b_3\|_{\dot{H}^k} \le \|\partial_2 b\|_{\dot{H}^k} + \|\partial_3 b\|_{\dot{H}^k}, \quad k = 1, 2, \ldots, \tag{3.14}$$

due to the divergence-free condition $\nabla \cdot b = 0$. This, together with Lemma 2.1, yields that (keeping in mind that $\partial_\nu^k = (\partial_2^k, \partial_3^k)$ with $k = 1, 2, 3$)

$$\begin{aligned}
H_{31} &= 3\int \partial_1 b_1 \partial_1^3 b \cdot \partial_1^3 u \, dx + 3\int \partial_1 b_2 \partial_2 \partial_1^2 b \cdot \partial_1^3 u \, dx \\
&\quad + 3\int \partial_1 b_3 \partial_3 \partial_1^2 b \cdot \partial_1^3 u \, dx + 3\int \partial_\nu b \cdot \nabla \partial_\nu^2 b \cdot \partial_\nu^3 u \, dx \\
&\le C\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_2 b\|_{L^\infty} \|\nabla \partial_2^2 b\|_{L^2} \|\partial_2^3 u\|_{L^2} + C\|\partial_3 b\|_{L^\infty} \|\nabla \partial_3^2 b\|_{L^2} \|\partial_3^3 u\|_{L^2} \\
&\le C\|(u,b)\|_{H^3} \left(\|(\partial_3 b, \partial_3 u)\|_{H^3}^2 + \|\partial_2 b\|_{H^3}^2 \right). \qquad (3.15)
\end{aligned}$$

Analogously to the derivations of (3.9) and (3.10), we deduce

$$\begin{aligned}
H_{32} &\le C\|\partial_1^2 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1^2 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^2 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_2^2 b\|_{L^6} \|\nabla \partial_2 b\|_{L^3} \|\partial_2^3 u\|_{L^2} + C\|\partial_3^2 b\|_{L^6} \|\nabla \partial_3 b\|_{L^3} \|\partial_3^3 u\|_{L^2} \\
&\le C\|(u,b)\|_{H^3} \Big(\|(\partial_3 b, \partial_3 u)\|_{H^3}^2 + \|\partial_2 b\|_{H^3}^2 \Big), \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
H_{33} &\le C\|\partial_1^3 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1^4 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_1^3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_3\|_{L^2}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1 b\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u\|_{L^2}^{\frac{1}{2}} \\
&\quad + C\|\partial_2^3 b\|_{L^2} \|\nabla b\|_{L^\infty} \|\partial_2^3 u\|_{L^2} + C\|\partial_3^3 b\|_{L^2} \|\nabla b\|_{L^\infty} \|\partial_3^3 u\|_{L^2} \\
&\le C\|(u,b)\|_{H^3} \left(\|(\partial_3 b, \partial_3 u)\|_{H^3}^2 + \|\partial_2 b\|_{H^3}^2 \right). \qquad (3.17)
\end{aligned}$$

Thus, putting (3.15), (3.16) and (3.17) into (3.13), we get

$$H_3 \le C\|(u,b)\|_{H^3} \left(\|(\partial_3 b, \partial_3 u)\|_{H^3}^2 + \|\partial_2 b\|_{H^3}^2 \right). \qquad (3.18)$$

In order to estimate H_4, we rewrite it in the form:

$$H_4 = -3\sum_{i=1}^{3} \int \partial_i u \cdot \nabla \partial_i^2 b \cdot \partial_i^3 b \, dx - 3\sum_{i=1}^{3} \int \partial_i^2 u \cdot \nabla \partial_i b \cdot \partial_i^3 b \, dx$$

$$-\sum_{i=1}^{3}\int \partial_i^3 u\cdot\nabla b\cdot\partial_i^3 b\,dx \triangleq \sum_{j=1}^{3} H_{4j}. \tag{3.19}$$

Using (3.6) and Lemma 2.1, we find

$$\begin{aligned}
H_{41} &= -3\int \partial_1 u_1\partial_1^3 b\cdot\partial_1^3 b\,dx - 3\int \partial_1 u_2\partial_2\partial_1^2 b\cdot\partial_1^3 b\,dx\\
&\quad - 3\int \partial_1 u_3\partial_3\partial_1^2 b\cdot\partial_1^3 b\,dx - 3\int \partial_\nu u\cdot\nabla\partial_\nu^2 b\cdot\partial_\nu^3 b\,dx\\
&\le C\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_1^2 u_1\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_3\partial_1 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1 u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1 u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_2 u\|_{L^\infty}\|\nabla\partial_2^2 b\|_{L^2}\|\partial_2^3 b\|_{L^2} + C\|\partial_3 u\|_{L^\infty}\|\nabla\partial_3^2 b\|_{L^2}\|\partial_3^3 b\|_{L^2}\\
&\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned} \tag{3.20}$$

In a similar manner,

$$\begin{aligned}
H_{42} &\le C\|\partial_1^2 u_1\|_{L^2}^{\frac12}\|\partial_1^3 u_1\|_{L^2}^{\frac12}\|\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1^2 u_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 u_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1^2 u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_2^2 u\|_{L^6}\|\nabla\partial_2 b\|_{L^3}\|\partial_2^3 b\|_{L^2} + C\|\partial_3^2 u\|_{L^6}\|\nabla\partial_3 b\|_{L^3}\|\partial_3^3 b\|_{L^2}\\
&\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right),
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
H_{43} &\le C\|\partial_1^3 u_1\|_{L^2}\|\partial_1 b\|_{L^2}^{\frac14}\|\partial_1^2 b\|_{L^2}^{\frac14}\|\partial_2\partial_1 b\|_{L^2}^{\frac14}\|\partial_2\partial_1^2 b\|_{L^2}^{\frac14}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1^3 u_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u_2\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2}^{\frac12}\|\partial_2\partial_1 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1^3 u_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u_3\|_{L^2}^{\frac12}\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_2^3 u\|_{L^2}\|\nabla b\|_{L^\infty}\|\partial_2^3 b\|_{L^2} + C\|\partial_3^3 u\|_{L^2}\|\nabla b\|_{L^\infty}\|\partial_3^3 b\|_{L^2}\\
&\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right).
\end{aligned} \tag{3.22}$$

Thus, substituting (3.20), (3.21) and (3.22) into (3.19) gives

$$H_4 \le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right). \tag{3.23}$$

Next, we estimate the term H_5, which can be written as follows:

$$H_5 = 3\sum_{i=1}^{3}\int \partial_i b\cdot\nabla\partial_i^2 u\cdot\partial_i^3 b\,dx + 3\sum_{i=1}^{3}\int \partial_i^2 b\cdot\nabla\partial_i u\cdot\partial_i^3 b\,dx + \sum_{i=1}^{3}\int \partial_i^3 b\cdot\nabla u\cdot\partial_i^3 b\,dx \triangleq \sum_{j=1}^{3} H_{5j}. \tag{3.24}$$

Similarly to the proof of (3.15), we deduce from (3.14) and Lemma 2.1 that

$$\begin{aligned} H_{51} &= 3\int \partial_1 b_1\partial_1^3 u\cdot\partial_1^3 b\,dx + 3\int \partial_1 b_2\partial_2\partial_1^2 u\cdot\partial_1^3 b\,dx \\ &\quad + 3\int \partial_1 b_3\partial_3\partial_1^2 u\cdot\partial_1^3 b\,dx + 3\int \partial_\nu b\cdot\nabla\partial_\nu^2 u\cdot\partial_\nu^3 b\,dx \\ &\le C\|\partial_1 b_1\|_{L^2}^{\frac12}\|\partial_1^2 b_1\|_{L^2}^{\frac12}\|\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_1 b_2\|_{L^2}^{\frac14}\|\partial_1^2 b_2\|_{L^2}^{\frac14}\|\partial_1\partial_2 b_2\|_{L^2}^{\frac14}\|\partial_2\partial_1^2 b_2\|_{L^2}^{\frac14}\|\partial_2\partial_1^2 u\|_{L^2}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_1 b_3\|_{L^2}^{\frac12}\|\partial_2\partial_1 b_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_2 b\|_{L^\infty}\|\nabla\partial_2^2 u\|_{L^2}\|\partial_2^3 b\|_{L^2} + C\|\partial_3 b\|_{L^\infty}\|\nabla\partial_3^2 u\|_{L^2}\|\partial_3^3 b\|_{L^2} \\ &\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} H_{52} &\le C\|\partial_1^2 b_1\|_{L^2}^{\frac12}\|\partial_1^3 b_1\|_{L^2}^{\frac12}\|\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_1^2 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1 u\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_1^2 b_3\|_{L^2}^{\frac12}\|\partial_2\partial_1^2 b_3\|_{L^2}^{\frac12}\|\partial_3\partial_1 u\|_{L^2}^{\frac12}\|\partial_3\partial_1^2 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12} \\ &\quad + C\|\partial_2^2 b\|_{L^6}\|\nabla\partial_2 u\|_{L^3}\|\partial_2^3 b\|_{L^2} + C\|\partial_3^2 b\|_{L^6}\|\nabla\partial_3 u\|_{L^3}\|\partial_3^3 b\|_{L^2} \\ &\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right). \end{aligned} \tag{3.26}$$

Let $u_h \triangleq (u_1,u_2)$ and $b_h \triangleq (b_1,b_2)$. By direct calculations, we have

$$\begin{aligned} H_{53} &= \int \partial_1^3 b_1\partial_1 u\cdot\partial_1^3 b\,dx + \int \partial_1^3 b_2\partial_2 u_1\partial_1^3 b_1\,dx + 2\int \left(\partial_1^3 b_2\right)^2\partial_2 u_2\,dx \\ &\quad - \int \left(\partial_1^3 b_2\right)^2\partial_2 u_2\,dx + 2\int \partial_1^3 b_2\partial_2 u_3\partial_1^3 b_3\,dx - \int \partial_1^3 b_2\partial_2 u_3\partial_1^3 b_3\,dx \\ &\quad + \int \partial_1^3 b_3\partial_3 u_h\cdot\partial_1^3 b_h\,dx - \int \partial_1^3 b_3\partial_1 u_1\partial_1^3 b_3\,dx - 4\int \partial_1^3 b_3\partial_2 u_2\partial_1^3 b_3\,dx \\ &\quad + 3\int \partial_1^3 b_3\partial_2 u_2\partial_1^3 b_3\,dx + \int \partial_2^3 b\cdot\nabla u\cdot\partial_2^3 b\,dx + \int \partial_3^3 b\cdot\nabla u\cdot\partial_3^3 b\,dx, \end{aligned}$$

and hence, it follows from (3.6), (3.14) and Lemma 2.1 that

$$\begin{aligned}
H_{53} \le\ & C\|\partial_1^3 b_1\|_{L^2}^{\frac12}\|\partial_1^4 b_1\|_{L^2}^{\frac12}\|\partial_1 u\|_{L^2}^{\frac12}\|\partial_3\partial_1 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b\|_{L^2}^{\frac12}\\
&+ C\|\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_2\partial_1 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\\
&+ C\|\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_2\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_3 u\|_{L^2}^{\frac12}\|\partial_3\partial_1 u\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b\|_{L^2}^{\frac12}\\
&+ C\|\partial_1^3 b_3\|_{L^2}\|\partial_2\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_1 u_1\|_{L^2}^{\frac12}\|\partial_1^2 u_1\|_{L^2}^{\frac12}\\
&+ C\|\partial_1^3 b_3\|_{L^2}\|\partial_2\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b_3\|_{L^2}^{\frac12}\|\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_2 u_2\|_{L^2}^{\frac12}\\
&+ C\|\nabla u\|_{L^\infty}\|\partial_2^3 b\|_{L^2}^2 + C\|\nabla u\|_{L^\infty}\|\partial_3^3 b\|_{L^2}^2 + E_1 + E_2 + E_3\\
\le\ & C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right) + E_1 + E_2 + E_3,
\end{aligned} \tag{3.27}$$

where

$$E_1 \triangleq 2\int \left(\partial_1^3 b_2\right)^2 \partial_2 u_2\ dx, \quad E_2 \triangleq -\int \partial_1^3 b_2 \partial_2 u_3 \partial_1^3 b_3\ dx,$$
$$E_3 \triangleq 3\int \left(\partial_1^3 b_3\right)^2 \partial_2 u_2\ dx.$$

Combining (3.25), (3.26), (3.27) with (3.24) gives

$$H_5 \le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right) + E_1 + E_2 + E_3. \tag{3.28}$$

Therefore, inserting (3.4), (3.12), (3.18), (3.23) and (3.28) into (3.3), we immediately obtain the following preliminary bound of $\dot{H}^3$-norms:

$$\begin{aligned}
&\frac12\frac{d}{dt}\|(\nabla^3 u,\nabla^3 b)\|_{L^2}^2 + \mu\|\partial_3\nabla^3 u\|_{L^2}^2 + \eta\left(\|\partial_2\nabla^3 b\|_{L^2}^2 + \|\partial_3\nabla^3 b\|_{L^2}^2\right)\\
&\quad\le C\|(u,b)\|_{H^3}\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2 + \|\partial_2 u\|_{H^2}^2\right) + \sum_{i=1}^3 D_i + \sum_{i=1}^3 E_i.
\end{aligned} \tag{3.29}$$

Step III. The estimates of D_1, D_2 and D_3

Clearly, to close the a priori estimates stated in (3.29), we still need to deal with D_i and E_i with $i = 1,2,3$, based on the special structure of (1.2).

Step III-1. The estimates of D_1 and D_3

We start with the estimate of D_1. First, it follows from (1.2) that

$$\partial_2 u_2 = \partial_t b_2 - \eta\partial_2^2 b_2 - \eta\partial_3^2 b_2 + u\cdot\nabla b_2 - b\cdot\nabla u_2, \tag{3.30}$$
$$\partial_2 u_3 = \partial_t b_3 - \eta\partial_2^2 b_3 - \eta\partial_3^2 b_3 + u\cdot\nabla b_3 - b\cdot\nabla u_3, \tag{3.31}$$

$$\partial_2 b_2 = \partial_t u_2 - \mu\partial_3^2 u_2 + u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p, \tag{3.32}$$

$$\partial_2 b_3 = \partial_t u_3 - \mu\partial_3^2 u_3 + u\cdot\nabla u_3 - b\cdot\nabla b_3 + \partial_3 p. \tag{3.33}$$

Plugging (3.30)–(3.31) into (3.32)–(3.33) gives

$$\begin{cases} \partial_2 u_2 = \partial_t(b_2 - \eta\partial_2 u_2) - \eta\partial_3^2 b_2 + \mu\eta\partial_2\partial_3^2 u_2 \\ \qquad\qquad -\eta\partial_2\left(u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p\right) + u\cdot\nabla b_2 - b\cdot\nabla u_2, \\ \partial_2 u_3 = \partial_t(b_3 - \eta\partial_2 u_3) - \eta\partial_3^2 b_3 + \mu\eta\partial_2\partial_3^2 u_3 \\ \qquad\qquad -\eta\partial_2\left(u\cdot\nabla u_3 - b\cdot\nabla b_3 + \partial_3 p\right) + u\cdot\nabla b_3 - b\cdot\nabla u_3. \end{cases} \tag{3.34}$$

Recalling the definition of D_1, by virtue of $(3.34)_1$ we obtain after integrating by parts that

$$\begin{aligned} D_1 &= 2\int \left(\partial_1^3 u_2\right)^2 \partial_t(b_2 - \eta\partial_2 u_2)\, dx - 2\eta\int \left(\partial_1^3 u_2\right)^2 \partial_3^2 b_2\, dx \\ &\quad + 2\mu\eta\int \left(\partial_1^3 u_2\right)^2 \partial_2\partial_3^2 u_2\, dx + 2\int \left(\partial_1^3 u_2\right)^2 (u\cdot\nabla b_2 - b\cdot\nabla u_2)\, dx \\ &\quad - 2\eta\int \left(\partial_1^3 u_2\right)^2 \partial_2\left(u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p\right) dx \\ &= 2\frac{d}{dt}\int \left(\partial_1^3 u_2\right)^2 (b_2 - \eta\partial_2 u_2)\, dx - 4\int \partial_1^3 u_2\partial_1^3\partial_t u_2(b_2 - \eta\partial_2 u_2)\, dx \\ &\quad + 4\eta\int \partial_3\partial_1^3 u_2\partial_1^3 u_2\partial_3 b_2\, dx - 4\mu\eta\int \partial_3\partial_1^3 u_2\partial_1^3 u_2\partial_3\partial_2 u_2\, dx \\ &\quad - 2\eta\int \left(\partial_1^3 u_2\right)^2 \partial_2\left(u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p\right) dx \\ &\quad + 2\int \left(\partial_1^3 u_2\right)^2 u\cdot\nabla b_2\, dx - 2\int \left(\partial_1^3 u_2\right)^2 b\cdot\nabla u_2\, dx \\ &\triangleq 2\frac{d}{dt}\int \left(\partial_1^3 u_2\right)^2 (b_2 - \eta\partial_2 u_2)\, dx + \sum_{j=1}^{6} D_{1j}. \end{aligned} \tag{3.35}$$

The terms on the right-hand side of (3.35) will be treated one by one. First, thanks to (3.32), it holds that

$$\begin{aligned} D_{11} &= -4\int \partial_1^3 u_2\partial_1^3\partial_2 b_2(b_2 - \eta\partial_2 u_2)\, dx + 4\mu\int \partial_3\partial_1^3 u_2\partial_3\partial_1^3 u_2(b_2 - \eta\partial_2 u_2)\, dx \\ &\quad + 4\mu\int \partial_1^3 u_2\partial_3\partial_1^3 u_2\partial_3 b_2\, dx - 4\mu\eta\int \partial_1^3 u_2\partial_3\partial_1^3 u_2\partial_3\partial_2 u_2\, dx \\ &\quad + 4\int \partial_1^3(u\cdot\nabla u_2 - b\cdot\nabla b_2 + \partial_2 p)\partial_1^3 u_2(b_2 - \eta\partial_2 u_2)\, dx \triangleq \sum_{j=1}^{5} D_{11j}, \end{aligned} \tag{3.36}$$

where the first four terms can be bounded as follows, using Lemma 2.1, Hölder's and Sobolev's inequalities.

$$\begin{aligned}\sum_{j=1}^{4} D_{11j} \le\ & C\|\partial_1^3\partial_2 b_2\|_{L^2}\|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_2 b_2\|_{H^1}^{\frac{1}{2}}\\ &+ C\|\partial_1^3 u_2\|_{L^2}\|\partial_2\partial_1^3 b_2\|_{L^2}\|\partial_2 u_2\|_{L^\infty}\\ &+ C\|\partial_3\partial_1^3 u_2\|_{L^2}\|\partial_3\partial_1^3 u_2\|_{L^2}\left(\|b_2\|_{L^\infty}+\|\partial_2 u_2\|_{L^\infty}\right)\\ &+ C\|\partial_1^3 u_2\|_{L^2}\|\partial_3\partial_1^3 u_2\|_{L^2}\left(\|\partial_3 b_2\|_{L^\infty}+\|\partial_3\partial_2 u_2\|_{L^\infty}\right)\\ &\le C\|(u,b)\|_{H^3}(\|\partial_2 u\|_{H^2}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 b\|_{H^3}^2).\end{aligned} \tag{3.37}$$

To deal with the term associated with the pressure, we observe that

$$p=(-\Delta)^{-1}\nabla\cdot(u\cdot\nabla u-b\cdot\nabla b)=(-\Delta)^{-1}(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i), \tag{3.38}$$

and hence

$$\begin{aligned}D_{115}=\ & 4\int(\partial_1^3 u\cdot\nabla u_2+3\partial_1^2 u\cdot\nabla\partial_1 u_2+3\partial_1 u\cdot\nabla\partial_1^2 u_2)\partial_1^3 u_2(b_2-\eta\partial_2 u_2)\,dx\\ &-4\int(\partial_1^3 b\cdot\nabla b_2+3\partial_1^2 b\cdot\nabla\partial_1 b_2+3\partial_1 b\cdot\nabla\partial_1^2 b_2)\partial_1^3 u_2(b_2-\eta\partial_2 u_2)\,dx\\ &+2\int u\cdot\nabla(\partial_1^3 u_2)^2(b_2-\eta\partial_2 u_2)\,dx-4\int b\cdot\nabla\partial_1^3 b_2\partial_1^3 u_2(b_2-\eta\partial_2 u_2)\,dx\\ &+4\int\partial_2\partial_1^3(-\Delta)^{-1}(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i)\partial_1^3 u_2(b_2-\eta\partial_2 u_2)\,dx\triangleq\sum_{j=1}^{5}F_j.\end{aligned} \tag{3.39}$$

Using (3.6), Lemma 2.2 and Sobolev's embedding inequality, one has

$$\begin{aligned}F_1=\ & 4\int\left(\partial_1^3 u\cdot\nabla u_2+3\partial_1 u\cdot\nabla\partial_1^2 u_2\right)\partial_1^3 u_2(b_2-\eta\partial_2 u_2)\ dx\\ &+12\int(\partial_1^2 u_1\partial_1^2 u_2+\partial_1^2 u_2\partial_2\partial_1 u_2+\partial_1^2 u_3\partial_3\partial_1 u_2)\partial_1^3 u_2\,(b_2-\eta\partial_2 u_2)\,dx\\ \le\ & C\left(\|(\partial_1^3 u,\partial_1^3 u_2)\|_{L^2}^2+\|(\nabla u_2,b_2,\partial_2 u_2)\|_{H^1}^2\right)\\ &\times\left(\|(\partial_3\partial_1^3 u,\partial_3\partial_1^3 u_2)\|_{L^2}^2+\|(\partial_2\nabla u_2,\partial_2 b_2,\partial_2^2 u_2)\|_{H^1}^2\right)\\ &+C\left(\|(\nabla\partial_1^2 u_2,\partial_1^3 u_2)\|_{L^2}^2+\|(\partial_1 u,b_2,\partial_2 u_2)\|_{H^1}^2\right)\\ &\times\left(\|(\partial_3\nabla\partial_1^2 u_2,\partial_3\partial_1^3 u_2)\|_{L^2}^2+\|(\partial_2\partial_1 u,\partial_2 b_2,\partial_2^2 u_2)\|_{H^1}^2\right)\\ &+C\|(b_2,\partial_2 u_2)\|_{L^\infty}\Big(\|\partial_1^2 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 u_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}}\\ &+\|\partial_1^2 u_\nu\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^2 u_\nu\|_{L^2}^{\frac{1}{2}}\|\partial_\nu\partial_1 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_\nu\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}}\Big)\|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\\ \le\ & C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 b\|_{H^3}^2\right),\end{aligned} \tag{3.40}$$

and analogously, by (3.14) one gets

$$F_2 \le C\|(u,b)\|_{H^3}^2 \left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_2 b\|_{H^3}^2 + \|\partial_3 b\|_{H^3}^2\right). \tag{3.41}$$

Since the Riesz operator $\partial_i(-\Delta)^{-\frac{1}{2}}$ with $i = 1,2,3$ is bounded in L^r for any $1 < r < \infty$, we see that

$$\begin{aligned}
&\|\partial_2\partial_1^3(-\Delta)^{-1}(\partial_i u_j\partial_j u_i - \partial_i b_j\partial_j b_i)\|_{L^2}\\
&\quad\le C\|\partial_2\partial_1\left(\partial_i u_j\partial_j u_i - \partial_i b_j\partial_j b_i\right)\|_{L^2}\\
&\quad\le C\|\partial_2\partial_1\nabla u\|_{L^2}\|\nabla u\|_{L^\infty} + C\|\partial_2\nabla u\|_{L^6}\|\partial_1\nabla u\|_{L^3}\\
&\qquad + C\|\partial_2\partial_1\nabla b\|_{L^2}\|\nabla b\|_{L^\infty} + C\|\partial_2\nabla b\|_{L^6}\|\partial_1\nabla b\|_{L^3}\\
&\quad\le C\|(u,b)\|_{H^3}\left(\|\partial_2 u\|_{H^2} + \|\partial_2 b\|_{H^2}\right),
\end{aligned} \tag{3.42}$$

from which, (2.5) and Lemma 2.1 it follows that

$$\begin{aligned}
F_5 &= 4\int \partial_2\partial_1^3(-\Delta)^{-1}(\partial_i u_j\partial_j u_i - \partial_i b_j\partial_j b_i)\partial_1^3 u_2(b_2 - \eta\partial_2 u_2)\,dx\\
&\le C\|\partial_2\partial_1^3(-\Delta)^{-1}(\partial_i u_j\partial_j u_i - \partial_i b_j\partial_j b_i)\|_{L^2}\\
&\quad\times\|\partial_1^3 u_2\|_{L^2_{x_1}L^2_{x_2}L^\infty_{x_3}}\left(\|b_2\|_{L^\infty_{x_1}L^\infty_{x_2}L^2_{x_3}} + \|\partial_2 u_2\|_{L^\infty_{x_1}L^\infty_{x_2}L^2_{x_3}}\right)\\
&\le C\|\partial_2\partial_1\left(\partial_i u_j\partial_j u_i - \partial_i b_j\partial_j b_i\right)\|_{L^2}\\
&\quad\times\|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}}\left(\|b_2\|_{H^1}^{\frac{1}{2}}\|\partial_2 b_2\|_{H^1}^{\frac{1}{2}} + \|\partial_2 u_2\|_{H^1}^{\frac{1}{2}}\|\partial_2^2 u_2\|_{H^1}^{\frac{1}{2}}\right)\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_2 b\|_{H^2}^2\right).
\end{aligned} \tag{3.43}$$

Thus, inserting (3.40), (3.41) and (3.43) into (3.39) shows

$$D_{115} \le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2 + \|\partial_3 u\|_{H^3}^2 + \|\partial_2 b\|_{H^2}^2\right) + F_3 + F_4,$$

which, combined with (3.37) and (3.36), yields

$$D_{11} \le C\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u,\partial_2 b)\|_{H^3}^2\right) + F_3 + F_4. \tag{3.44}$$

For D_{12}, D_{13} and D_{16}, we infer from Lemmas 2.1 and 2.2 that

$$\begin{aligned}
&D_{12} + D_{13} + D_{16}\\
&\quad= 4\eta\int \partial_3\partial_1^3 u_2\partial_1^3 u_2\partial_3 b_2\,dx - 4\mu\eta\int \partial_3\partial_1^3 u_2\partial_1^3 u_2\partial_3\partial_2 u_2\,dx\\
&\qquad - 2\int \partial_1^3 u_2\partial_1^3 u_2(b\cdot\nabla u_2)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le C\|\partial_3\partial_1^3 u_2\|_{L^2}\|\partial_1^3 u_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 u_2\|_{L^2}^{\frac12}\\
&\quad\times\left(\|\partial_3 b_2\|_{H^1}^{\frac12}\|\partial_2\partial_3 b_2\|_{H^1}^{\frac12}+\|\partial_3\partial_2 u_2\|_{H^1}^{\frac12}\|\partial_3\partial_2^2 u_2\|_{H^1}^{\frac12}\right)\\
&\quad+C\left(\|\partial_1^3 u_2\|_{L^2}^2+\|(b,\nabla u_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 u_2\|_{L^2}^2+\|(\partial_2 b,\partial_2\nabla u_2)\|_{H^1}^2\right)\\
&\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \qquad (3.45)
\end{aligned}
$$

In order to estimate D_{14}, we first deal with the pressure term. Indeed, similarly to the derivation of (3.42), it is easily seen from (3.38) that

$$
\begin{aligned}
\|\partial_2^2 p\|_{L^\infty_{x_1,x_2}L^2_{x_3}}&\le C\|\partial_2^2 p\|_{L^2}^{\frac14}\|\partial_2^3 p\|_{L^2}^{\frac14}\|\partial_1\partial_2^2 p\|_{L^2}^{\frac14}\|\partial_1\partial_2^3 p\|_{L^2}^{\frac14}\\
&\le C\|\partial_2(u\cdot\nabla u-b\cdot\nabla b)\|_{L^2}^{\frac14}\|\partial_2(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i)\|_{L^2}^{\frac12}\\
&\quad\times\|\partial_2^2(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i)\|_{L^2}^{\frac14}. \qquad (3.46)
\end{aligned}
$$

By (2.1), (2.2) and (2.4), we have

$$
\begin{aligned}
&\|\partial_2(u\cdot\nabla u)\|_{L^2}^{\frac14}\|\partial_2(\partial_i u_j\partial_j u_i)\|_{L^2}^{\frac12}\|\partial_2^2(\partial_i u_j\partial_j u_i)\|_{L^2}^{\frac14}\\
&\le C\left(\|\partial_2 u\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\nabla u\|_{L^\infty_{x_1,x_2}L^2_{x_3}}+\|\partial_2\nabla u\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|u\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\right)^{\frac14}\\
&\quad\times C\left(\|\partial_2\nabla u\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\nabla u\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\right)^{\frac12}\\
&\quad\times C\left(\|\partial_2^2\nabla u\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\nabla u\|_{L^\infty_{x_1,x_2}L^2_{x_3}}+\|\partial_2\nabla u\|_{L^6}\|\partial_2\nabla u\|_{L^3}\right)^{\frac14}\\
&\le C\Big(\|\partial_2 u\|_{L^2}^{\frac12}\|\partial_3\partial_2 u\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}^{\frac12}\|\partial_2\nabla u\|_{H^1}^{\frac12}\\
&\qquad+\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_3\partial_2\nabla u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12}\|\partial_2 u\|_{H^1}^{\frac12}\Big)^{\frac14}\\
&\quad\times C\left(\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\partial_3\partial_2\nabla u\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}^{\frac12}\|\partial_2\nabla u\|_{H^1}^{\frac12}\right)^{\frac12}\\
&\quad\times C\left(\|\partial_2^2\nabla u\|_{L^2}^{\frac12}\|\partial_3\partial_2^2\nabla u\|_{L^2}^{\frac12}\|\nabla u\|_{H^1}^{\frac12}\|\partial_2\nabla u\|_{H^1}^{\frac12}+\|\partial_2\nabla u\|_{H^1}^2\right)^{\frac14}\\
&\le C\|u\|_{H^3}^{\frac12}\|\partial_2 u\|_{H^2}^{\frac{11}{8}}\|\partial_3 u\|_{H^3}^{\frac18}. \qquad (3.47)
\end{aligned}
$$

The other terms in (3.46) admit similar bounds. Hence, we deduce

$$
\|\partial_2^2 p\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\le C\|(u,b)\|_{H^3}^{\frac12}\|(\partial_2 u,\partial_2 b)\|_{H^2}^{\frac{11}{8}}\|(\partial_3 u,\partial_3 b)\|_{H^3}^{\frac18}. \qquad (3.48)
$$

In view of (3.48), Lemmas 2.1 and 2.2, we find

$$
D_{14}=-2\eta\int\left(\partial_1^3 u_2\right)^2\left(\partial_2 u\cdot\nabla u_2-\partial_2 b\cdot\nabla b_2-b\cdot\nabla\partial_2 b_2\right)\,dx
$$

$$
\begin{aligned}
&- 2\eta \int \left(\partial_1^3 u_2\right)^2 \partial_2^2 p \, dx - 2\eta \int \left(\partial_1^3 u_2\right)^2 u \cdot \nabla \partial_2 u_2 \, dx \\
&\le C \left(\|\partial_1^3 u_2\|_{L^2}^2 + \|(\partial_2 u, \nabla u_2, \partial_2 b, \nabla b_2, b, \nabla \partial_2 b_2)\|_{H^1}^2 \right) \\
&\quad \times \left(\|\partial_3 \partial_1^3 u_2\|_{L^2}^2 + \|(\partial_2^2 u, \partial_2 \nabla u_2, \partial_2^2 b, \partial_2 \nabla b_2, \partial_2 b, \partial_2^2 \nabla b_2)\|_{H^1}^2 \right) \\
&\quad + C \|\partial_1^3 u_2\|_{L^2} \|\partial_1^3 u_2\|_{L^2_{x_1,x_2} L^\infty_{x_3}} \|\partial_2^2 p\|_{L^\infty_{x_1,x_2} L^2_{x_3}} + \Xi_1 \\
&\le C \|(u,b)\|_{H^3}^2 \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \right) + \Xi_1 \\
&\quad + C \|\partial_1^3 u_2\|_{L^2}^{\frac{3}{2}} \|\partial_3 \partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|(u,b)\|_{H^3}^{\frac{1}{2}} \|(\partial_2 u, \partial_2 b)\|_{H^2}^{\frac{11}{8}} \|(\partial_3 u, \partial_3 b)\|_{H^3}^{\frac{1}{8}} \\
&\quad \le C \|(u,b)\|_{H^3}^2 \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \right) + \Xi_1,
\end{aligned} \tag{3.49}
$$

where

$$
\Xi_1 \triangleq -2\eta \int \left(\partial_1^3 u_2\right)^2 u \cdot \nabla \partial_2 u_2 \, dx.
$$

A key and amazing consequence of the above estimates is that

$$
\begin{aligned}
D_{15} + F_3 + \Xi_1 = 2 \int \left(\partial_1^3 u_2\right)^2 u \cdot \nabla b_2 \, dx + 2 \int u \cdot \nabla (\partial_1^3 u_2)^2 (b_2 - \eta \partial_2 u_2) \, dx \\
- 2\eta \int \left(\partial_1^3 u_2\right)^2 u \cdot \nabla \partial_2 u_2 \, dx = 0,
\end{aligned} \tag{3.50}
$$

and consequently, it follows from (3.35), (3.44), (3.45), (3.49) and (3.50) that

$$
\begin{aligned}
D_1 \le 2 \frac{d}{dt} \int \left(\partial_1^3 u_2\right)^2 (b_2 - \eta \partial_2 u_2) \, dx \\
+ C \|(u,b)\|_{H^3}^2 \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \right) + F_4,
\end{aligned} \tag{3.51}
$$

where

$$
F_4 = -4 \int b \cdot \nabla \partial_1^3 b_2 \partial_1^3 u_2 (b_2 - \eta \partial_2 u_2) \, dx.
$$

It remains to estimate F_4, which will be tackled with the term E_1 in H_{53}. To do this, we use $(3.34)_1$ to get that

$$
\begin{aligned}
E_1 &= 2 \int \left(\partial_1^3 b_2\right)^2 \partial_t (b_2 - \eta \partial_2 u_2) \, dx - 2\eta \int \left(\partial_1^3 b_2\right)^2 \partial_3^2 b_2 \, dx \\
&\quad + 2\mu\eta \int \left(\partial_1^3 b_2\right)^2 \partial_2 \partial_3^2 u_2 \, dx + 2 \int \left(\partial_1^3 b_2\right)^2 (u \cdot \nabla b_2 - b \cdot \nabla u_2) \, dx \\
&\quad - 2\eta \int \left(\partial_1^3 b_2\right)^2 \partial_2 (u \cdot \nabla u_2 - b \cdot \nabla b_2 + \partial_2 p) \, dx \\
&= 2 \frac{d}{dt} \int \left(\partial_1^3 b_2\right)^2 (b_2 - \eta \partial_2 u_2) \, dx - 4 \int \partial_1^3 \partial_t b_2 \partial_1^3 b_2 (b_2 - \eta \partial_2 u_2) \, dx
\end{aligned}
$$

$$
\begin{aligned}
&- 2\eta \int \left(\partial_1^3 b_2\right)^2 \partial_3^2 b_2 \, dx + 2\mu\eta \int \left(\partial_1^3 b_2\right)^2 \partial_2 \partial_3^2 u_2 \, dx \\
&- 2\eta \int \left(\partial_1^3 b_2\right)^2 \partial_2 \left(u \cdot \nabla u_2 - b \cdot \nabla b_2 + \partial_2 p\right) dx \\
&+ 2 \int \left(\partial_1^3 b_2\right)^2 \left(u \cdot \nabla b_2 - b \cdot \nabla u_2\right) dx \\
&\triangleq 2\frac{d}{dt} \int \left(\partial_1^3 b_2\right)^2 \left(b_2 - \eta \partial_2 u_2\right) \, dx + \sum_{i=1}^{5} E_{1i}.
\end{aligned} \tag{3.52}
$$

In view of (3.30), we have

$$
\begin{aligned}
E_{11} = &-4 \int \partial_1^3 \left(\partial_2 u_2 + \eta \partial_2^2 b_2 + \eta \partial_3^2 b_2\right) \partial_1^3 b_2 \left(b_2 - \eta \partial_2 u_2\right) dx \\
&+ 4 \int \partial_1^3 \left(u \cdot \nabla b_2\right) \partial_1^3 b_2 \left(b_2 - \eta \partial_2 u_2\right) dx \\
&- 4 \int \partial_1^3 \left(b \cdot \nabla u_2\right) \partial_1^3 b_2 \left(b_2 - \eta \partial_2 u_2\right) dx \triangleq \sum_{i=1}^{3} E_{11i}.
\end{aligned} \tag{3.53}
$$

Based upon integration by parts, we have

$$
\begin{aligned}
E_{111} = &\, 4 \int \partial_1^3 \left(u_2 + \eta \partial_2 b_2\right) \partial_2 \partial_1^3 b_2 \left(b_2 - \eta \partial_2 u_2\right) dx \\
&+ 4 \int \partial_1^3 \left(u_2 + \eta \partial_2 b_2\right) \partial_1^3 b_2 \left(\partial_2 b_2 - \eta \partial_2^2 u_2\right) dx \\
&+ 4\eta \int \left(\partial_3 \partial_1^3 b_2\right)^2 \left(b_2 - \eta \partial_2 u_2\right) dx \\
&+ 4\eta \int \partial_1^3 \partial_3 b_2 \partial_1^3 b_2 \left(\partial_3 b_2 - \eta \partial_3 \partial_2 u_2\right) dx,
\end{aligned}
$$

so that, by Lemma 2.1 we deduce

$$
\begin{aligned}
E_{111} \le &\, C \|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_2\|_{L^2} \| \left(b_2, \partial_2 u_2\right) \|_{H^1}^{\frac{1}{2}} \|\partial_2 \left(b_2, \partial_2 u_2\right) \|_{H^1}^{\frac{1}{2}} \\
&+ C \|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 b_2\|_{L^2}^{\frac{1}{2}} \\
&+ C \|\partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_1^3 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 u_2\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^2 u_2\|_{L^2}^{\frac{1}{2}} \\
&+ C \|\partial_2 \partial_1^3 b_2\|_{L^2} \|\partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 b_2\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 u_2\|_{H^1}^{\frac{1}{2}} \|\partial_3 \partial_2^2 u_2\|_{H^1}^{\frac{1}{2}} \\
&+ C \left(\|\partial_3 \partial_1^3 b_2\|_{L^2}^2 + \|\partial_2 \partial_1^3 b_2\|_{L^2}^2\right) \| \left(b_2, \partial_2 u_2\right) \|_{L^\infty} \\
&+ C \left(\|\partial_2 \partial_1^3 b_2\|_{L^2} + \|\partial_3 \partial_1^3 b_2\|_{L^2}\right) \|\partial_1^3 b_2\|_{L^2} \| \left(\partial_2 b_2, \partial_3 b_2, \partial_3 \partial_2 u_2\right) \|_{L^\infty} \\
\le &\, C \|(u, b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right).
\end{aligned} \tag{3.54}
$$

By straightforward calculation, we have

$$\begin{aligned}
&E_{112}+E_{113}\\
&\quad=4\int(\partial_1^3u\cdot\nabla b_2+3\partial_1^2u\cdot\nabla\partial_1b_2+3\partial_1u\cdot\nabla\partial_1^2b_2)\partial_1^3b_2(b_2-\eta\partial_2u_2)\,dx\\
&\qquad-4\int(\partial_1^3b\cdot\nabla u_2+3\partial_1^2b\cdot\nabla\partial_1u_2+3\partial_1b\cdot\nabla\partial_1^2u_2)\partial_1^3b_2(b_2-\eta\partial_2u_2)\,dx\\
&\qquad-2\int(\partial_1^3b_2)^2u\cdot\nabla(b_2-\eta\partial_2u_2)\,dx-4\int b\cdot\nabla\partial_1^3u_2\partial_1^3b_2\,(b_2-\eta\partial_2u_2)\,dx\\
&\quad\triangleq\sum_{j=1}^{4}R_j.
\end{aligned}\tag{3.55}$$

In terms of Lemmas 2.1 and 2.2, we obtain in a similar manner as the derivation of (3.40) that

$$\begin{aligned}
R_1&\le C\left(\|(\partial_1^3u,\partial_1^3b_2,\partial_1^2\nabla b)\|_{L^2}^2+\|(\nabla b_2,b_2,\partial_2u_2,\partial_1u)\|_{H^1}^2\right)\\
&\qquad\times\left(\|\partial_3(\partial_1^3u,\partial_1^3b_2,\partial_1^2\nabla b)\|_{L^2}^2+\|\partial_2(\nabla b_2,b_2,\partial_2u_2,\partial_1u)\|_{H^1}^2\right)\\
&\quad+C\|\,(b_2,\partial_2u_2)\,\|_{L^\infty}\Big(\|\partial_1^2u_1\|_{L^2}^{\frac12}\|\partial_1^3u_1\|_{L^2}^{\frac12}\|\partial_1^2b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^2b_2\|_{L^2}^{\frac12}\\
&\qquad+\|\partial_1^2u_\nu\|_{L^2}^{\frac12}\|\partial_2\partial_1^2u_\nu\|_{L^2}^{\frac12}\|\partial_\nu\partial_1b_2\|_{L^2}^{\frac12}\|\partial_\nu\partial_1^2b_2\|_{L^2}^{\frac12}\Big)\|\partial_1^3b_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3b_2\|_{L^2}^{\frac12}\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|\partial_3u\|_{H^3}^2+\|\partial_2b\|_{H^3}^2\right).
\end{aligned}\tag{3.56}$$

Analogously,

$$R_2\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|\partial_3u\|_{H^3}^2+\|\partial_2b\|_{H^3}^2+\|\partial_3b\|_{H^3}^2\right),\tag{3.57}$$

and

$$\begin{aligned}
R_3&\le C\left(\|(\partial_1^3b_2,\partial_1^3b_2)\|_{L^2}^2+\|(u,\nabla b_2,\nabla\partial_2u_2)\|_{H^1}^2\right)\\
&\qquad\times\left(\|\partial_2(\partial_1^3b_2,\partial_1^3b_2)\|_{L^2}^2+\|\partial_3(u,\nabla b_2,\nabla\partial_2u_2)\|_{H^1}^2\right)\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|\partial_2b\|_{H^3}^2+\|\partial_3b\|_{H^3}^2\right).
\end{aligned}\tag{3.58}$$

It follows from (3.53), (3.54), (3.55), (3.56), (3.57) and (3.58), we have

$$E_{11}\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_2b,\partial_3b)\|_{H^3}^2\right)+R_4,\tag{3.59}$$

where

$$R_4=-4\int b\cdot\nabla\partial_1^3u_2\partial_1^3b_2(b_2-\eta\partial_2u_2)dx.$$

For E_{12}, E_{13} and E_{15}, an application of Lemmas 2.1 and 2.2 yields

$$\begin{aligned}
&E_{12}+E_{13}+E_{15}\\
&\le C\|\partial_1^3 b_2\|_{L^2}\|\partial_2\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_3^2 b_2\|_{L^2}^{\frac12}\|\partial_1\partial_3^2 b_2\|_{L^2}^{\frac12}\\
&\quad + C\|\partial_1^3 b_2\|_{L^2}\|\partial_2\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 u_2\|_{L^2}^{\frac12}\|\partial_1\partial_2\partial_3^2 u_2\|_{L^2}^{\frac12}\\
&\quad + C\left(\|\partial_1^3 b_2\|_{L^2}^2+\|(u,\nabla b_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 b_2\|_{L^2}^2+\|\partial_2(u,\nabla b_2)\|_{H^1}^2\right)\\
&\quad + C\left(\|\partial_1^3 b_2\|_{L^2}^2+\|(b,\nabla u_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 b_2\|_{L^2}^2+\|\partial_2(b,\nabla u_2)\|_{H^1}^2\right)\\
&\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \qquad (3.60)
\end{aligned}$$

We proceed to estimate E_{14}. Similarly to the estimate of (3.49), we infer from (3.48) and Lemma 2.2 that

$$\begin{aligned}
E_{14}&\le C\left(\|\partial_1^3 b_2\|_{L^2}^2+\|(\partial_2 u,\nabla u_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 b_2\|_{L^2}^2+\|\partial_2(\partial_2 u,\nabla u_2)\|_{H^1}^2\right)\\
&\quad + C\|\partial_1^3 b_2\|_{L^2}\|\partial_2\partial_1^3 b_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3 b_2\|_{L^2}^{\frac12}\|\nabla\partial_2 u_2\|_{L^2}^{\frac12}\|\partial_1\nabla\partial_2 u_2\|_{L^2}^{\frac12}\|u\|_{L^\infty}\\
&\quad + C\left(\|\partial_1^3 b_2\|_{L^2}^2+\|(\partial_2 b,\nabla b_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 b_2\|_{L^2}^2+\|\partial_2(\partial_2 b,\nabla b_2)\|_{H^1}^2\right)\\
&\quad + C\left(\|\partial_1^3 b_2\|_{L^2}^2+\|(b,\nabla\partial_2 b_2)\|_{H^1}^2\right)\left(\|\partial_3\partial_1^3 b_2\|_{L^2}^2+\|\partial_2(b,\partial_2\nabla b_2)\|_{H^1}^2\right)\\
&\quad + C\|\partial_1^3 b_2\|_{L^2}\|\partial_1^3 b_2\|_{L^2_{x_1}L^2_{x_2}L^\infty_{x_3}}\|\partial_2^2 p\|_{L^\infty_{x_1}L^\infty_{x_2}L^2_{x_3}}\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right)\\
&\quad + C\|\partial_1^3 b_2\|_{L^2}^{\frac32}\|\partial_3\partial_1^3 b_2\|_{L^2}^{\frac12}\|(u,b)\|_{H^3}^{\frac12}\|(\partial_2 u,\partial_2 b)\|_{H^2}^{\frac{11}{8}}\|(\partial_3 u,\partial_3 b)\|_{H^3}^{\frac18}\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \qquad (3.61)
\end{aligned}$$

Thus, plugging (3.59), (3.60), (3.61) into (3.52), we have

$$\begin{aligned}
E_1&\le 2\frac{d}{dt}\int\left(\partial_1^3 b_2\right)^2(b_2-\eta\partial_2 u_2)\,dx\\
&\quad + C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right)+R_4,
\end{aligned}$$

which, combined with (3.51), yields

$$\begin{aligned}
D_1+E_1&\le 2\frac{d}{dt}\int\left(\left(\partial_1^3 u_2\right)^2+\left(\partial_1^3 b_2\right)^2\right)(b_2-\eta\partial_2 u_2)\,dx\\
&\quad + C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right), \qquad (3.62)
\end{aligned}$$

since it follows from Lemmas 2.1 and 2.2 that

$$
\begin{aligned}
R_4+F_4 &= -4\int b\cdot\nabla(\partial_1^3u_2\partial_1^3b_2)(b_2-\eta\partial_2u_2)dx = 4\int \partial_1^3u_2\partial_1^3b_2 b\cdot\nabla(b_2-\eta\partial_2u_2)dx\\
&\le C\left(\|(\partial_1^3u_2,\partial_1^3b_2)\|_{L^2}^2+\|(b,\nabla b_2)\|_{H^1}^2\right)\left(\|\partial_3(\partial_1^3u_2,\partial_1^3b_2)\|_{L^2}^2+\|\partial_2(b,\nabla b_2)\|_{H^1}^2\right)\\
&\quad + C\|\partial_1^3u_2\|_{L^2}^{\frac12}\|\partial_3\partial_1^3u_2\|_{L^2}^{\frac12}\|\partial_1^3b_2\|_{L^2}^{\frac12}\|\partial_2\partial_1^3b_2\|_{L^2}^{\frac12}\|\partial_2\nabla u_2\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla u_2\|_{L^2}^{\frac12}\|b\|_{L^\infty}\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|\partial_3u\|_{H^3}^2+\|\partial_2b\|_{H^3}^2+\|\partial_3b\|_{H^3}^2\right).
\end{aligned}
$$

In the exactly same way as the treatments of D_1 and E_1, we have

$$
\begin{aligned}
D_3+E_3 &\le 3\frac{d}{dt}\int\left(\left(\partial_1^3u_3\right)^2+\left(\partial_1^3b_3\right)^2\right)(b_2-\eta\partial_2u_2)\,dx\\
&\quad + C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_2b,\partial_3b)\|_{H^3}^2\right). \qquad (3.63)
\end{aligned}
$$

Step III-2. The estimate of D_2

Based upon $(3.34)_2$ and integration by parts, we see that

$$
\begin{aligned}
D_2 &= -\frac{d}{dt}\int\partial_1^3u_2\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx+\int\partial_1^3u_2\partial_1^3\partial_tu_3(b_3-\eta\partial_2u_3)\,dx\\
&\quad+\int\partial_1^3\partial_tu_2\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx-\eta\int\partial_3\partial_1^3u_2\partial_1^3u_3\partial_3b_3\,dx\\
&\quad-\eta\int\partial_1^3u_2\partial_3\partial_1^3u_3\partial_3b_3\,dx+\mu\eta\int\partial_3\partial_1^3u_2\partial_1^3u_3\partial_3\partial_2u_3\,dx\\
&\quad+\mu\eta\int\partial_1^3u_2\partial_3\partial_1^3u_3\partial_3\partial_2u_3\,dx-\int\partial_1^3u_2\partial_1^3u_3u\cdot\nabla b_3\,dx\\
&\quad+\int\partial_1^3u_2\partial_1^3u_3b\cdot\nabla u_3\,dx+\eta\int\partial_1^3u_2\partial_1^3u_3\partial_2\left(u\cdot\nabla u_3-b\cdot\nabla b_3+\partial_3p\right)dx\\
&\triangleq -\frac{d}{dt}\int\partial_1^3u_2\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx+\sum_{j=1}^{9}D_{2j}. \qquad (3.64)
\end{aligned}
$$

In view of (3.32) and (3.33), one has

$$
\begin{aligned}
D_{21}+D_{22} &= \int\partial_1^3u_2\partial_1^3\partial_tu_3(b_3-\eta\partial_2u_3)\,dx+\int\partial_1^3\partial_tu_2\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx\\
&= \int\partial_1^3u_2\partial_1^3\partial_2b_3(b_3-\eta\partial_2u_3)\,dx-\mu\int\partial_3\partial_1^3u_2\partial_3\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx\\
&\quad-\mu\int\partial_1^3u_2\partial_3\partial_1^3u_3\partial_3b_3\,dx+\mu\eta\int\partial_1^3u_2\partial_3\partial_1^3u_3\partial_3\partial_2u_3\,dx\\
&\quad-\int\partial_1^3(u\cdot\nabla u_3-b\cdot\nabla b_3+\partial_3p)\partial_1^3u_2(b_3-\eta\partial_2u_3)\,dx\\
&\quad+\int\partial_1^3\partial_2b_2\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx-\mu\int\partial_3\partial_1^3u_2\partial_3\partial_1^3u_3(b_3-\eta\partial_2u_3)\,dx
\end{aligned}
$$

$$-\mu\int \partial_3\partial_1^3u_2\partial_1^3u_3\partial_3b_3\ dx+\mu\eta\int \partial_3\partial_1^3u_2\partial_1^3u_3\partial_3\partial_2u_3\ dx$$
$$-\int \partial_1^3(u\cdot\nabla u_2-b\cdot\nabla b_2+\partial_2p)\partial_1^3u_3(b_3-\eta\partial_2u_3)\ dx\triangleq\sum_{j=1}^{10}L_j. \tag{3.65}$$

Similarly to the estimate of (3.37) and (3.39), we have

$$\sum_{i=1}^{4}L_i+\sum_{i=6}^{9}L_i\le C\|(u,b)\|_{H^3}\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_3b,\partial_2b)\|_{H^3}^2\right), \tag{3.66}$$

and

$$L_5+L_{10}\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|\left(\partial_3u,\partial_2b,\partial_3b\right)\|_{H^3}^2\right)+S_1+S_2, \tag{3.67}$$

where

$$S_1=-\int u\cdot\nabla(\partial_1^3u_3\partial_1^3u_2)(b_3-\eta\partial_2u_3)dx,$$
$$S_2=\int\left(b\cdot\nabla\partial_1^3b_3\partial_1^3u_2+b\cdot\nabla\partial_1^3b_2\partial_1^3u_3\right)(b_3-\eta\partial_2u_3)dx.$$

Plugging (3.66) and (3.67) into (3.65) gives

$$D_{21}+D_{22}\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\times\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_2b,\partial_3b)\|_{H^3}^2\right)+S_1+S_2. \tag{3.68}$$

Analogously to the estimates of D_{12},D_{13},D_{16} in (3.45) and D_{14} in (3.49), we have

$$D_{23}+D_{24}+D_{25}+D_{26}+D_{28}\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_2b,\partial_3b)\|_{H^3}^2\right), \tag{3.69}$$

and

$$D_{29}\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2u\|_{H^2}^2+\|(\partial_3u,\partial_2b,\partial_3b)\|_{H^3}^2\right)+\Xi_2, \tag{3.70}$$

where

$$\Xi_2=\eta\int\partial_1^3u_2\partial_1^3u_3u\cdot\nabla\partial_2u_3\ dx.$$

Similarly to that in (3.50), it is easily checked that

$$D_{27}+S_1+\Xi_2=0. \tag{3.71}$$

Thanks to (3.68), (3.69), (3.70) and (3.71), we conclude from (3.64) that

$$D_2 \le -\frac{d}{dt}\int \left(\partial_1^3 u_2 \partial_1^3 u_3\right)\left(b_3 - \eta\partial_2 u_3\right)\,dx + S_2 + \left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right). \tag{3.72}$$

The treatment of S_2 strongly relies on the estimate of E_2 in (3.27). Similarly to the derivation of (3.52), we have by $(3.34)_1$ that

$$\begin{aligned} E_2 =& -\frac{d}{dt}\int \partial_1^3 b_2 \partial_1^3 b_3 (b_3 - \eta\partial_2 u_3)\,dx + \int \partial_1^3\partial_t b_2 \partial_1^3 b_3\,(b_3 - \eta\partial_2 u_3)\,dx \\ &+ \int \partial_1^3 b_2 \partial_1^3 \partial_t b_3\,(b_3 - \eta\partial_2 u_3)\;dx + \eta\int \partial_1^3 b_2 \partial_1^3 b_3 \partial_3^2 b_3\;dx \\ &- \mu\eta\int \partial_1^3 b_2 \partial_1^3 b_3 \partial_2\partial_3^2 u_3\;dx - \int \partial_1^3 b_2 \partial_1^3 b_3\,(u\cdot\nabla b_3 - b\cdot\nabla u_3)\;dx \\ &+ \eta\int \partial_1^3 b_2 \partial_1^3 b_3 \partial_2\,(u\cdot\nabla u_3 - b\cdot\nabla b_3 + \partial_3 p)\,dx \\ \triangleq& -\frac{d}{dt}\int \partial_1^3 b_2 \partial_1^3 b_3 (b_3 - \eta\partial_2 u_3)\,dx + \sum_{i=1}^{6} E_{2i}. \end{aligned} \tag{3.73}$$

By (3.30) and (3.31), we have

$$\begin{aligned} E_{21} + E_{22} =& \int \partial_1^3\partial_t b_2 \partial_1^3 b_3\,(b_3 - \eta\partial_2 u_3)\;dx + \int \partial_1^3 b_2 \partial_1^3\partial_t b_3\,(b_3 - \eta\partial_2 u_3)\,dx \\ =& \int \partial_1^3\left(\partial_2 u_2 + \eta\partial_2^2 b_2 + \eta\partial_3^2 b_2\right)\partial_1^3 b_3\,(b_3 - \eta\partial_2 u_3)\;dx \\ &+ \int \partial_1^3\left(\partial_2 u_3 + \eta\partial_2^2 b_3 + \eta\partial_3^2 b_3\right)\partial_1^3 b_2\,(b_3 - \eta\partial_2 u_3)\,dx \\ &- \int \left[\partial_1^3\left(u\cdot\nabla b_2\right)\partial_1^3 b_3 + \partial_1^3\left(u\cdot\nabla b_3\right)\partial_1^3 b_2\right](b_3 - \eta\partial_2 u_3)\,dx \\ &+ \int \left[\partial_1^3\left(b\cdot\nabla u_2\right)\partial_1^3 b_3 + \partial_1^3\left(b\cdot\nabla u_3\right)\partial_1^3 b_2\right](b_3 - \eta\partial_2 u_3)\,dx \\ \triangleq& \sum_{i=1}^{4} E_{21i}. \end{aligned} \tag{3.74}$$

Analogously to the estimate of E_{111} in (3.54), by Lemma 2.1 we obtain after integrating by parts and

$$E_{211} + E_{212} \le C\|(u,b)\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right). \tag{3.75}$$

Note that

$$\begin{aligned}
E_{213}+E_{214} =& -\int (\partial_1^3 u\cdot\nabla b_2+3\partial_1^2 u\cdot\nabla\partial_1 b_2+3\partial_1 u\cdot\nabla\partial_1^2 b_2)\partial_1^3 b_3(b_3-\eta\partial_2 u_3)\,dx\\
&-\int (\partial_1^3 u\cdot\nabla b_3+3\partial_1^2 u\cdot\nabla\partial_1 b_3+3\partial_1 u\cdot\nabla\partial_1^2 b_3)\partial_1^3 b_2(b_3-\eta\partial_2 u_3)\,dx\\
&+\int (\partial_1^3 b\cdot\nabla u_2+3\partial_1^2 b\cdot\nabla\partial_1 u_2+3\partial_1 b\cdot\nabla\partial_1^2 u_2)\partial_1^3 b_3(b_3-\eta\partial_2 u_3)\,dx\\
&+\int (\partial_1^3 b\cdot\nabla u_3+3\partial_1^2 b\cdot\nabla\partial_1 u_3+3\partial_1 b\cdot\nabla\partial_1^2 u_3)\partial_1^3 b_2(b_3-\eta\partial_2 u_3)\,dx\\
&-\int \left(u\cdot\nabla\partial_1^3 b_2\partial_1^3 b_3+u\cdot\nabla\partial_1^3 b_3\partial_1^3 b_2\right)(b_3-\eta\partial_2 u_3)dx\\
&+\int \left(b\cdot\nabla\partial_1^3 u_2\partial_1^3 b_3+b\cdot\nabla\partial_1^3 u_3\partial_1^3 b_2\right)(b_3-\eta\partial_2 u_3)dx\\
\triangleq&\sum_{i=1}^{6} M_i. \qquad (3.76)
\end{aligned}$$

In a similar manner as that used in the derivation of (3.56), we find

$$\sum_{i=1}^{4} M_i \le C\|\,(u,b)\,\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \qquad (3.77)$$

Integrating by parts and using the divergence-free condition $\nabla\cdot u=0$, we infer from Lemma 2.2 that

$$\begin{aligned}
M_5 =& \int \partial_1^3 b_2\partial_1^3 b_3 u\cdot\nabla(b_3-\eta\partial_2 u_3)\,dx\\
&\le C\left(\|(\partial_1^3 b_2,\partial_1^3 b_3)\|_{L^2}^2+\|(u,\nabla b_3,\nabla\partial_2 u_3)\|_{H^1}^2\right)\\
&\qquad\times\left(\|\partial_2(\partial_1^3 b_2,\partial_1^3 b_3)\|_{L^2}^2+\|\partial_3(u,\nabla b_3,\nabla\partial_2 u_3)\|_{H^1}^2\right)\\
&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \qquad (3.78)
\end{aligned}$$

Based on (3.75), (3.76), (3.77) and (3.78), we know from (3.74) that

$$\begin{aligned}
E_{21}+E_{22} \le& C(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2)\\
&\times(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2)+M_6, \qquad (3.79)
\end{aligned}$$

where

$$M_6=\int\left(b\cdot\nabla\partial_1^3 u_2\partial_1^3 b_3+b\cdot\nabla\partial_1^3 u_3\partial_1^3 b_2\right)(b_3-\eta\partial_2 u_3)\,dx.$$

Analogously to the derivation of (3.60), we have by Lemmas 2.1 and 2.2 that

$$E_{23}+E_{24}+E_{25}\le C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\times\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \tag{3.80}$$

It remains to bound E_{26}. Indeed, it follows from (3.38) and the L^r-estimates ($1<r<\infty$) of Riesz operator that

$$\begin{aligned}\|\partial_2\partial_3 p\|_{L^\infty_{x_1,x_2}L^2_{x_3}}&\le C\|\partial_2\partial_3 p\|_{L^2}^{\frac14}\|\partial_2^2\partial_3 p\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3 p\|_{L^2}^{\frac14}\|\partial_1\partial_2^2\partial_3 p\|_{L^2}^{\frac14}\\&\le C\|\partial_2(u\cdot\nabla u-b\cdot\nabla b)\|_{L^2}^{\frac14}\|\partial_2(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i)\|_{L^2}^{\frac12}\\&\quad\times\|\partial_2^2(\partial_i u_j\partial_j u_i-\partial_i b_j\partial_j b_i)\|_{L^2}^{\frac14},\end{aligned}$$

so that, by (3.48) we see that

$$\|\partial_2\partial_3 p\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\le C\|(u,b)\|_{H^3}^{\frac12}\|(\partial_2 u,\partial_2 b)\|_{H^2}^{\frac{11}{8}}\|(\partial_3 u,\partial_3 b)\|_{H^3}^{\frac18}.$$

Hence, similarly to the estimate of (3.61), we find

$$E_{26}\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right). \tag{3.81}$$

Now, putting (3.79), (3.80), (3.81) into (3.73) gives

$$\begin{aligned}E_2\le&-\frac{d}{dt}\int\partial_1^3 b_2\partial_1^3 b_3(b_3-\eta\partial_2 u_3)\,dx+M_6\\&+C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right),\end{aligned}$$

which, combining with (3.72), yields

$$\begin{aligned}D_2+E_2\le&-\frac{d}{dt}\int\left(\partial_1^3 u_2\partial_1^3 u_3+\partial_1^3 b_2\partial_1^3 b_3\right)(b_3-\eta\partial_2 u_3)\,dx\\&+C\left(\|(u,b)\|_{H^3}+\|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right),\end{aligned} \tag{3.82}$$

since it is easily deduced in a similar manner as the treatment of R_4+F_4 that

$$\begin{aligned}M_6+S_2&=\int b\cdot\nabla\left(\partial_1^3 u_2\partial_1^3 b_3+\partial_1^3 u_3\partial_1^3 b_2\right)(b_3-\eta\partial_2 u_3)dx\\&=-\int\left(\partial_1^3 u_2\partial_1^3 b_3+\partial_1^3 u_3\partial_1^3 b_2\right)b\cdot\nabla(b_3-\eta\partial_2 u_3)dx\\&\le C\|(u,b)\|_{H^3}^2\left(\|\partial_2 u\|_{H^2}^2+\|\partial_3 u\|_{H^3}^2+\|\partial_2 b\|_{H^3}^2+\|\partial_3 b\|_{H^3}^2\right).\end{aligned}$$

Step IV. The estimate of $\mathcal{E}_1(t)$

Now, inserting (3.62), (3.63) and (3.82) into (3.29), we conclude that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla^3 u, \nabla^3 b)\|_{L^2}^2 + \mu\|\partial_3\nabla^3 u\|_{L^2}^2 + \eta\left(\|\partial_2\nabla^3 b\|_{L^2}^2 + \|\partial_3\nabla^3 b\|_{L^2}^2\right)\\
&\quad\le 2\frac{d}{dt}\int\left(\left(\partial_1^3 u_2\right)^2 + \left(\partial_1^3 b_2\right)^2\right)(b_2 - \eta\partial_2 u_2)\,dx\\
&\qquad - \frac{d}{dt}\int\left(\partial_1^3 u_2\partial_1^3 u_3 + \partial_1^3 b_2\partial_1^3 b_3\right)(b_3 - \eta\partial_2 u_3)\,dx\\
&\qquad + 3\frac{d}{dt}\int\left(\left(\partial_1^3 u_3\right)^2 + \left(\partial_1^3 b_3\right)^2\right)(b_2 - \eta\partial_2 u_2)\,dx\\
&\qquad + C\left(\|(u,b)\|_{H^3} + \|(u,b)\|_{H^3}^2\right)\left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right),
\end{aligned}$$

which, added to (3.2) and integrated over $(0,t)$, we obtain

$$\mathcal{E}_1(t) \le \left(\mathcal{E}_1(0) + \mathcal{E}_1^{\frac{3}{2}}(0)\right) + C\mathcal{E}_1^{\frac{3}{2}}(t) + C\left(\mathcal{E}_1^{\frac{1}{2}}(t) + \mathcal{E}_1(t)\right)\left(\mathcal{E}_2(t) + \mathcal{E}_1(t)\right),$$

which, combined with Cauchy-Schwarz's inequality, gives rise to (3.1). The proof of Lemma 3.1 is therefore complete. □

3.2. A priori estimates (II)

In the subsection, we study the dissipation generated by the background magnetic field and prove the estimate of $\mathcal{E}_2(t)$.

Lemma 3.2. *For any $t \ge 0$, it holds that*

$$\mathcal{E}_2(t) \le C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{\frac{3}{2}}(t) + C\mathcal{E}_2^{\frac{3}{2}}(t). \tag{3.83}$$

Proof. It suffices to establish the estimates of the following two items:

$$\int_0^t \|\partial_2 u(\tau)\|_{L^2}^2 d\tau \quad \text{and} \quad \int_0^t \|\nabla^2\partial_2 u(\tau)\|_{L^2}^2 d\tau,$$

which will be achieved by using the special structure of equation $(1.2)_2$,

$$\partial_2 u = \partial_t b + u\cdot\nabla b - \eta\partial_2^2 b - \eta\partial_3^2 b - b\cdot\nabla u. \tag{3.84}$$

Step I. The estimate of $\|\partial_2 u(t)\|_{L^2}$

Multiplying (3.84) by $\partial_2 u$ in L^2 and integrating by parts, we get

$$\begin{aligned}\|\partial_2 u\|_{L^2}^2 &= \int \partial_2 u \cdot \partial_t b \, dx + \int u \cdot \nabla b \cdot \partial_2 u \, dx \\ &\quad - \eta \int \left(\partial_2^2 b + \partial_3^2 b\right) \cdot \partial_2 u \, dx - \int b \cdot \nabla u \cdot \partial_2 u \, dx \\ &\triangleq J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.85}$$

In terms of $(1.2)_1$ and the fact that $\nabla \cdot b = 0$, we deduce after integrating by parts that

$$\begin{aligned} J_1 &= \frac{d}{dt} \int \partial_2 u \cdot b \, dx + \int \partial_2 b \cdot \left(\partial_2 b + \mu \partial_3^2 u + b \cdot \nabla b - u \cdot \nabla u\right) dx \\ &\leq \frac{d}{dt} \int \partial_2 u \cdot b \, dx + C\|\partial_2 b\|_{H^3}^2 + C\|\partial_3 u\|_{H^3}^2 \\ &\quad + C\|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 b\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 b\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\leq \frac{d}{dt} \int \partial_2 u \cdot b \, dx + C\|(\partial_3 u, \partial_2 b)\|_{H^3}^2 \\ &\quad + C\|(u,b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right), \end{aligned} \tag{3.86}$$

and similarly,

$$\begin{aligned} J_2 + J_4 &\leq C\|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\nabla b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C\|\partial_2 u\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 u\|_{L^2}^{\frac{1}{2}} \|b\|_{L^2}^{\frac{1}{2}} \|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla u\|_{L^2}^{\frac{1}{2}} \\ &\leq C\|(u,b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right). \end{aligned} \tag{3.87}$$

Finally, it is easily seen that

$$J_3 \leq \frac{1}{2}\|\partial_2 u\|_{L^2}^2 + C\|\left(\partial_2 b, \partial_3 b\right)\|_{H^3}^2,$$

from which, (3.86), (3.87) and (3.85), we know

$$\begin{aligned} \|\partial_2 u\|_{L^2}^2 &\leq 2\frac{d}{dt} \int \partial_2 u \cdot b \, dx + C\|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \\ &\quad + C\|(u,b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2\right). \end{aligned} \tag{3.88}$$

Step II. The estimate of $\|\nabla^2\partial_2 u(t)\|_{L^2}$

In order to estimate $\|\nabla^2\partial_2 u\|_{L^2}$, we apply ∇^2 to (3.84) and multiply it by $\nabla^2\partial_2 u$ in L^2 to get that

$$\begin{aligned}\|\nabla^2\partial_2 u\|_{L^2}^2 =& \int \nabla^2\partial_2 u\cdot\partial_t\nabla^2 b\,dx + \int \nabla^2(u\cdot\nabla b)\cdot\nabla^2\partial_2 u\,dx\\ &-\eta\int\nabla^2\left(\partial_2^2 b+\partial_3^2 b\right)\cdot\nabla^2\partial_2 u\,dx - \int\nabla^2(b\cdot\nabla u)\cdot\nabla^2\partial_2 u\,dx\\ \triangleq& W_1+W_2+W_3+W_4.\end{aligned}\tag{3.89}$$

Owing to $(1.2)_1$ and $\nabla\cdot b=0$, we see that (3.89)

$$\begin{aligned}W_1 =& \frac{d}{dt}\int\nabla^2\partial_2 u\cdot\nabla^2 b\,dx + \int\nabla^2\partial_2 b\cdot\nabla^2\left(\partial_2 b+\mu\partial_3^2 u\right)dx\\ &+\int\nabla^2\partial_2 b\cdot\nabla^2\left(b\cdot\nabla b-u\cdot\nabla u\right)dx\\ \triangleq& \frac{d}{dt}\int\nabla^2\partial_2 u\cdot\nabla^2 b\,dx + W_{11}+W_{12},\end{aligned}\tag{3.90}$$

where the second term on the right-hand side can be easily bounded by

$$W_{11}\le C\|\partial_2 b\|_{H^3}^2 + C\|\partial_3 u\|_{H^3}^2.\tag{3.91}$$

Integrating by parts and Lemma 2.1, we have

$$\begin{aligned}W_{12} =& \int\nabla^2\partial_2 b\cdot\left(\nabla^2 b_i\cdot\partial_i b+2\nabla b_i\cdot\partial_i\nabla b+b_i\partial_i\nabla^2 b\right)dx\\ &-\int\nabla^2\partial_2 b\cdot\left(\nabla^2 u_i\cdot\partial_i u+2\nabla u_i\cdot\partial_i\nabla u+u_i\partial_i\nabla^2 u\right)dx\\ \le& C\|\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\nabla b\|_{L^2}^{\frac12}\|\partial_2\nabla b\|_{L^2}^{\frac12}\|\nabla^2 b\|_{L^2}^{\frac12}\|\partial_3\nabla^2 b\|_{L^2}^{\frac12}\\ &+C\|\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|b\|_{L^2}^{\frac12}\|\partial_2 b\|_{L^2}^{\frac12}\|\nabla^3 b\|_{L^2}^{\frac12}\|\partial_3\nabla^3 b\|_{L^2}^{\frac12}\\ &+C\|\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\nabla u\|_{L^2}^{\frac12}\|\partial_2\nabla u\|_{L^2}^{\frac12}\|\nabla^2 u\|_{L^2}^{\frac12}\|\partial_3\nabla^2 u\|_{L^2}^{\frac12}\\ &+C\|\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|\partial_1\partial_2\nabla^2 b\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12}\|\partial_2 u\|_{L^2}^{\frac12}\|\nabla^3 u\|_{L^2}^{\frac12}\|\partial_3\nabla^3 u\|_{L^2}^{\frac12}\\ \le& C\|(u,b)\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right),\end{aligned}$$

which, together with (3.91) and (3.90), implies that

$$\begin{aligned}W_1 \le& \frac{d}{dt}\int\nabla^2\partial_2 u\cdot\nabla^2 b\,dx + C\|\partial_2 b\|_{H^3}^2+C\|\partial_3 u\|_{H^3}^2\\ &+C\|(u,b)\|_{H^3}\left(\|\partial_2 u\|_{H^2}^2+\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^3}^2\right).\end{aligned}\tag{3.92}$$

Analogously, we deduce from Lemma 2.1 that

$$\begin{aligned} W_2 + W_4 &= \int \nabla^2 \partial_2 u \cdot \left(\nabla^2 u_i \cdot \partial_i b + 2\nabla u_i \cdot \partial_i \nabla b + u_i \partial_i \nabla^2 b \right) dx \\ &\quad - \int \nabla^2 \partial_2 u \cdot \left(\nabla^2 b_i \cdot \partial_i u + 2\nabla b_i \cdot \partial_i \nabla u + b_i \partial_i \nabla^2 u \right) dx \\ &\leq C \|\partial_2 \nabla^2 u\|_{L^2} \|\nabla b\|_{H^1}^{\frac{1}{2}} \|\partial_2 \nabla b\|_{H^1}^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla^2 u\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_2 \nabla^2 u\|_{L^2} \|\nabla u\|_{H^1}^{\frac{1}{2}} \|\partial_2 \nabla u\|_{H^1}^{\frac{1}{2}} \|\nabla^2 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla^2 b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_2 \nabla^2 u\|_{L^2} \|u\|_{H^1}^{\frac{1}{2}} \|\partial_2 u\|_{H^1}^{\frac{1}{2}} \|\nabla^3 b\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla^3 b\|_{L^2}^{\frac{1}{2}} \\ &\quad + C \|\partial_2 \nabla^2 u\|_{L^2} \|b\|_{H^1}^{\frac{1}{2}} \|\partial_2 b\|_{H^1}^{\frac{1}{2}} \|\nabla^3 u\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla^3 u\|_{L^2}^{\frac{1}{2}} \\ &\leq C \|(u,b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \right), \end{aligned} \tag{3.93}$$

and finally,

$$W_3 \leq \frac{1}{2} \|\partial_2 \nabla^2 u\|_{L^2}^2 + C \| \left(\partial_2 b, \partial_3 b\right) \|_{H^3}^2. \tag{3.94}$$

Hence, inserting (3.92), (3.93) and (3.94) into (3.89), we arrive at

$$\begin{aligned} \|\partial_2 \nabla^2 u\|_{L^2} \leq 2 \frac{d}{dt} \int \nabla^2 \partial_2 u \cdot \nabla^2 b \, dx + C \| \left(\partial_2 b, \partial_3 b, \partial_3 u\right) \|_{H^3}^2 \\ + C \|(u,b)\|_{H^3} \left(\|\partial_2 u\|_{H^2}^2 + \|(\partial_3 u, \partial_2 b, \partial_3 b)\|_{H^3}^2 \right). \end{aligned} \tag{3.95}$$

Step III. The estimate of $\mathcal{E}_2(t)$

Now, adding up (3.88) and (3.95), we obtain after integrating it over $(0,t)$ that

$$\int_0^t \|\partial_2 u\|_{H^2}^2 d\tau \leq C\mathcal{E}_1(0) + C\mathcal{E}_1(t) + C\mathcal{E}_1^{\frac{1}{2}}(t) \left(\mathcal{E}_1(t) + \mathcal{E}_2(t)\right)$$

which, together with Cauchy-Schwarz's inequality, leads to the desired estimate (3.83) immediately. □

3.3. Proof of Theorem 1.1

With Lemmas 3.1 and 3.2 at hand, we can now prove the global stability result stated in Theorem 1.1 by applying the bootstrapping argument (see, e.g. [37]).

Proof of Theorem 1.1. Since the local well-posedness of smooth solutions to the problem (1.2) can be established by the standard approach (see, e.g. [26]), it suffices to establish the global bounds of the solutions.

Indeed, since it holds that $\mathcal{E}(0) = \mathcal{E}_1(0) = \|(u_0, b_0)\|_{H^3}^2$ for $\mathcal{E}(t) = \mathcal{E}_1(t) + \mathcal{E}_2(t)$, we easily deduce from (3.1) and (3.83) that

$$\mathcal{E}(t) \leq C_1 \left(\mathcal{E}(0) + \mathcal{E}(0)^{\frac{3}{2}} \right) + C_2 \mathcal{E}(t)^{\frac{3}{2}} + C_3 \mathcal{E}(t)^2. \tag{3.96}$$

Then, based upon (3.96), an application of the bootstrap argument leads to the stability result established in Theorem 1.1, provided the initial data $\|(u_0, b_0)\|_{H^3}^2$ is chosen to be sufficiently small such that

$$C_1 \left(\mathcal{E}(0) + \mathcal{E}(0)^{\frac{3}{2}} \right) \leq \frac{1}{4} \min \left\{ \frac{1}{16 C_2^2}, \ \frac{1}{4 C_3} \right\}. \tag{3.97}$$

In fact, if we make the ansatz that for $0 < t \leq \infty$,

$$\mathcal{E}(t) \leq \min \left\{ \frac{1}{16 C_2^2}, \ \frac{1}{4 C_3} \right\}, \tag{3.98}$$

then (3.96) implies

$$\mathcal{E}(t) \leq C_1 \left(\mathcal{E}(0) + \mathcal{E}(0)^{\frac{3}{2}} \right) + \frac{1}{2} \mathcal{E}(t),$$

which, together with (3.97), yields

$$\mathcal{E}(t) \leq 2 C_1 \left(\mathcal{E}(0) + \mathcal{E}(0)^{\frac{3}{2}} \right) \leq \frac{1}{2} \min \left\{ \frac{1}{16 C_2^2}, \ \frac{1}{4 C_3} \right\}.$$

The bootstrapping argument then asserts that (3.98) actually holds for all time. The proof of Theorem 1.1 is therefore complete. □

4. Decay rates and proof of Theorem 1.2

4.1. Estimates of nonlinear terms

This section aims to prove the decay rates of the global solutions, based on Proposition 2.1. To do this, we first deal with the nonlinear term N_1,

$$\widehat{N_1} = \mathbb{P} \, \widehat{(b \cdot \nabla b - u \cdot \nabla u)} = (\widehat{N_{11}}, \widehat{N_{12}}, \widehat{N_{13}}). \tag{4.1}$$

In the following, we only consider the estimate of $\mathbb{P} \, \widehat{(b \cdot \nabla b)}$, since the other term $\mathbb{P} \, \widehat{(u \cdot \nabla u)}$ can be treated in the same way. In fact,

$$\mathbb{P} \, \widehat{(b \cdot \nabla b)} = \sum_{k=1}^{3} i \xi_k \widehat{b_k b} + \sum_{k,l=1}^{3} i \xi |\xi|^{-2} i \xi_k i \xi_l \widehat{b_k b_l}$$

$$= \left(\mathbb{P}_1\widehat{(b\cdot\nabla b)}, \mathbb{P}_2\widehat{(b\cdot\nabla b)}, \mathbb{P}_3\widehat{(b\cdot\nabla b)}\right),$$

where

$$\begin{aligned}\mathbb{P}_j\widehat{(b\cdot\nabla b)} &\triangleq \sum_{k=1}^{3} i\xi_k\widehat{b_k b_j} + \sum_{k,l=1}^{3} i\xi_j|\xi|^{-2}i\xi_k i\xi_l\widehat{b_k b_l},\\ &= \widehat{(b\cdot\nabla b_j)} - \partial_j\Delta^{-1}\widehat{\nabla\cdot(b\cdot\nabla b)}, \quad j=1,2,3. \end{aligned} \tag{4.2}$$

Next, we present a more concise expression of $\mathbb{P}_1\widehat{(b\cdot\nabla b)}$. Let $\xi_\nu = (\xi_2,\xi_3)$ and $|\xi|^2 = \xi_1^2 + |\xi_\nu|^2$. Then, by direct calculations we have

$$\begin{aligned}\mathbb{P}_1\widehat{(b\cdot\nabla b)} &= \sum_{k=1}^{3} i\xi_k\widehat{b_k b_1} + \sum_{k,l=1}^{3} i\xi_1|\xi|^{-2}i\xi_k i\xi_l\widehat{b_k b_l}\\ &= \sum_{k=1}^{3} i\xi_k\widehat{b_k b_1} - \sum_{k,l=1}^{3} i\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\\ &= \sum_{k=1}^{3} i\xi_k\widehat{b_k b_1} - \sum_{k=1}^{3} i\xi_1^2|\xi|^{-2}\xi_k\widehat{b_k b_1} - \sum_{k=1}^{3}\sum_{l=2}^{3} i\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\\ &= \sum_{k=1}^{3} i\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1} - \sum_{k=1}^{3}\sum_{l=2}^{3} i\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}. \end{aligned} \tag{4.3}$$

The proof of decay rates is based on the bootstrapping argument and the following ansatz with $k=0,1$, $i=2,3$ and $l=1,2$:

$$\begin{aligned}
&\|\partial_1^k(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}}, && \|\partial_i(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1},\\
&\|(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{4}}, && \|\partial_i(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{5}{4}},\\
&\|\partial_i\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{2}}, && \|\partial_l\partial_2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{12}},\\
&\|\partial_1\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-1}, && \|\partial_1^2(u,b)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{3}{8}},\\
&\|\partial_i\partial_3(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{7}{4}}, && \|\partial_3^3(u_1,b_1)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{9}{4}},\\
&\|\partial_l\partial_2\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{12}}, && \|\partial_1^2\partial_3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{1}{2}},\\
&\|\partial_1\partial_3^2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{5}{4}}, && \|\partial_2\partial_3^2(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{11}{6}},\\
&\|\partial_3^3(u,b)(t)\|_{L^2} \le C_0\varepsilon(1+t)^{-\frac{23}{12}}, &&
\end{aligned} \tag{4.4}$$

where C_0 is an absolutely positive constant to be determined later.

By exploiting the integral expression of the solutions and the upper bound of the kernel function, we can show that the half upper bounds of the solutions in (4.4) can be

halved, provided C_0 is chosen to be large enough and ε is chosen to be suitably small. Then the half upper bounds in (4.4) are indeed justifiable by applying the bootstrapping argument. The verification is a long and tedious process involving repeated applications of various anisotropic inequalities and the upper bounds on the kernel functions. To begin, we first derive the decay rates of the nonlinear terms, which plays a key role in the proof of Theorem 1.2.

Lemma 4.1. *Assume that (u, b) is a smooth solution of the problem (1.2), satisfying (4.4). Then, there exists an absolute positive constant $C > 0$, such that*

$$\left\| \|b \cdot \nabla b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{8}}, \tag{4.5}$$

$$\left\| \|\partial_1 (b \cdot \nabla b)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{4}}, \tag{4.6}$$

$$\|\partial_1 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{37}{24}}, \tag{4.7}$$

$$\|\partial_2 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}}, \tag{4.8}$$

$$\|\partial_3 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{13}{6}}, \tag{4.9}$$

$$\left\| \|\partial_1^2 (b \cdot \nabla b)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq CC_0\varepsilon (1+t)^{-\frac{1}{2}} \|\partial_\nu b\|_{H^2} + CC_0\varepsilon^2 (1+t)^{-\frac{7}{8}}, \tag{4.10}$$

$$\|\partial_l\partial_h (b \cdot \nabla b)\|_{L^2} \leq CC_0\varepsilon^2 (1+t)^{-\frac{11}{12}}, \quad \forall\, l, h \in \{1, 2\}, \tag{4.11}$$

$$\|\partial_1\partial_3 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{3}}, \tag{4.12}$$

$$\|\partial_2\partial_3 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}}, \tag{4.13}$$

$$\|\partial_3^2 (b \cdot \nabla b)\|_{L^2} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{29}{12}}. \tag{4.14}$$

Proof. Using the Minkowski inequality (2.2) and the ansatz in (4.4), we have

$$\begin{aligned}
\left\| \|b \cdot \nabla b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} &\leq \left\| \|b_1\partial_1 b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} + \left\| \|b_\nu\partial_\nu b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \\
&\leq \left\| \|b_1\partial_1 b\|_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} + \left\| \|b_\nu\partial_\nu b\|_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} \\
&\leq \left\| \|b_1\|_{L^\infty_{x_1}} \|\partial_1 b\|_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} + \left\| \|b_\nu\|_{L^\infty_{x_1}} \|\partial_\nu b\|_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} \\
&\leq C\|b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b\|_{L^2} + C\|b_\nu\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_\nu\|_{L^2}^{\frac{1}{2}} \|\partial_\nu b\|_{L^2} \\
&\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{8}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{2}} \\
&\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{8}},
\end{aligned}$$

where we have used the fact that $\nabla \cdot b = 0$ to infer from (4.4) that

$$\|\partial_1 b_1\|_{L^2} \le \|\partial_2 b_2\|_{L^2} + \|\partial_3 b_3\|_{L^2} \le CC_0\varepsilon(1+t)^{-1}. \tag{4.15}$$

In a similar manner,

$$\begin{aligned}
&\Big\| \|\partial_1 (b\cdot\nabla b)\|_{L^1_{x_2,x_3}} \Big\|_{L^2_{x_1}} \\
&\le C\|\partial_\nu b\|_{L^2}\|\partial_1 b_\nu\|_{L^2}^{\frac12}\|\partial_1^2 b_\nu\|_{L^2}^{\frac12} + C\|\partial_\nu\partial_1 b\|_{L^2}\|b_\nu\|_{L^2}^{\frac12}\|\partial_1 b_\nu\|_{L^2}^{\frac12} \\
&\quad + C\|\partial_1 b_1\|_{L^2}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1^2 b\|_{L^2}^{\frac12} + C\|\partial_1^2 b\|_{L^2}\|b_1\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac12} \\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac54}.
\end{aligned} \tag{4.16}$$

To proceed, we first estimate $\|b\|_{L^\infty}$. It is easy to derive from the anisotropic Sobolev embedding inequality and (4.4) that

$$\begin{aligned}
\|b\|_{L^\infty} &\le C\|b\|_{L^2}^{\frac18}\|\partial_1 b\|_{L^2}^{\frac18}\|\partial_2 b\|_{L^2}^{\frac18}\|\partial_3 b\|_{L^2}^{\frac18} \\
&\quad\times \|\partial_1\partial_2 b\|_{L^2}^{\frac18}\|\partial_1\partial_3 b\|_{L^2}^{\frac18}\|\partial_2\partial_3 b\|_{L^2}^{\frac18}\|\partial_1\partial_2\partial_3 b\|_{L^2}^{\frac18} \\
&\le CC_0\varepsilon(1+t)^{-\frac{11}{12}}.
\end{aligned} \tag{4.17}$$

Using the similar inequalities as that in (4.15), we have by (4.4) that

$$\begin{aligned}
\|b_1\|_{L^\infty} &\le C\|b_1\|_{L^2}^{\frac18}\|\partial_1 b_1\|_{L^2}^{\frac18}\|\partial_2 b_1\|_{L^2}^{\frac18}\|\partial_3 b_1\|_{L^2}^{\frac18} \\
&\quad\times \|\partial_1\partial_2 b_1\|_{L^2}^{\frac18}\|\partial_1\partial_3 b_1\|_{L^2}^{\frac18}\|\partial_2\partial_3 b_1\|_{L^2}^{\frac18}\|\partial_1\partial_2\partial_3 b_1\|_{L^2}^{\frac18} \\
&\le CC_0\varepsilon(1+t)^{-\frac76}.
\end{aligned} \tag{4.18}$$

With the help of (4.4), (4.17) and (4.18), we obtain

$$\begin{aligned}
\|b\cdot\nabla b\|_{L^2} &\le C\|b_1\|_{L^\infty}\|\partial_1 b\|_{L^2} + C\|b_\nu\|_{L^\infty}\|\partial_\nu b\|_{L^2} \\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac53},
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
\|\partial_1(b\cdot\nabla b)\|_{L^2} &\le \|b_1\|_{L^\infty}\|\partial_1^2 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_1\partial_\nu b\|_{L^2} + \|\partial_1 b\cdot\nabla b\|_{L^2} \\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{37}{24}} + \|\partial_1 b_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_1 b\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \\
&\quad + \|\partial_1 b_\nu\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_\nu b\|_{L^2_{x_1,x_3}L^\infty_{x_2}} \\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{37}{24}},
\end{aligned} \tag{4.20}$$

where we have used the following anisotropic Sobolev inequalities for $i,j,k\in\{1,2,3\}$ and $i\neq j\neq k$:

$$\begin{aligned}
&\|f\|_{L^2_{x_i,x_j}L^\infty_{x_k}} \le C\|f\|_{L^2}^{\frac{1}{2}}\|\partial_k f\|_{L^2}^{\frac{1}{2}},\\
&\|f\|_{L^\infty_{x_i,x_j}L^2_{x_k}} \le C\|f\|_{L^2}^{\frac{1}{4}}\|\partial_i f\|_{L^2}^{\frac{1}{4}}\|\partial_j f\|_{L^2}^{\frac{1}{4}}\|\partial_i\partial_j f\|_{L^2}^{\frac{1}{4}}.
\end{aligned} \tag{4.21}$$

Similarly to the derivation of (4.20), we have

$$\begin{aligned}
\|\partial_2 (b\cdot\nabla b)\|_{L^2} \le{}& \|b_1\|_{L^\infty}\|\partial_1\partial_2 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_2\partial_\nu b\|_{L^2}\\
&+ \|\partial_2 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_1 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&+ \|\partial_2 b_\nu\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_\nu b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\
\le{}& CC_0^2\varepsilon^2(1+t)^{-\frac{11}{6}},
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_3(b\cdot\nabla b)\|_{L^2} \le{}& \|b_1\|_{L^\infty}\|\partial_1\partial_3 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_\nu\partial_3 b\|_{L^2}\\
&+ \|\partial_3 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_1 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&+ \|\partial_3 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
\le{}& CC_0^2\varepsilon^2(1+t)^{-\frac{13}{6}}.
\end{aligned}$$

Using the divergence-free condition $\nabla\cdot b = 0$ and (4.15) again, we deduce from Hölder inequality and (4.4) that

$$\begin{aligned}
&\Big\|\|\partial_1^2(b\cdot\nabla b)\|_{L^1_{x_2,x_3}}\Big\|_{L^2_{x_1}}\\
&\le C\|\partial_\nu b\|_{L^2}\|\partial_1^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 b\|_{L^2}^{\frac{1}{2}} + C\|\partial_1 b\|_{L^2}\|\partial_\nu\partial_1 b\|_{L^2}^{\frac{1}{2}}\|\partial_\nu\partial_1^2 b\|_{L^2}^{\frac{1}{2}}\\
&\quad + C\|\partial_\nu\partial_1^2 b\|_{L^2}\|b_\nu\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_\nu\|_{L^2}^{\frac{1}{2}} + C\|\partial_1^3 b\|_{L^2}\|b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\\
&\le CC_0\varepsilon(1+t)^{-\frac{1}{2}}\|\partial_\nu b\|_{H^2} + CC_0\varepsilon^2(1+t)^{-\frac{7}{8}}.
\end{aligned} \tag{4.22}$$

For $l,h\in\{1,2\}$, by (4.15) one infers from (4.4) and (4.17) in a similar manner as the derivation of (4.20) that

$$\begin{aligned}
\|\partial_l\partial_h(b\cdot\nabla b)\|_{L^2} \le{}& \|b\|_{L^\infty}\|b\|_{H^3} + \|\partial_l\partial_h b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+ \|\partial_l\partial_h b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&+ \|\partial_l b_1\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_h\partial_1 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&+ \|\partial_l b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_h\partial_\nu b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&+ \|\partial_h b_1\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_l\partial_\nu b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&+ \|\partial_h b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_l\partial_\nu b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}
\end{aligned}$$

$$\leq CC_0\varepsilon^2 (1+t)^{-\frac{11}{12}}.$$

Using (4.17), (4.18), the anisotropic inequalities (4.21) and the analogous inequality as that in (4.15), we infer from (4.4) that

$$\begin{aligned}
&\|\partial_1\partial_3(b\cdot\nabla b)\|_{L^2}\\
&\quad\leq \|b_1\|_{L^\infty}\|\partial_1^2\partial_3 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_1\partial_3\partial_\nu b\|_{L^2}\\
&\qquad + \|\partial_1\partial_3 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}} + \|\partial_1\partial_3 b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\qquad + \|\partial_1 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_1\partial_3 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}} + \|\partial_1 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_3\partial_\nu b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\qquad + \|\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}} + \|\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1\partial_\nu b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad\leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{3}},\\
&\|\partial_2\partial_3(b\cdot\nabla b)\|_{L^2}\\
&\quad\leq \|b_1\|_{L^\infty}\|\partial_1\partial_2\partial_3 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_2\partial_3\partial_\nu b\|_{L^2}\\
&\qquad + \|\partial_2\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\nabla b\|_{L^\infty_{x_2,x_3}L^2_{x_1}} + \|\partial_2 b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3\nabla b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&\qquad + \|\partial_3 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2\nabla b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad\leq CC_0^2\varepsilon^2(1+t)^{-\frac{11}{6}},
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_3^2(b\cdot\nabla b)\|_{L^2}\\
&\quad\leq \|b_1\|_{L^\infty}\|\partial_1\partial_3^2 b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_3^2\partial_\nu b\|_{L^2}\\
&\qquad + \|\partial_3^2 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}} + \|\partial_3^2 b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\qquad + 2\|\partial_3\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_3\partial_1 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}} + 2\|\partial_3\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_3\partial_\nu b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad\leq CC_0^2\varepsilon^2(1+t)^{-\frac{29}{12}}.
\end{aligned}$$

Thus, collecting the above estimates together ends the proof of Lemma 4.1. □

4.2. Decay rates of (u,b)

With Lemma 4.1 at hand, we are now ready to justify the decay estimate of the solution (u,b). For brevity, we will focus on the decay rates of u, since the same decay rates of b can be achieved in a similar manner due to the resemblance between (2.7) and (2.8). The proof will be divided into several steps by considering the different-order derivatives of the solutions. We only focus on the large-time behavior on the interval $t\geq 1$, since the short-time boundedness of the solutions on $0\leq t\leq 1$ is trivial, due to Theorem 1.1. To begin, we first observe from Proposition 2.1 that

$$|\widehat{K_1}|, |\widehat{K_2}| \le Ce^{-c\xi_\nu^2 t}, \quad \text{if} \quad \xi \in \Omega_1 \cup \Omega_{21} \cup \Omega_{22}, \tag{4.23}$$

$$|\widehat{K_1}|, |\widehat{K_2}| \le C\left(e^{-ct} + e^{-c\xi_\nu^2 t}\right), \quad \text{if} \quad \xi \in \Omega_{23}. \tag{4.24}$$

Step I. The decay estimates of $\|u\|_{L^2}$

It follows from Plancherel's Theorem and (2.7) that

$$\begin{aligned}\|u\|_{L^2} = \|\widehat{u}\|_{L^2} \le& \left\|\widehat{K_1}(t)\widehat{u_0}\right\|_{L^2} + \left\|\widehat{K_2}(t)\widehat{b_0}\right\|_{L^2} \\ &+ \int_0^t \left\|\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} d\tau \\ &+ \int_0^t \left\|\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \mathrm{I}_m. \end{aligned} \tag{4.25}$$

First, using Corollary 2.1 and Plancherel's theorem, we see that for any $t \ge 1$,

$$\begin{aligned}\mathrm{I}_1 + \mathrm{I}_2 &\le C\|e^{-c\xi_\nu^2 t}(\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct}\|(\widehat{u_0}, \widehat{b_0})\|_{L^2} \\ &\le C(1+t)^{-\frac{1}{2}}\left(\|(u_0, b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|(u_0, b_0)\|_{L^2}\right). \end{aligned} \tag{4.26}$$

Keeping the definitions of N_1 and N_2 in mind, by (4.23) and (4.24) we have

$$\begin{aligned}\mathrm{I}_3 + \mathrm{I}_4 \le& C\int_0^t \left\|e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau + C\int_0^t \left\|e^{-c\xi_\nu^2(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2} d\tau \\ &+ C\int_0^t \left\|e^{-c(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau + C\int_0^t \left\|e^{-c(t-\tau)}\widehat{u\cdot\nabla u}\right\|_{L^2} d\tau \\ &+ C\int_0^t \left\|e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla u}\right\|_{L^2} d\tau + C\int_0^t \left\|e^{-c\xi_\nu^2(t-\tau)}\widehat{u\cdot\nabla b}\right\|_{L^2} d\tau \\ &+ C\int_0^t \left\|e^{-c(t-\tau)}\widehat{b\cdot\nabla u}\right\|_{L^2} d\tau + C\int_0^t \left\|e^{-c(t-\tau)}\widehat{u\cdot\nabla b}\right\|_{L^2} d\tau \\ \triangleq& \mathrm{I}_{34}^1 + \ldots + \mathrm{I}_{34}^8, \end{aligned} \tag{4.27}$$

where we have used the fact that the Leray projection operator $\mathbb{P}$ is bounded in L^2 (see [35]). In the next, we only deal with I_{34}^1 and I_{34}^3, since the other terms can be treated in a similar manner, due to their analogous structures.

By Corollary 2.1 with $\alpha = 0$ and Lemmas 2.4–2.5, we infer from (4.5) and (4.19) that

$$\begin{aligned}
\mathrm{I}_{34}^1 + \mathrm{I}_{34}^3 \le\ & C \int_0^{t-1} \left\| e^{-c\xi_\nu^2(t-\tau)} \widehat{b \cdot \nabla b} \right\|_{L^2} d\tau + C \int_{t-1}^{t} \left\| e^{-c\xi_\nu^2(t-\tau)} \widehat{b \cdot \nabla b} \right\|_{L^2} d\tau \\
& + C \int_0^t e^{-c(t-\tau)} \|b \cdot \nabla b\|_{L^2} d\tau \\
\le\ & C \int_0^t (1+t-\tau)^{-\frac{1}{2}} \left\| \|b \cdot \nabla b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} d\tau + C \int_0^t e^{-c(t-\tau)} \|b \cdot \nabla b\|_{L^2} d\tau \\
\le\ & C C_0^2 \varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{8}} d\tau + C C_0^2 \varepsilon^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{5}{3}} d\tau \\
\le\ & C C_0^2 \varepsilon^2 (1+t)^{-\frac{1}{2}}, \quad \forall\ t \ge 1,
\end{aligned} \tag{4.28}$$

where we have used the simple facts that $t - \tau \ge 1$ for $\tau \in [0, t-1]$ with $t \ge 1$ and

$$1 \le e^c e^{-c(t-\tau)} \le C e^{-c(t-\tau)}, \quad \forall\ \tau \in [t-1, t],\ \ t \ge 1.$$

Analogously, the other terms in (4.27) admit the same decay rates. Thus, in view of (4.28), we deduce from (4.27) that

$$\mathrm{I}_3 + \mathrm{I}_4 \le C C_0^2 \varepsilon^2 (1+t)^{-\frac{1}{2}}, \quad \forall\ t \ge 1. \tag{4.29}$$

Collecting (4.25), (4.26) and (4.29) together yields

$$\|u(t)\|_{L^2} \le C_1 \varepsilon (1+t)^{-\frac{1}{2}} + C_2 C_0^2 \varepsilon^2 (1+t)^{-\frac{1}{2}}. \tag{4.30}$$

Now, if C_0 and ε are chosen to be such that

$$C_0 \ge 4C_1 \quad \text{and} \quad 0 < \varepsilon \le (4C_2 C_0)^{-1},$$

then it readily follows from (4.30) that

$$\|u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon (1+t)^{-\frac{1}{2}}, \quad \forall\ t \ge 1.$$

Step II. The decay estimates of $\|\partial_i u\|_{L^2}$ with $i \in \{1,2,3\}$

For $i \in \{1,2,3\}$, we have by Plancherel's Theorem and (2.7) that

$$\|\partial_i u\|_{L^2} = \|\widehat{\partial_i u}\|_{L^2} = \||\xi_i| \widehat{u}\|_{L^2}$$

$$\leq \left\| |\xi_i| \widehat{K_1}(t) \widehat{u_0} \right\|_{L^2} + \left\| |\xi_i| \widehat{K_2}(t) \widehat{b_0} \right\|_{L^2} + \int_0^t \left\| |\xi_i| \widehat{K_1}(t-\tau) \widehat{N_1}(\tau) \right\|_{L^2} d\tau + \int_0^t \left\| |\xi_i| \widehat{K_2}(t-\tau) \widehat{N_2}(\tau) \right\|_{L^2} d\tau \triangleq \sum_{m=1}^{3} \mathrm{II}_{i,m}. \tag{4.31}$$

Step II-1. The decay estimate of $\|\partial_1 u\|_{L^2}$

Similarly to the derivation of (4.26) for I_1 and I_2 in Step I-1, by Plancherel's Theorem and Corollary 2.1 we infer from (4.31) that for $i = 1$,

$$\begin{aligned} \mathrm{II}_{1,1} + \mathrm{II}_{1,2} &\leq \left\| \widehat{K_1}(t) \widehat{\partial_1 u_0} \right\|_{L^2} + \left\| \widehat{K_2}(t) \widehat{\partial_1 b_0} \right\|_{L^2} \\ &\leq C \| e^{-c\xi_\nu^2 t} (\widehat{\partial_1 u_0}, \widehat{\partial_1 b_0}) \|_{L^2} + C e^{-ct} \left\| (\widehat{\partial_1 u_0}, \widehat{\partial_1 b_0}) \right\|_{L^2} \\ &\leq C(1+t)^{-\frac{1}{2}} \left(\| (\partial_1 u_0, \partial_1 b_0) \|_{L^2_{x_1} L^1_{x_2,x_3}} + \| (\partial_1 u_0, \partial_1 b_0) \|_{L^2} \right) \\ &\leq C\varepsilon (1+t)^{-\frac{1}{2}}. \end{aligned} \tag{4.32}$$

From now on, analogously to treatment of (4.27) in Step I-1, for simplicity we only consider the terms associated with $b \cdot \nabla b$ in N_1 and N_2. So, to deal with $\mathrm{II}_{1,3}$ and $\mathrm{II}_{1,4}$, by (4.23) and (4.24) we deduce from Plancherel's Theorem, Corollary 2.1, (4.6) and (4.7) that

$$\begin{aligned} &\mathrm{II}_{1,3} + \mathrm{II}_{1,4} \\ &\triangleq \int_0^t \left\| \xi_1 \widehat{K_1}(t-\tau) \widehat{b \cdot \nabla b} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_1 \widehat{K_2}(t-\tau) \widehat{b \cdot \nabla b} \right\|_{L^2} d\tau \\ &\leq \int_0^t \left\| \widehat{K_1}(t-\tau) \partial_1 \widehat{(b \cdot \nabla b)} \right\|_{L^2} d\tau + \int_0^t \left\| \widehat{K_2}(t-\tau) \partial_1 \widehat{(b \cdot \nabla b)} \right\|_{L^2} d\tau \\ &\leq C \int_0^t \left\| e^{-c\xi_\nu^2 (t-\tau)} \partial_1 \widehat{(b \cdot \nabla b)} \right\|_{L^2} d\tau + C \int_0^t e^{-c(t-\tau)} \left\| \partial_1 \widehat{(b \cdot \nabla b)} \right\|_{L^2} d\tau \\ &\leq C \int_0^t \left[(1+t-\tau)^{-\frac{1}{2}} \| \partial_1 (b \cdot \nabla b) \|_{L^2_{x_1} L^1_{x_2,x_3}} + e^{-c(t-\tau)} \| \partial_1 (b \cdot \nabla b) \|_{L^2} \right] d\tau \\ &\leq C C_0^2 \varepsilon^2 \int_0^t \left[(1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{5}{4}} + e^{-c(t-\tau)} (1+\tau)^{-\frac{37}{24}} \right] d\tau \end{aligned}$$

$$\leq CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}}. \tag{4.33}$$

In view of (4.32) and (4.33), by choosing C_0 large enough and ε suitably small, we know

$$\|\partial_1 u(t)\|_{L^2} \leq \left(C\varepsilon + CC_0^2\varepsilon^2\right)(1+t)^{-\frac{1}{2}} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}}.$$

Step II-2. The decay estimates of $\|\partial_i u\|_{L^2}$ with $i \in \{2,3\}$

Analogously to the treatment of (4.27) in Step I-1, we focus our attention on the estimates of the terms involving $b\cdot\nabla b$. Using Corollary 2.1 with $\alpha = 1$, Lemmas 2.4 and 2.5, we deduce from (4.31), (4.5), (4.8) and (4.9) that for $i \in \{2,3\}$,

$$\begin{aligned}
\|\partial_i u\|_{L^2} &\leq C\|\xi_i e^{-c\xi_\nu^2 t}(\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct}\left\|(\widehat{\partial_i u_0}, \widehat{\partial_i b_0})\right\|_{L^2} \\
&\quad + C\int_0^t \left\|\xi_i e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau + C\int_0^t e^{-c(t-\tau)}\left\|\partial_i\widehat{(b\cdot\nabla b)}\right\|_{L^2} d\tau \\
&\leq C(1+t)^{-1}\left(\|(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|(\partial_i u_0, \partial_i b_0)\|_{L^2}\right) \\
&\quad + C\int_0^t \left[(1+t-\tau)^{-1}\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}} + e^{-c(t-\tau)}\|\partial_i(b\cdot\nabla b)\|_{L^2}\right] d\tau \\
&\leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-\frac{11}{8}}\, d\tau \\
&\quad + CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{11}{6}}\, d\tau \\
&\leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2(1+t)^{-1},
\end{aligned}$$

from which it follows that

$$\|\partial_i u(t)\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1}, \quad \forall\ i \in \{2,3\},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III. The decay estimates of $\|\partial_i\partial_j u\|_{L^2}$ with $i,j \in \{1,2,3\}$

In terms of (2.7), we see that for $i,j \in \{1,2,3\}$,

$$\begin{aligned}
\|\partial_i\partial_j u\|_{L^2} &= \|\widehat{\partial_{ij}u}\|_{L^2} = \|\xi_i\xi_j\widehat{u}\|_{L^2} \\
&\leq \left\|\xi_i\xi_j\widehat{K_1}(t)\widehat{u_0}\right\|_{L^2} + \left\|\xi_i\xi_j\widehat{K_2}(t)\widehat{b_0}\right\|_{L^2}
\end{aligned}$$

$$+\int_0^t \left\|\xi_i\xi_j\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} d\tau$$

$$+\int_0^t \left\|\xi_i\xi_j\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \mathrm{III}_{ij,m}. \tag{4.34}$$

Step III-1. The decay estimates of $\|\partial_3\partial_j u\|_{L^2}$ with $j \in \{2,3\}$

Since $|\xi_3\xi_j| \le \xi_\nu^2$ for $j \in \{2,3\}$, using (4.23), (4.24) and Corollary 2.1 with $\alpha = 2$, we deduce from (4.34) with $i = 3$ and $j \in \{2,3\}$ that

$$\begin{aligned}\mathrm{III}_{3j,1} + \mathrm{III}_{3j,2} &\le C\|\xi_3\xi_j e^{-c\xi_\nu^2 t}(\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct}\left\|\xi_3\xi_j(\widehat{u_0}, \widehat{b_0})\right\|_{L^2}\\ &\le C(1+t)^{-\frac{3}{2}}\left(\|(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|\partial_3\partial_j(u_0,b_0)\|_{L^2}\right)\\ &\le C\varepsilon(1+t)^{-\frac{3}{2}}.\end{aligned} \tag{4.35}$$

Thanks to (2.10) and (2.12)–(2.14), to bound $\mathrm{III}_{3j,3}$ and $\mathrm{III}_{3j,4}$, it suffices to consider the following two items with $j \in \{2,3\}$:

$$\mathrm{III}_{3j,3} \triangleq \int_0^t \left\|\xi_3\xi_j e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau,$$

$$\mathrm{III}_{3j,4} \triangleq \int_0^t \left\|\xi_3\xi_j e^{-c(1+\xi_3^2)(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau.$$

It is worth mentioning here that the factor $e^{-\xi_3^2 t}$ is taken into account (see (2.14)), which is different from the previous one in (4.24). However, this is of particular importance for the decay estimates involving the x_3-direction. Using Corollary 2.1 and Lemmas 2.4–2.5, we can make use of (4.5) and (4.9) to bound $\mathrm{III}_{3j,3}$ as follows,

$$\begin{aligned}\mathrm{III}_{3j,3} &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}}\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau\\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\|\partial_3(b\cdot\nabla b)\|_{L^2} d\tau\\ &\le CC_0^2\varepsilon^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{11}{8}} d\tau\end{aligned}$$

$$+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{13}{6}}\, d\tau$$

$$\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{2}}, \quad \forall\, j \in \{2,3\}. \tag{4.36}$$

Due to (4.8), (4.9) and the fact that $|\xi_3| e^{-\xi_3^2 t} \leq Ct^{-1/2}$, we see that for $j \in \{2,3\}$,

$$\begin{aligned}
\mathrm{III}_{3j,4} &\leq C \int_0^t |\xi_3| e^{-c\xi_3^2(t-\tau)} e^{-c(t-\tau)} \|\partial_j (b\cdot\nabla b)\|_{L^2} d\tau \\
&\leq CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{6}}\, d\tau \\
&\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}},
\end{aligned} \tag{4.37}$$

since Lemma 2.5, together with Hölder inequality, shows that for $m_1 \in (1,2)$, $\frac{1}{m_1}+\frac{1}{m_2}=1$ and $q>0$,

$$\begin{aligned}
&\int_0^t (t-\tau)^{-\frac{1}{2}} e^{-c(t-\tau)} (1+\tau)^{-q}\, d\tau \\
&\leq \left(\int_0^t (t-\tau)^{-\frac{m_1}{2}} e^{-\frac{cm_1}{2}(t-\tau)} d\tau \right)^{\frac{1}{m_1}} \left(\int_0^t e^{-\frac{cm_2}{2}(t-\tau)} (1+\tau)^{-qm_2}\, d\tau \right)^{\frac{1}{m_2}} \\
&\leq C (1+t)^{-q}.
\end{aligned} \tag{4.38}$$

Thus, collecting (4.35), (4.36) and (4.37) together implies that for $\forall\, j \in \{2,3\}$,

$$\|\partial_3\partial_j u(t)\|_{L^2} \leq \left(C\varepsilon + CC_0^2\varepsilon^2\right)(1+t)^{-\frac{3}{2}} \leq \frac{C_0}{2}\varepsilon (1+t)^{-\frac{3}{2}},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III-2. The decay estimates of $\|\partial_2\partial_l u\|_{L^2}$ with $l=1,2$

In view of (4.23), (4.24) and Corollary 2.1 with $\alpha=1$, we infer from (4.34) with $i=2$ and $j=1$ that

$$\begin{aligned}
\mathrm{III}_{21,1} + \mathrm{III}_{21,2} &\leq C\|\xi_2\xi_1 e^{-c\xi_\nu^2 t} (\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct} \left\| \xi_2\xi_1 (\widehat{u_0}, \widehat{b_0}) \right\|_{L^2} \\
&\leq C(1+t)^{-1} \left(\|\partial_1 (u_0, b_0)\|_{L^2_{x_1} L^1_{x_2,x_3}} + \|\partial_2\partial_1 (u_0, b_0)\|_{L^2} \right) \\
&\leq C\varepsilon (1+t)^{-1}.
\end{aligned} \tag{4.39}$$

For $\mathrm{III}_{21,3}$ and $\mathrm{III}_{21,4}$, we can deal with the terms involving $b\cdot\nabla b$ as follows:

$$\begin{aligned}\mathrm{III}_{21,3}+\mathrm{III}_{21,4}&\le C\int_0^t(1+t-\tau)^{-1}\|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\&\quad+C\int_0^t e^{-c(t-\tau)}\|\partial_2\partial_1(b\cdot\nabla b)\|_{L^2}d\tau\\&\le CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-1}(1+\tau)^{-\frac{5}{4}}d\tau\\&\quad+CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{11}{12}}d\tau\\&\le CC_0^2\varepsilon^2(1+t)^{-\frac{11}{12}},\end{aligned}\tag{4.40}$$

where we have used (4.6), (4.11) Lemmas 2.4 and 2.5. So, combining (4.39) with (4.40) gives

$$\|\partial_2\partial_1 u(t)\|_{L^2}\le C\varepsilon(1+t)^{-1}+CC_0^2\varepsilon^2(1+t)^{-\frac{11}{12}},\quad\forall\ t\ge 1.$$

The decay estimate of $\|\partial_2^2 u\|_{L^2}$ can be achieved in a similar manner by applying Corollary 2.1 with $\alpha=2$, and the decay rate is of the same order (i.e. 11/12), since it is completely determined by the estimate of $\|\partial_2^2(b\cdot\nabla b)\|_{L^2}$ in (4.11). As a result, by choosing C_0 large enough and ε suitably small, we arrive at

$$\|\partial_2\partial_l u(t)\|_{L^2}\le\frac{C_0}{2}\varepsilon(1+t)^{-\frac{11}{12}},\quad\forall\ l\in\{1,2\}.$$

Step III-3. The decay estimate of $\|\partial_3\partial_1 u\|_{L^2}$

Similarly to that in Step III-2, by (4.23), (4.24) and (4.12) we deduce from (4.34) with $i=3$ and $j=1$ that

$$\begin{aligned}\|\partial_3\partial_1 u\|_{L^2}&\le C(1+t)^{-1}\left(\|\partial_1(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}}+\|\partial_1\partial_3(u_0,b_0)\|_{L^2}\right)\\&\quad+C\int_0^t(1+t-\tau)^{-1}\|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\&\quad+C\int_0^t e^{-c(t-\tau)}\|\partial_3\partial_1(b\cdot\nabla b)\|_{L^2}d\tau\end{aligned}$$

$$\leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-1}(1+\tau)^{-\frac{5}{4}}\,d\tau$$

$$+ CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{5}{3}}\,d\tau$$

$$\leq C\varepsilon(1+t)^{-1} + CC_0^2\varepsilon^2(1+t)^{-1} \leq \frac{C_0}{2}\varepsilon(1+t)^{-1},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III-4. The decay estimate of $\|\partial_1^2 u\|_{L^2}$

In order to estimate $\|\partial_1^2 u\|_{L^2}$, by (4.23), (4.24) and Corollary 2.1 with $\alpha = 0$ we first infer from (4.34) with $i = j = 1$ that

$$\begin{aligned}\mathrm{III}_{11,1} + \mathrm{III}_{11,2} &\leq C\|\xi_1^2 e^{-c\xi_\nu^2 t}(\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct}\|\xi_1^2(\widehat{u_0}, \widehat{b_0})\|_{L^2} \\ &\leq C(1+t)^{-\frac{1}{2}}\left(\|\partial_1^2(u_0, b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|\partial_1^2(u_0, b_0)\|_{L^2}\right) \\ &\leq C\varepsilon(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{4.41}$$

Similarly to the derivation of (4.28), it follows from (4.23), (4.24), (4.10), (4.11), Corollary 2.1 and Lemma 2.4 that for any $\delta \in (0,1)$,

$$\begin{aligned}\mathrm{III}_{11,3} + \mathrm{III}_{11,4} &\leq C\int_0^t (1+t-\tau)^{-\frac{1}{2}}\|\partial_1^2(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau \\ &\quad + C\int_0^t e^{-c(t-\tau)}\|\partial_1^2(b\cdot\nabla b)\|_{L^2}d\tau \\ &\leq CC_0\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{7}{8}}d\tau \\ &\quad + CC_0\varepsilon\int_0^t (1+t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}\|\partial_\nu b\|_{H^2}d\tau \\ &\quad + CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{11}{12}}d\tau \\ &\leq CC_0\varepsilon^2(1+t)^{-\frac{3}{8}} + CC_0\varepsilon^2\left(\int_0^t (1+t-\tau)^{-1}(1+\tau)^{-1}d\tau\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq CC_0\varepsilon^2(1+t)^{-\frac{3}{8}} + CC_0\varepsilon^2(1+t)^{-\frac{1-\delta}{2}} + CC_0\varepsilon^2(1+t)^{-\frac{11}{12}}. \quad (4.42)$$

Thus, by choosing $\delta \in (0, 1/4]$, we conclude from (4.41)–(4.42) that

$$\|\partial_1^2 u(t)\|_{L^2} \leq C_3\varepsilon(1+t)^{-\frac{1}{2}} + C_4C_0\varepsilon^2(1+t)^{-\frac{3}{8}}, \quad \forall\, t \geq 1.$$

So, if C_0 and ε are chosen to be such that

$$C_0 \geq 4C_3 \quad \text{and} \quad 0 < \varepsilon \leq (4C_4)^{-1},$$

then it holds that

$$\|\partial_1^2 u(t)\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{3}{8}}, \quad \forall\, t \geq 1.$$

Step IV. The decay estimates of $\|\partial_i\partial_j\partial_k u\|_{L^2}$ with $i, j, k \in \{1, 2, 3\}$

For $i, j, k \in \{1, 2, 3\}$, it readily follows from (2.7) that

$$\begin{aligned}
\|\partial_i\partial_j\partial_k u\|_{L^2} &= \|\widehat{\partial_{ijk}u}\|_{L^2} = \|\xi_i\xi_j\xi_k\widehat{u}\|_{L^2} \\
&\leq \left\|\xi_i\xi_j\xi_k\widehat{K_1}(t)\widehat{u_0}\right\|_{L^2} + \left\|\xi_i\xi_j\xi_k\widehat{K_2}(t)\widehat{b_0}\right\|_{L^2} \\
&\quad + \int_0^t \left\|\xi_i\xi_j\xi_k\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} d\tau \\
&\quad + \int_0^t \left\|\xi_i\xi_j\xi_k\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \mathrm{IV}_{ijk,m}. \quad (4.43)
\end{aligned}$$

Step IV-1. The decay estimates of $\|\partial_l\partial_2\partial_3 u\|_{L^2}$ with $l \in \{1, 2\}$

Taking $i = 1, j = 2$ and $k = 3$ in (4.43), by (4.23), (4.24) and Corollary 2.1 with $\alpha = 2$ we derive from (4.6) that

$$\begin{aligned}
\mathrm{IV}_{123,1} + \mathrm{IV}_{123,2} &\leq C\|\xi_1\xi_2\xi_3 e^{-c\xi_\nu^2 t}(\widehat{u_0}, \widehat{b_0})\|_{L^2} + Ce^{-ct}\left\|\xi_1\xi_2\xi_3(\widehat{u_0}, \widehat{b_0})\right\|_{L^2} \\
&\leq C(1+t)^{-\frac{3}{2}}\|\partial_1(u_0, b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + Ce^{-ct}\|\partial_1\partial_2\partial_3(u_0, b_0)\|_{L^2} \\
&\leq C\varepsilon(1+t)^{-\frac{3}{2}}. \quad (4.44)
\end{aligned}$$

Using (2.14), (4.23) and (4.38), we deduce from (4.6), (4.11) and Corollary 2.1 in a similar manner as the derivation of (4.37) that

$$\begin{aligned}
\mathrm{IV}_{123,3}+\mathrm{IV}_{123,4} &\le C\int_0^t \left\|\xi_1\xi_2\xi_3 e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau \\
&\quad + C\int_0^t \left\|\xi_1\xi_2\xi_3 e^{-c(1+\xi_3^2)(t-\tau)}\widehat{b\cdot\nabla b}\right\|_{L^2} d\tau \\
&\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}} \|\partial_1\partial_2(b\cdot\nabla b)\|_{L^2} d\tau \\
&\le CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{5}{4}} d\tau \\
&\quad + CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{11}{12}} d\tau \\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}} + CC_0\varepsilon^2(1+t)^{-\frac{11}{12}},
\end{aligned}$$

which, together with (4.44), gives

$$\begin{aligned}
\|\partial_1\partial_2\partial_3 u(t)\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}} + CC_0\varepsilon^2(1+t)^{-\frac{11}{12}} \\
&\le C\varepsilon(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2(1+t)^{-\frac{11}{12}} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{11}{12}}, \qquad (4.45)
\end{aligned}$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Analogously, applying Corollary 2.1 with $\alpha = 3$ to the terms $\||\xi_2\xi_2\xi_3|e^{-c\xi_\nu^2 t}\widehat{u_0}\|_{L^2}$ and $\||\xi_2\xi_2\xi_3|e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}\|_{L^2}$, we can make use of (4.5) and (4.11) to obtain the same estimate as that in (4.45) for $\|\partial_2\partial_2\partial_3 u(t)\|_{L^2}$. Hence,

$$\|\partial_l\partial_2\partial_3 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-\frac{11}{12}}, \quad \forall\, l\in\{1,2\}.$$

Step IV-2. The decay estimate of $\|\partial_1^2\partial_3 u\|_{L^2}$

To derive the decay estimate of $\|\partial_1^2\partial_3 u\|_{L^2}$, we first infer from (2.14) and (4.23) and Corollary 2.1 with $\alpha = 1$ that

$$\begin{aligned}
\|\partial_1^2\partial_3 u\|_{L^2} &\le C(1+t)^{-1}\left(\|\partial_1^2(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|\partial_1^2\partial_3(u_0,b_0)\|_{L^2}\right) \\
&\quad + C\int_0^t \left\|\xi_1^2\xi_3 e^{-c\xi_\nu^2(t-\tau)}\widehat{b\cdot\nabla b}(\tau)\right\|_{L^2} d\tau
\end{aligned}$$

$$+C\int_0^t \left\| \xi_1^2\xi_3 e^{-c(1+\xi_3^2)(t-\tau)}\widehat{b\cdot\nabla b}(\tau)\right\|_{L^2} d\tau$$
$$\leq C\varepsilon(1+t)^{-1} + \mathrm{IV}_{113,3} + \mathrm{IV}_{113,4}. \tag{4.46}$$

Noting that

$$\int_0^t \|\partial_\nu b\|_{H^2}^2 d\tau \leq C\varepsilon^2$$

due to Theorem 1.1, by (4.10), (4.11), (4.38), Corollary 2.1 with $\alpha = 1$ and Lemma 2.4, we obtain in a manner similar to the derivation of (4.28) that

$$\begin{aligned}
\mathrm{IV}_{113,3} + \mathrm{IV}_{113,4} &\leq C\int_0^t (1+t-\tau)^{-1} \|\partial_1^2 (b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_1^2 (b\cdot\nabla b)\|_{L^2} d\tau \\
&\leq CC_0\varepsilon \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{1}{2}} \|\partial_\nu b\|_{H^2} d\tau \\
&\quad + CC_0\varepsilon^2 \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-\frac{7}{8}} d\tau \\
&\quad + CC_0\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{12}} d\tau \\
&\leq CC_0\varepsilon^2 (1+t)^{-\frac{7}{8}+\delta} + CC_0\varepsilon^2 (1+t)^{-\frac{11}{12}} \\
&\quad + CC_0\varepsilon^2 \left(\int_0^t (1+t-\tau)^{-2} (1+\tau)^{-1} d\tau \right)^{\frac{1}{2}} \\
&\leq CC_0\varepsilon^2 (1+t)^{-\frac{1}{2}},
\end{aligned} \tag{4.47}$$

provided δ is chosen to be such that $0 < \delta \leq \frac{3}{8}$. Combining with (4.46) and (4.47) gives rise to

$$\|\partial_1^2\partial_3 u(t)\|_{L^2} \leq C\varepsilon(1+t)^{-1} + CC_0\varepsilon^2 (1+t)^{-\frac{1}{2}} \leq \frac{C_0}{2}\varepsilon (1+t)^{-\frac{1}{2}},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step IV-3. The decay estimate of $\|\partial_1\partial_3^2 u\|_{L^2}$

Similarly to the arguments in Step IV-2, choosing C_0 large enough and ε suitably small, we deduce from (4.6) and (4.12) that

$$\begin{aligned}\|\partial_1\partial_3^2 u\|_{L^2} \le{}& C(1+t)^{-\frac{3}{2}}\left(\|\partial_1(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}}+\|\partial_1\partial_3^2(u_0,b_0)\|_{L^2}\right)\\&+C\int_0^t(1+t-\tau)^{-\frac{3}{2}}\|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2x_3}}d\tau\\&+C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_1\partial_3(b\cdot\nabla b)\|_{L^2}d\tau\\\le{}& C\varepsilon(1+t)^{-\frac{3}{2}}+CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{5}{4}}d\tau\\&+CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{5}{3}}d\tau\\\le{}&\left(C\varepsilon+CC_0^2\varepsilon^2\right)(1+t)^{-\frac{5}{4}}\le\frac{C_0}{2}\varepsilon(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Step IV-4. The decay estimate of $\|\partial_2\partial_3^2 u\|_{L^2}$

In view of (4.5), (4.9), (4.14), (4.38) and Corollary 2.1 with $\alpha=3$, we have from (4.43), (4.23) and (4.24) that

$$\begin{aligned}\|\partial_2\partial_3^2 u\|_{L^2} \le{}& C\varepsilon(1+t)^{-2}+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-2}\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\|\partial_3^2(b\cdot\nabla b)\|_{L^2}d\tau\\&+C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_2\partial_3(b\cdot\nabla b)\|_{L^2}d\tau\\\le{}& C\varepsilon(1+t)^{-2}+CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-2}(1+\tau)^{-\frac{11}{8}}d\tau\end{aligned}$$

$$
\begin{aligned}
&+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{29}{12}}\, d\tau \\
&+ CC_0^2\varepsilon^2 \int_{0}^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{6}}\, d\tau \\
&\le C\varepsilon(1+t)^{-2} + CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}},
\end{aligned}
$$

from which we obtain after choosing C_0 large and ε suitably small that

$$
\|\partial_2\partial_3^2 u(t)\|_{L^2} \le C\varepsilon(1+t)^{-2} + CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}} \le \frac{C_0}{2}\varepsilon (1+t)^{-\frac{11}{6}}.
$$

Step IV-5. The decay estimate of $\|\partial_3^3 u\|_{L^2}$

Analogously to that in Step IV-4, by (4.5) and (4.14) we have

$$
\begin{aligned}
\|\partial_3^3 u\|_{L^2} &\le C\varepsilon(1+t)^{-2} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|b\cdot\nabla b\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_3^2 (b\cdot\nabla b)\|_{L^2} d\tau \\
&\quad + C\int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_3^2 (b\cdot\nabla b)\|_{L^2} d\tau \\
&\le C\varepsilon(1+t)^{-2} + CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} (1+\tau)^{-\frac{11}{8}} d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{29}{12}} d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{29}{12}} d\tau \\
&\le C\varepsilon(1+t)^{-\frac{23}{12}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{23}{12}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{29}{12}},
\end{aligned}
$$

so that, choosing C_0 large enough and ε suitably small, we find

$$
\|\partial_3^3 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon (1+t)^{-\frac{23}{12}}, \quad \forall\, t \ge 1.
$$

4.3. Enhanced decay rates of (u_1, b_1)

In this subsection, we aim to improve the decay rates of (u_1, b_1) by applying Corollaries 2.1 and 2.2.

Part I. Improved decay rate of $\|u_1\|_{L^2}$

We start with the improved decay estimate of $\|u_1\|_{L^2}$. First, it follows from Plancherel's Theorem and (2.7) that

$$\begin{aligned}\|u_1\|_{L^2} = \|\widehat{u_1}\|_{L^2} \le & \left\|\widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2} + \left\|\widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2} \\ & + \int_0^t \left\|\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2} d\tau \\ & + \int_0^t \left\|\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \Lambda_{1,m} \end{aligned} \tag{4.48}$$

Since $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, by (4.23) and (4.24) we can make use of Corollary 2.2 with $\beta = 0$ and Plancherel's Theorem to bound the first two terms on the right-hand side of (4.48) as follows,

$$\begin{aligned}\Lambda_{1,1} + \Lambda_{1,2} &\le C\|e^{-c\xi_\nu^2 t}(\widehat{u_{10}}, \widehat{b_{10}})\|_{L^2} + Ce^{-ct}\|(\widehat{u_{10}}, \widehat{b_{10}})\|_{L^2} \\ &\le C(1+t)^{-\frac{3}{4}} \left(\|(u_0, b_0)\|_{L^1} + \|(u_0, b_0)\|_{L^2}\right) \\ &\le C\varepsilon(1+t)^{-\frac{3}{4}}. \end{aligned} \tag{4.49}$$

Since $N_{11} = \mathbb{P}_1(b \cdot \nabla b) + \mathbb{P}_1(u \cdot \nabla u)$, it holds that

$$\begin{aligned}\Lambda_{1,3} \le & \int_0^t \left\|\widehat{K_1}(t-\tau)\mathbb{P}_1\widehat{(b \cdot \nabla b)}\right\|_{L^2} d\tau \\ & + \int_0^t \left\|\widehat{K_1}(t-\tau)\mathbb{P}_1\widehat{(u \cdot \nabla u)}\right\|_{L^2} d\tau \triangleq \Lambda_{1,31} + \Lambda_{1,32}. \end{aligned} \tag{4.50}$$

Clearly, it suffices to consider the first term $\Lambda_{1,31}$, since the second one can be treated in the same manner due to their mathematical resemblance. To do this, we first infer from (4.3) that

$$\Lambda_{1,31} \le C \int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2 |\xi|^{-2} \xi_k \widehat{b_k b_1} \right\|_{L^2} d\tau$$

$$+C\int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau$$

$$+C\int_0^t \left\| e^{-c(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2|\xi|^{-2}\xi_k \widehat{b_k b_1} \right\|_{L^2} d\tau$$

$$+C\int_0^t \left\| e^{-c(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \Lambda_{1,31m}. \tag{4.51}$$

Due to (4.4), (4.17), (4.18) and the fact that $\nabla \cdot b = 0$, one has

$$\begin{aligned} \|b\cdot\nabla b_1\|_{L^2} &\le \|b_1\|_{L^\infty}\|\partial_1 b_1\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_\nu b_1\|_{L^2} \\ &\le \|b_1\|_{L^\infty}\|\partial_\nu b\|_{L^2} + \|b_\nu\|_{L^\infty}\|\partial_\nu b_1\|_{L^2} \\ &\le CC_0^2\varepsilon^2 (1+t)^{-\frac{13}{6}}. \end{aligned} \tag{4.52}$$

Thus, using Corollary 2.1 with $\alpha = 1$, (4.52) and Lemma 2.4, we see that for any $t \ge 1$,

$$\begin{aligned} \Lambda_{1,311} &\le C\int_0^t (1+t-\tau)^{-1} \left\| \|bb_1\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} d\tau \\ &\quad + C\int_0^t e^{-c(t-\tau)}\|b\cdot\nabla b_1\|_{L^2} d\tau \\ &\le CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-1}(1+\tau)^{-\frac{11}{8}} d\tau \\ &\quad + CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)}(1+\tau)^{-\frac{13}{6}} d\tau \\ &\le CC_0^2\varepsilon^2 (1+t)^{-1} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{4}}, \end{aligned} \tag{4.53}$$

where we have used (2.1), (2.2) and the divergence-free condition $\nabla \cdot b = 0$ to get from (4.4) that

$$\begin{aligned} \left\| \|bb_1\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} &\le \left\| \|bb_1\|_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} \le \left\| \|b\|_{L^2_{x_1}} \|b_1\|_{L^\infty_{x_1}} \right\|_{L^1_{x_2,x_3}} \\ &\le C \left\| \|b\|_{L^2_{x_1}} \|b_1\|^{\frac12}_{L^2_{x_1}} \|\partial_1 b_1\|^{\frac12}_{L^2_{x_1}} \right\|_{L^1_{x_2,x_3}} \le C\|b\|_{L^2}\|b_1\|^{\frac12}_{L^2}\|\partial_1 b_1\|^{\frac12}_{L^2} \\ &\le C\|b\|_{L^2}\|b_1\|^{\frac12}_{L^2}\|\partial_\nu b\|^{\frac12}_{L^2} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{8}}. \end{aligned} \tag{4.54}$$

Similarly, since it follows from (4.4) that

$$\left\| \|bb_\nu\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \le \|b\|_{L^2} \|b_\nu\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_\nu\|_{L^2}^{\frac{1}{2}} \le CC_0^2\varepsilon^2 (1+t)^{-1} \tag{4.55}$$

and

$$\|(\partial_2(bb_2), \partial_3(bb_3))\|_{L^2} \le C\|b\|_{L^\infty} \|\partial_\nu b\|_{L^2} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{23}{12}}, \tag{4.56}$$

by choosing $0 < \delta \le 1/4$, we can bound $\Lambda_{1,312}$ as follows,

$$\begin{aligned}
\Lambda_{1,312} &\le C \int_0^t (1+t-\tau)^{-1} \left\| \|(bb_2, bb_3)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} \left(\|\partial_2(bb_2)\|_{L^2} + \|\partial_3(bb_3)\|_{L^2} \right) d\tau \\
&\le CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-1} d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{23}{12}} d\tau \\
&\le CC_0^2\varepsilon^2 (1+t)^{-1+\delta} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{4.57}$$

Next, by Lemma 2.5, (4.52) and (4.56) we have

$$\begin{aligned}
\Lambda_{1,313} + \Lambda_{1,314} &\le C \int_0^t e^{-c(t-\tau)} \left(\|b \cdot \nabla b_1\|_{L^2} + \|\partial_2(bb_2)\|_{L^2} + \|\partial_3(bb_3)\|_{L^2} \right) d\tau \\
&\le CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{23}{12}} d\tau \le CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{4}}.
\end{aligned} \tag{4.58}$$

Thus, based upon (4.53), (4.57) and (4.58), we conclude from (4.51) that

$$\Lambda_{1,31} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{4}}, \quad \forall\, t \ge 1.$$

The same estimate also holds for $\Lambda_{1,32}$. Thus, it follows from (4.50) that

$$\Lambda_{1,3} \le CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{4}}, \quad \forall\, t \ge 1. \tag{4.59}$$

We proceed to estimate $\Lambda_{1,4}$. Noting that

$$\begin{aligned} N_{21} &= b\cdot\nabla u_1 - u\cdot\nabla b_1 = \nabla\cdot(bu_1 - ub_1) \\ &= \partial_2(b_2u_1 - u_2b_1) + \partial_3(b_3u_1 - u_3b_1), \end{aligned} \tag{4.60}$$

we have from (4.23) and (4.24) that

$$\begin{aligned} \Lambda_{1,4} \le & \int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)}\xi_2\widehat{b_2u_1}(\tau)\right\|_{L^2} d\tau + \int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)}\xi_2\widehat{u_2b_1}(\tau)\right\|_{L^2} d\tau \\ & + \int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)}\xi_3\widehat{b_3u_1}(\tau)\right\|_{L^2} d\tau + \int_0^t \left\| e^{-c\xi_\nu^2(t-\tau)}\xi_3\widehat{u_3b_1}(\tau)\right\|_{L^2} d\tau \\ & + \int_0^t \left\| e^{-c(t-\tau)}\xi_2\widehat{b_2u_1}(\tau)\right\|_{L^2} d\tau + \int_0^t \left\| e^{-c(t-\tau)}\xi_2\widehat{u_2b_1}(\tau)\right\|_{L^2} d\tau \\ & + \int_0^t \left\| e^{-c(t-\tau)}\xi_3\widehat{b_3u_1}(\tau)\right\|_{L^2} d\tau + \int_0^t \left\| e^{-c(t-\tau)}\xi_3\widehat{u_3b_1}(\tau)\right\|_{L^2} d\tau \\ \triangleq & \ \Lambda_{1,41} + \ldots + \Lambda_{1,48}. \end{aligned} \tag{4.61}$$

It is easily checked that the estimates (4.54) and (4.52) hold for bu_1 and $\partial_\nu(b_\nu u_1)$, respectively. So, analogously to the derivations of (4.53) and (4.58), we have

$$\begin{aligned} & \Lambda_{1,41} + \Lambda_{1,43} + \Lambda_{1,45} + \Lambda_{1,47} \\ & \le C\int_0^t \left[(1+t-\tau)^{-1}\|b_\nu u_1\|_{L^2_{x_1}L^1_{x_2,x_3}} + e^{-c(t-\tau)}\|\partial_\nu(b_\nu u_1)\|_{L^2}\right] d\tau \\ & \le CC_0^2\varepsilon^2\int_0^t \left[(1+t-\tau)^{-1}(1+\tau)^{-\frac{11}{8}} + e^{-c(t-\tau)}(1+\tau)^{-\frac{13}{6}}\right] d\tau \\ & \le CC_0^2\varepsilon^2\,(1+t)^{-1}. \end{aligned} \tag{4.62}$$

The terms involving u_2b_1 and u_3b_1 can be treated similarly and admit the same decay estimate as that in (4.62). Consequently,

$$\Lambda_{1,4} \le CC_0^2\varepsilon^2(1+t)^{-1}, \quad \forall\, t \ge 1,$$

which, together with (4.48), (4.49) and (4.59), yields

$$\|u_1(t)\|_{L^2} \le C_5\varepsilon(1+t)^{-\frac{3}{4}} + C_6C_0^2\varepsilon^2\,(1+t)^{-\frac{3}{4}}, \quad \forall\, t \ge 1. \tag{4.63}$$

Thus, if C_0 and ε are chosen to be such that

$$C_0 \geq 4C_5 \quad \text{and} \quad 0 < \varepsilon \leq (4C_6C_0)^{-1},$$

then one infers from (4.63) that

$$\|u_1(t)\|_{L^2} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{3}{4}}, \quad \forall\, t \geq 1.$$

Part II. Improved decay rates of $\|\partial_\nu u_1\|_{L^2}$ with $\nu \in \{2,3\}$

Let $\xi_\nu = (\xi_2, \xi_3)$ and $\xi_\nu^2 = \xi_2^2 + \xi_3^2$. This step aims to improve the decay estimate of $\|\partial_\nu u_1\|_{L^2}$ by applying Corollaries 2.1, 2.2 and (4.3). First, it is easy to see that

$$\begin{aligned}
\|\partial_\nu u_1\|_{L^2} &= \|\widehat{\partial_\nu u_1}\|_{L^2} = \|\xi_\nu \widehat{u_1}\|_{L^2} \\
&\leq \left\|\xi_\nu \widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2} + \left\|\xi_\nu \widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2} \\
&\quad + \int_0^t \left\|\xi_\nu \widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2} d\tau \\
&\quad + \int_0^t \left\|\xi_\nu \widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \Lambda_{2,m}.
\end{aligned} \tag{4.64}$$

Since $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, it follows from Corollary 2.2 with $\beta = 1$ that

$$\begin{aligned}
\Lambda_{2,1} + \Lambda_{2,2} &\leq C\||\xi_\nu| e^{-c\xi_\nu^2 t}(\widehat{u_{10}}, \widehat{b_{10}})\|_{L^2} + Ce^{-ct}\|\partial_\nu(\widehat{u_{10}}, \widehat{b_{10}})\|_{L^2} \\
&\leq C(1+t)^{-\frac{5}{4}}\left(\|(u_0, b_0)\|_{L^1} + \|\partial_\nu(u_0, b_0)\|_{L^2}\right) \\
&\leq C\varepsilon(1+t)^{-\frac{5}{4}}.
\end{aligned} \tag{4.65}$$

To estimate $\Lambda_{2,3}$, similarly to the treatment of (4.50) in Part I, we only deal with the term associated with $\widehat{\mathbb{P}_1(b\cdot\nabla b)}$ in $\widehat{N_{11}}$, which is still denoted by $\Lambda_{2,3}$ for simplicity. By (4.23) and (4.24), we infer from (4.3) that

$$\begin{aligned}
\Lambda_{2,3} &\leq C\int_0^t \left\|\xi_\nu e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2 |\xi|^{-2} \xi_k \widehat{b_k b_1}\right\|_{L^2} d\tau \\
&\quad + C\int_0^t \left\|\xi_\nu e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1 |\xi|^{-2} \xi_k \xi_l \widehat{b_k b_l}\right\|_{L^2} d\tau \\
&\quad + C\int_0^t \left\|\xi_\nu e^{-c(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2 |\xi|^{-2} \xi_k \widehat{b_k b_1}\right\|_{L^2} d\tau
\end{aligned}$$

$$+C\int_0^t \left\| \xi_\nu e^{-c(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \Lambda_{2,3m}. \tag{4.66}$$

Using Corollary 2.1 with $\alpha = 2$, Lemmas 2.4-2.5, (4.8), (4.9), and (4.54), we obtain

$$\begin{aligned}
\Lambda_{2,31} + \Lambda_{2,33} + \Lambda_{2,34} &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \left\| \|bb_1\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} \left\| \partial_\nu \widehat{(b\cdot\nabla b)} \right\|_{L^2} d\tau \\
&\le CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-\frac{11}{6}} d\tau \\
&\le CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{4}}, \quad \forall\, t \ge 1.
\end{aligned} \tag{4.67}$$

The treatment of $\Lambda_{2,32}$ is slightly different. Indeed, noting that

$$\begin{aligned}
\Lambda_{2,32} &\le C\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \left\| \xi_\nu e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \\
&\le C\int_0^{\frac{t}{2}} \left\| \xi_\nu^2 e^{-c\xi_\nu^2(t-\tau)} \widehat{bb_\nu} \right\|_{L^2} d\tau + C\int_{\frac{t}{2}}^t \left\| \xi_\nu e^{-c\xi_\nu^2(t-\tau)} \widehat{\partial_\nu(bb_\nu)} \right\|_{L^2} d\tau,
\end{aligned}$$

we infer from Corollary 2.1, (4.55) and (4.56) that for any $0 < \delta < 1$,

$$\begin{aligned}
\Lambda_{2,32} &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} \left\| \|bb_\nu\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} d\tau \\
&\quad + C\int_{\frac{t}{2}}^t \left\| \xi_\nu e^{-c\xi_\nu^2(t-\tau)} \right\|_{L^\infty} \|\partial_\nu(bb_\nu)\|_{L^2} d\tau \\
&\le CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-1} d\tau
\end{aligned}$$

$$+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{23}{12}}\, d\tau$$
$$\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{2}+\delta} + CC_0^2\varepsilon^2 (1+\tau)^{-\frac{17}{12}},$$

and hence, we easily deduce by choosing $0 < \delta \leq 1/4$ that

$$\Lambda_{2,32} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{4}}. \tag{4.68}$$

Thus, substituting (4.67) and (4.68) into (4.66) gives

$$\Lambda_{2,3} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{4}}. \tag{4.69}$$

For $\Lambda_{2,4}$, analogously to treatment of (4.61) in Part I, we only deal with the terms associated with $\widehat{b_2u_1}$ and $\widehat{b_3u_1}$ in $\widehat{N_{21}}$. By (4.60) we have

$$\Lambda_{2,4} \leq \int_0^t \left\| \xi_\nu e^{-c\xi_\nu^2(t-\tau)} \xi_2 \widehat{b_2u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_\nu e^{-c\xi_\nu^2(t-\tau)} \xi_3 \widehat{b_3u_1} \right\|_{L^2} d\tau$$
$$+ \int_0^t \left\| \xi_\nu e^{-c(t-\tau)} \xi_2 \widehat{b_2u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_\nu e^{-c(t-\tau)} \xi_3 \widehat{b_3u_1} \right\|_{L^2} d\tau d\tau.$$

The estimate of $\Lambda_{2,4}$ is based on the estimates of $\|\partial_i\partial_2(b_2u_1)\|_{L^2}$ and $\|\partial_i\partial_3(b_3u_1)\|_{L^2}$ with $i = 2, 3$. Noting that (4.4), together with (4.21), implies

$$\|\partial_\nu b\|_{L^2_{x_1,x_3}L^\infty_{x_2}} \leq C\|\partial_\nu b\|_{L^2}^{\frac{1}{2}} \|\partial_2\partial_\nu b\|_{L^2}^{\frac{1}{2}} \leq CC_0\varepsilon(1+t)^{-\frac{23}{24}} \tag{4.70}$$

and

$$\|\partial_\nu u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \leq C\|\partial_\nu u_1\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_\nu u_1\|_{L^2}^{\frac{1}{4}} \|\partial_3\partial_\nu u_1\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_3\partial_\nu u_1\|_{L^2}^{\frac{1}{4}}$$
$$\leq CC_0\varepsilon(1+t)^{-\frac{29}{24}}. \tag{4.71}$$

Thus, it follows from (4.70), (4.71), (4.17), (4.18) and (4.4) that for $\nu \in \{2, 3\}$,

$$\|\partial_2\partial_\nu (b_2u_1)\|_{L^2} + \|\partial_3\partial_\nu (b_3u_1)\|_{L^2}$$
$$\leq C\|b\|_{L^\infty} (\|\partial_2\partial_\nu u_1\|_{L^2} + \|\partial_3\partial_\nu u_1\|_{L^2})$$
$$+ C\|u_1\|_{L^\infty} (\|\partial_2\partial_\nu b\|_{L^2} + \|\partial_3\partial_\nu b\|_{L^2})$$
$$+ C\|\partial_\nu u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \|\partial_\nu b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}$$
$$\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{6}}. \tag{4.72}$$

With the help of (4.54) and (4.72), similarly to the derivation of (4.62), we deduce

$$\begin{aligned}\Lambda_{2,4} &\le C\int_0^t (1+t-\tau)^{-\frac{3}{2}}\left(\Big\|\|b_2u_1\|_{L^1_{x_2,x_3}}\Big\|_{L^2_{x_1}}+\Big\|\|b_3u_1\|_{L^1_{x_2,x_3}}\Big\|_{L^2_{x_1}}\right)d\tau\\ &\quad + C\int_0^t e^{-c(t-\tau)}\left(\|\partial_\nu\partial_2\left(b_2u_1\right)\|_{L^2}+\|\partial_\nu\partial_3\left(b_3u_1\right)\|_{L^2}\right)d\tau\\ &\le CC_0^2\varepsilon^2\int_0^t\left[(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{11}{8}}+e^{-c(t-\tau)}(1+\tau)^{-\frac{11}{6}}\right]d\tau\\ &\le CC_0^2\varepsilon^2(1+t)^{-\frac{11}{8}}. \end{aligned}\tag{4.73}$$

Plugging (4.65), (4.69) and (4.73) into (4.64), choosing C_0 large enough and ε suitably small, we get that for $\nu\in\{2,3\}$,

$$\|\partial_\nu u_1(t)\|_{L^2}\le\left(C\varepsilon+CC_0^2\varepsilon^2\right)(1+t)^{-\frac{5}{4}}\le\frac{C_0}{2}\varepsilon(1+t)^{-\frac{5}{4}}.$$

Part III. Improved decay rates of $\|\partial_3\partial_i u_1\|_{L^2}$ with $i\in\{2,3\}$

This step aims to improve the decay estimates of $\|\partial_3\partial_i u_1\|_{L^2}$ with $i=2,3$. First of all, we have by (2.7) that

$$\begin{aligned}\|\partial_3\partial_i u_1\|_{L^2}&=\|\widehat{\partial_3\partial_i u_1}\|_{L^2}=\|\xi_3\xi_i\widehat{u_1}\|_{L^2}\\ &\le\Big\|\xi_3\xi_i\widehat{K_1}(t)\widehat{u_{10}}\Big\|_{L^2}+\Big\|\xi_3\xi_i\widehat{K_2}(t)\widehat{b_{10}}\Big\|_{L^2}\\ &\quad+\int_0^t\Big\|\xi_3\xi_i\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\Big\|_{L^2}d\tau\\ &\quad+\int_0^t\Big\|\xi_3\xi_i\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\Big\|_{L^2}d\tau\triangleq\sum_{m=1}^4\Lambda_{3,m}. \end{aligned}\tag{4.74}$$

For $i\in\{2,3\}$, we can apply (4.23), (4.24) and Corollary 2.2 with $\beta=2$ to get

$$\begin{aligned}\Lambda_{3,1}+\Lambda_{3,2}&\le C\||\xi_\nu|^2e^{-c\xi_\nu^2t}\widehat{u_{10}}\|_{L^2}+Ce^{-ct}\|\xi_3\xi_i\widehat{u_{10}}\|_{L^2}\\ &\le C(1+t)^{-\frac{7}{4}}\left(\|u_0\|_{L^1}+\|\partial_3\partial_iu_0\|_{L^2}\right)\le C\varepsilon(1+t)^{-\frac{7}{4}}. \end{aligned}\tag{4.75}$$

Analogously to that in Part II, to deal with $\Lambda_{3,3}$ and $\Lambda_{3,4}$, it suffices to consider the terms associated with $\widehat{\mathbb{P}_1(b\cdot\nabla b)}$ in $\widehat{N_{11}}$ and $\widehat{b_2u_1},\widehat{b_3u_1}$ in $\widehat{N_{21}}$ (still denoted by $\Lambda_{3,3}$ and $\Lambda_{3,4}$). First, it follows from (4.3) and Proposition 2.1 that

$$\begin{aligned}\Lambda_{3,3} &\triangleq \int_0^t \left\|\xi_3\xi_i\widehat{K_1}(t-\tau)\mathbb{P}_1\widehat{(b\cdot\nabla b)}\right\|_{L^2} d\tau\\ &\le C\int_0^t \left\|\xi_3\xi_i e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3 \xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2} d\tau\\ &\quad + C\int_0^t \left\|\xi_3\xi_i e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3 \xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\right\|_{L^2} d\tau\\ &\quad + C\int_0^t \left\|\xi_3\xi_i e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3 \xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2} d\tau\\ &\quad + C\int_0^t \left\|\xi_3\xi_i e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3 \xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\right\|_{L^2} d\tau\\ &\triangleq \Lambda_{3,31}+\Lambda_{3,32}+\Lambda_{3,33}+\Lambda_{3,34}. \end{aligned} \tag{4.76}$$

To bound $\Lambda_{3,31}$, we first observe from (4.21) and (4.4) that

$$\begin{aligned}\|\partial_3(b\cdot\nabla b_1)\|_{L^2} &\le \|b\|_{L^\infty}\|\partial_3\nabla b_1\|_{L^2} + C\|\partial_3 b\|_{L^2}^{\frac12}\|\partial_2\partial_3 b\|_{L^2}^{\frac12}\\ &\quad\times\|\nabla b_1\|_{L^2}^{\frac14}\|\partial_1\nabla b_1\|_{L^2}^{\frac14}\|\partial_3\nabla b_1\|_{L^2}^{\frac14}\|\partial_1\partial_3\nabla b_1\|_{L^2}^{\frac14}\\ &\le CC_0^2\varepsilon^2(1+t)^{-\frac73}, \end{aligned} \tag{4.77}$$

which, together with (4.54) and Corollary 2.1, shows that

$$\begin{aligned}\Lambda_{3,31} &\le \left(\int_0^{\frac t2}+\int_{\frac t2}^t\right)\left\|\xi_3\xi_i e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2} d\tau\\ &\le C\int_0^{\frac t2}\left\||\xi_\nu|^3 e^{-c\xi_\nu^2(t-\tau)}\widehat{bb_1}\right\|_{L^2} d\tau\\ &\quad + C\int_{\frac t2}^t\left\|\xi_\nu e^{-c\xi_\nu^2(t-\tau)}\partial_3\widehat{(b\cdot\nabla b_1)}\right\|_{L^2} d\tau\\ &\le C\int_0^{\frac t2}(1+t-\tau)^{-2}\|bb_1\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \end{aligned}$$

$$
\begin{aligned}
&+ C \int_{\frac{t}{2}}^{t} \left\| \xi_\nu e^{-c\xi_\nu^2 (t-\tau)} \right\|_{L^\infty} \|\partial_3 (b \cdot \nabla b_1)\|_{L^2} d\tau \\
&\leq C C_0^2 \varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} (1+\tau)^{-\frac{11}{8}} \, d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{7}{3}} \, d\tau \\
&\leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{7}{4}} .
\end{aligned} \tag{4.78}
$$

In a similar manner, by choosing $0 < \delta \leq 1/4$ we deduce that for $i \in \{2, 3\}$,

$$
\begin{aligned}
\Lambda_{3,32} &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|b b_\nu\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_3 \partial_\nu (b b_\nu)\|_{L^2} d\tau \\
&\leq C C_0^2 \varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} (1+\tau)^{-1} \, d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{7}{3}} \, d\tau \\
&\leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{7}{4}} ,
\end{aligned} \tag{4.79}
$$

where we have used (4.55) and the following inequality (due to (4.4)):

$$
\begin{aligned}
\|\partial_3 \partial_\nu (b b_\nu)\|_{L^2} &\leq C \|b\|_{L^\infty} \|\partial_3 \partial_\nu b\|_{L^2} + C \||\partial_3 b||\partial_\nu b|\|_{L^2} \\
&\leq C \|b\|_{L^\infty} \|\partial_3 \partial_\nu b\|_{L^2} + C \|\partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times \|\partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_3 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_3 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \\
&\leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{7}{3}} .
\end{aligned}
$$

Similarly to that in (4.37), by (4.8), (4.9) and (4.38) we obtain

$$
\Lambda_{3,33} + \Lambda_{3,34} \leq C \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_i (b \cdot \nabla b)\|_{L^2} d\tau
$$

$$\leq CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau\ )^{-\frac{11}{6}}d\tau$$
$$\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{7}{4}}\,,\quad \forall\ i\in\{2,3\}. \tag{4.80}$$

Dragging (4.78), (4.79) and (4.80) into (4.76), we see that for $i\in\{2,3\}$,

$$\Lambda_{3,3}\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{7}{4}}\,. \tag{4.81}$$

Next, we deal with $\Lambda_{3,4}$. First, we make use of (4.4), (4.17), (4.18) and the divergence-free condition $\nabla\cdot u=0$ in a manner as that in (4.15) to get

$$\begin{aligned}
&\|\partial_2\partial_3(b_2u_1)\|_{L^2}+\|\partial_3^2(b_3u_1)\|_{L^2}\\
&\quad\leq\|b\|_{L^\infty}\|\partial_\nu\partial_3u_1\|_{L^2}+\|u_1\|_{L^\infty}\|\partial_\nu\partial_3b\|_{L^2}\\
&\qquad+\|\partial_2u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3b_2\|_{L^\infty_{x_1,x_2}L^2_{x_3}}+\|\partial_3u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2b_2\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\qquad+2\|\partial_3u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3b_3\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&\quad\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{125}{48}}\,.
\end{aligned}\tag{4.82}$$

Analogously to the derivations of (4.36) and (4.37), using (4.54), (4.72), (4.82), Proposition 2.1 and Corollary 2.1, we infer from the definition of $\widehat{N_{21}}$ in (4.60) that

$$\begin{aligned}
\Lambda_{3,4}&\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-2}\left\|\|(b_2u_1,b_3u_1)\|_{L^1_{x_2x_3}}\right\|_{L^2_{x_1}}d\tau\\
&\quad+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\left\|\left(\partial_2\partial_3(b_2u_1),\partial_3^2(b_3u_1)\right)\right\|_{L^2}d\tau\\
&\quad+C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\left\|\left(\partial_i\partial_2(b_2u_1),\partial_i\partial_3(b_3u_1)\right)\right\|_{L^2}d\tau\\
&\leq CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-2}(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&\quad+CC_0^2\varepsilon^2\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{125}{48}}\,d\tau\\
&\quad+CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{11}{6}}\,d\tau
\end{aligned}$$

$$\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{7}{4}}\,. \tag{4.83}$$

Thus, plugging (4.75), (4.81) and (4.83) into (4.74), we see that for $i\in\{2,3\}$,

$$\|\partial_3\partial_i u_1(t)\|_{L^2}\leq \left(C\varepsilon+CC_0^2\varepsilon^2\right)(1+t)^{-\frac{7}{4}}\leq \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{7}{4}}\,,$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Part IV. Improved decay rate of $\|\partial_3^3 u_1\|_{L^2}$

The enhanced decay rate of $\|\partial_3^3 u_1\|_{L^2}$ needs more subtle analysis. To do so, we first infer from (2.7) that

$$\begin{aligned}\|\partial_3^3 u_1\|_{L^2}&=\|\widehat{\partial_3^3 u_1}\|_{L^2}=\|\xi_3^3\widehat{u_1}\|_{L^2}\\&\leq\left\|\xi_3^3\widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2}+\left\|\xi_3^3\widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2}\\&\quad+\int_0^t\left\|\xi_3^3\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2}d\tau\\&\quad+\int_0^t\left\|\xi_3^3\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2}d\tau\triangleq\sum_{m=1}^4\Lambda_{4,m},\end{aligned}\tag{4.84}$$

where the first two terms can be easily bounded by Corollary 2.2 with $\beta=3$,

$$\begin{aligned}\Lambda_{4,1}+\Lambda_{4,2}&\leq C\|\xi_3^3e^{-c\xi_\nu^2 t}\widehat{u_{10}}\|_{L^2}+Ce^{-ct}\left\|\xi_3^3\widehat{u_{10}}\right\|_{L^2}\\&\leq C(1+t)^{-\frac{9}{4}}\left(\|u_0\|_{L^1}+\|\partial_3^3u_0\|_{L^2}\right)\leq C\varepsilon(1+t)^{-\frac{9}{4}}.\end{aligned}\tag{4.85}$$

Next, we consider the term $\widehat{\mathbb{P}_1(b\cdot\nabla b)}$ in $\widehat{N_{11}}$, which is still denoted by $\Lambda_{4,3}$ for notational convenience. It follows from (4.3), (4.23) and (2.14) that

$$\begin{aligned}\Lambda_{4,3}&\leq C\int_0^t\left\|\xi_3^3e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_kb_1}\right\|_{L^2}d\tau\\&\quad+C\int_0^t\left\|\xi_3^3e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_kb_l}\right\|_{L^2}d\tau\\&\quad+C\int_0^t\left\|\xi_3^3e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_kb_1}\right\|_{L^2}d\tau\\&\quad+C\int_0^t\left\|\xi_3^3e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_kb_l}\right\|_{L^2}d\tau\end{aligned}$$

$$\triangleq \Lambda_{4,31} + \Lambda_{4,32} + \Lambda_{4,33} + \Lambda_{4,34}. \tag{4.86}$$

Similarly to the derivation of (4.68), by (4.54) we have

$$\begin{aligned}
\Lambda_{4,31} &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} \|bb_1\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_3^2 (b \cdot \nabla b_1)(\tau)\|_{L^2} d\tau \\
&\leq C C_0^2 \varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{4}} d\tau \\
&\leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{9}{4}},
\end{aligned} \tag{4.87}$$

since it follows from (4.4) and the divergence-free condition $\nabla \cdot b = 0$ that

$$\begin{aligned}
\|\partial_3^2 (b \cdot \nabla b_1)\|_{L^2} &\leq \|b\|_{L^\infty} \|\nabla \partial_3^2 b_1\|_{L^2} + \|\partial_3^2 b\|_{L^2_{x_1,x_3} L^\infty_{x_2}} \|\nabla b_1\|_{L^\infty_{x_1,x_3} L^2_{x_2}} \\
&\quad + \|\nabla \partial_3 b_1\|_{L^2_{x_1,x_2} L^\infty_{x_3}} \|\partial_3 b\|_{L^\infty_{x_1,x_2} L^2_{x_3}} \\
&\leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{11}{4}}.
\end{aligned}$$

In a similar manner, applying Corollary 2.1 with $\alpha = 4$, by (4.55) we have

$$\begin{aligned}
\Lambda_{4,32} &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} \|(bb_2, bb_3)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} \left\| \xi_\nu e^{-c\xi_\nu^2 (t-\tau)} \right\|_{L^\infty} \|(\partial_3^3 (bb_2), \partial_3^3 (bb_3))\|_{L^2} d\tau \\
&\leq C C_0^2 \varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-1} d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{17}{6}} d\tau
\end{aligned}$$

$$\leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{2}+\delta} + CC_0^2\varepsilon^2(1+\tau)^{-\frac{7}{3}}$$
$$\leq CC_0^2\varepsilon^2(1+t)^{-\frac{9}{4}}, \tag{4.88}$$

provided $0<\delta\leq 1/4$. Here, we have used (4.17) and (4.4) to get that

$$\begin{aligned}
&\|\partial_3^3(bb_2)\|_{L^2} + \|\partial_3^3(bb_3)\|_{L^2}\\
&\leq C\|b\|_{L^\infty}\|\partial_3^3 b\|_{L^2} + C\|\partial_3^2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_3 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\leq C\|b\|_{L^\infty}\|\partial_3^3 b\|_{L^2} + C\|\partial_3^2 b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_3^2 b\|_{L^2}^{\frac{1}{2}}\\
&\qquad\times\|\partial_3 b\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3 b\|_{L^2}^{\frac{1}{4}}\|\partial_3^2 b\|_{L^2}^{\frac{1}{4}}\|\partial_2\partial_3^2 b\|_{L^2}^{\frac{1}{4}}\\
&\leq CC_0^2\varepsilon^2(1+t)^{-\frac{17}{6}}.
\end{aligned}$$

Due to (4.14) and (4.38), it is easily seen that

$$\begin{aligned}
\Lambda_{4,33}+\Lambda_{4,34} &\leq C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\left\|\partial_3^2(b\cdot\nabla b)\right\|_{L^2}d\tau\\
&\leq CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{29}{12}}d\tau\\
&\leq CC_0^2\varepsilon^2(1+t)^{-\frac{29}{12}},
\end{aligned}$$

which, together with (4.87), (4.88) and (5.36), yields

$$\Lambda_{4,3}\leq CC_0^2\varepsilon^2(1+t)^{-\frac{9}{4}},\quad \forall\ t\geq 1. \tag{4.89}$$

Analogously to that in Part II, we only deal with the terms $\widehat{b_2u_1}, \widehat{b_3u_1}$ in $\Lambda_{4,4}$, which are still marked by $\Lambda_{4,4}$. Due to (4.4), (4.17) and (4.18), one has

$$\begin{aligned}
&\|\partial_3^3(b_2u_1)\|_{L^2} + \|\partial_3^3(b_3u_1)\|_{L^2}\\
&\leq C\|b\|_{L^\infty}\|\partial_3^3 u_1\|_{L^2} + C\|\partial_3 u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3^2 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&\quad + C\|u_1\|_{L^\infty}\|\partial_3^3 b\|_{L^2} + C\|\partial_3^2 u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&\leq CC_0^2\varepsilon^2(1+t)^{-\frac{37}{12}},
\end{aligned}$$

and

$$\begin{aligned}
&\|\partial_2\partial_3^2(b_2u_1)\|_{L^2} + \|\partial_3^3(b_3u_1)\|_{L^2}\\
&\leq \|b\|_{L^\infty}\left(\|\partial_2\partial_3^2 u_1\|_{L^2} + \|\partial_3^3 u_1\|_{L^2}\right) + \|u_1\|_{L^\infty}\left(\|\partial_2\partial_3^2 b_2\|_{L^2} + \|\partial_3^3 b_3\|_{L^2}\right)\\
&\quad + C\|\partial_3^2 b_2\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_2 u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} + C\|\partial_3 u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2\partial_3 b_2\|_{L^2_{x_1,x_2}L^\infty_{x_3}}
\end{aligned}$$

$$
\begin{aligned}
&+C\|\partial_2\partial_3 u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 b_2\|_{L^\infty_{x_1,x_2}L^2_{x_3}}+C\|\partial_3^2 u_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_2 b_2\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\
&+C\|\partial_3 u_1\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_3^2 b_3\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_3^2 u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 b_3\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{11}{4}},
\end{aligned}
$$

which, together with (4.54), (4.38) and (4.72), yields

$$
\begin{aligned}
\Lambda_{4,4}&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{2}}\Big\|\|(b_2u_1,b_3u_1)\|_{L^1_{x_2,x_3}}\Big\|_{L^2_{x_1}}d\tau\\
&\quad+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\big\|\big(\partial_3^3(b_2u_1),\partial_3^3(b_3u_1)\big)\big\|_{L^2}d\tau\\
&\quad+C\int_0^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\big\|\big(\partial_2\partial_3^2(b_2u_1),\partial_3^3(b_3u_1)\big)\big\|_{L^2}d\tau\\
&\le CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{2}}(1+\tau)^{-\frac{11}{8}}d\tau\\
&\quad+CC_0^2\varepsilon^2\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{11}{4}}d\tau\\
&\quad+CC_0^2\varepsilon^2\int_0^{t}e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{37}{12}}d\tau\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{9}{4}}.
\end{aligned}
\tag{4.90}
$$

Thus, inserting (4.85), (4.89) and (4.90) into (4.84), we obtain by choosing C_0 large enough and ε suitably small that

$$
\|\partial_3^3 u_1(t)\|_{L^2}\le\big(C\varepsilon+CC_0^2\varepsilon^2\big)(1+t)^{-\frac{9}{4}}\le\frac{C_0}{2}\varepsilon(1+t)^{-\frac{9}{4}}.
$$

Proof of Theorem 1.2. Based on (4.4) and all the estimates established in Subsections 4.2 and 4.3, we immediately arrive at the desired decay rates of the solutions stated in Theorem 1.2 by applying the bootstrapping arguments. □

5. Improved decay rates and proof of Theorem 1.3

5.1. Estimates of nonlinear terms

This section is devoted to the proof of Theorem 1.3. It is worth pointing out that the following decay rates of (u,b) stated in Theorem 1.2 are optimal and cannot be improved any more,

$$\begin{aligned}
&\|\partial_1^k(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{1}{2}}, \qquad &&\|\partial_i(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-1},\\
&\|(u_1,b_1)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{3}{4}}, &&\|\partial_i(u_1,b_1)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{5}{4}},\\
&\|\partial_i\partial_3(u_1,b_1)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{7}{4}}, &&\|\partial_3^3(u_1,b_1)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{9}{4}},
\end{aligned}\tag{5.1}$$

where $k\in\{0,1\}$ and $i\in\{2,3\}$. Moreover, under the conditions of Theorem 1.3 it has been shown in [24] that for any $t\ge 0$,

$$\|(u,b)(t)\|_{H^4}^2+\int_0^T\left(\|(\partial_3 u,\partial_2 b,\partial_3 b)\|_{H^4}^2+\|\partial_2 u\|_{H^3}^2\right)dt\le\varepsilon^2,\tag{5.2}$$

provided $\varepsilon>0$ is sufficiently small. In addition to (5.1), to prove Theorem 1.3, we make the following further ansatz:

$$\begin{aligned}
&\|\partial_1\partial_i(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-1}, &&\|\partial_i\partial_j(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{3}{2}},\\
&\|\partial_1^2(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{1}{2}}, &&\|\partial_1\partial_i\partial_3(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{3}{2}},\\
&\|\partial_l\partial_2^2(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-1}, &&\|\partial_1^2\partial_i(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-1},\\
&\|\partial_i\partial_j\partial_3(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-2}, &&\|\partial_1^3(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{3}{8}},\\
&\|\partial_1^3\partial_3(u,b)(t)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{1}{2}}, &&\|\partial_1^2\partial_3^2(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{11}{8}},\\
&\|\partial_l\partial_h\partial_2\partial_3(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-1}, &&\|\partial_1\partial_3^3(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-2},\\
&\|\partial_l\partial_2\partial_3^2(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-2}, &&\|\partial_i\partial_3^3(u,b)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{5}{2}}
\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}
&\|\partial_i\partial_j(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{7}{4}}, &&\|\partial_i\partial_3^2(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{9}{4}},\\
&\|\partial_i\partial_3^3(u_1,b_1)\|_{L^2}\le C_0\varepsilon(1+t)^{-\frac{11}{4}},
\end{aligned}\tag{5.4}$$

where $k\in\{0,1\}$, $i,j\in\{2,3\}$ and $l,h\in\{1,2\}$ and C_0 will be specified later. By means of the method of spectral analysis, we can show that the bounds in the ansatz are indeed halved, provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Then the bootstrapping argument asserts that the half upper bounds in the ansatz indeed hold for all time.

We begin with the following estimates of nonlinear terms, which are mathematically crucial for the entire analysis.

Lemma 5.1. *Assume that (u, b) is a smooth solution of the problem (1.2), satisfying (5.1) and (5.3). Then, there exists an absolute positive constant C, such that*

$$\left\| \|b \cdot \nabla b\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{11}{8}}, \tag{5.5}$$

$$\left\| \|\partial_1 (b \cdot \nabla b)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{11}{8}}, \tag{5.6}$$

$$\|\partial_1 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{15}{8}}, \tag{5.7}$$

$$\|\partial_i (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{19}{8}}, \quad \forall\, i \in \{2,3\}, \tag{5.8}$$

$$\left\| \|\partial_1^2 (b \cdot \nabla b)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{5}{4}}, \tag{5.9}$$

$$\|\partial_1^2 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{7}{4}}, \tag{5.10}$$

$$\|\partial_l \partial_2 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-2}, \quad \forall\, l \in \{1,2\}, \tag{5.11}$$

$$\|\partial_1 \partial_3 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{19}{8}}, \tag{5.12}$$

$$\|\partial_i \partial_3 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{23}{8}}, \quad \forall\, i \in \{2,3\}, \tag{5.13}$$

$$\left\| \|\partial_1^3 (b \cdot \nabla b)\|_{L^1_{x_2,x_3}} \right\|_{L^2_{x_1}} \leq C C_0 \varepsilon (1+t)^{-\frac{1}{2}} \|\partial_\nu b\|_{H^3} + C C_0 \varepsilon^2 (1+t)^{-\frac{7}{8}}, \tag{5.14}$$

$$\|\partial_l \partial_g \partial_h (b \cdot \nabla b)\|_{L^2} \leq C C_0 \varepsilon^2 (1+t)^{-1}, \quad \forall\, l,g,h \in \{1,2\}, \tag{5.15}$$

$$\|\partial_1^2 \partial_3 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{15}{8}}, \tag{5.16}$$

$$\|\partial_l \partial_2 \partial_3 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-2}, \quad \forall\, l \in \{1,2\}, \tag{5.17}$$

$$\|\partial_1 \partial_3^2 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{21}{8}}, \tag{5.18}$$

$$\|\partial_2 \partial_3^2 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-3}, \tag{5.19}$$

$$\|\partial_3^3 (b \cdot \nabla b)\|_{L^2} \leq C C_0^2 \varepsilon^2 (1+t)^{-\frac{27}{8}}. \tag{5.20}$$

Proof. The estimates of (5.5) and (5.6) can be shown in the same way as that used for (4.5) and (4.6), using the decay rate of $\|\partial_1^2 (u,b)\|_{L^2} \leq C(1+t)^{-1/2}$ stated in (5.3). To proceed, we need to improve the estimates of $\|b\|_{L^\infty}$ and $\|b_1\|_{L^\infty}$. Similarly to that in Lemma 4.1, by (5.1) and (5.3) we have

$$\begin{aligned} \|b\|_{L^\infty} \leq\ & C \|b\|_{L^2}^{\frac{1}{8}} \|\partial_1 b\|_{L^2}^{\frac{1}{8}} \|\partial_2 b\|_{L^2}^{\frac{1}{8}} \|\partial_3 b\|_{L^2}^{\frac{1}{8}} \\ & \times \|\partial_1 \partial_2 b\|_{L^2}^{\frac{1}{8}} \|\partial_1 \partial_3 b\|_{L^2}^{\frac{1}{8}} \|\partial_2 \partial_3 b\|_{L^2}^{\frac{1}{8}} \|\partial_1 \partial_2 \partial_3 b\|_{L^2}^{\frac{1}{8}} \end{aligned}$$

$$\leq CC_0\varepsilon\,(1+t)^{-1}\,, \tag{5.21}$$

and

$$\begin{aligned}\|b_1\|_{L^\infty} &\leq C\|b_1\|_{L^2}^{\frac{1}{8}}\|\partial_1 b_1\|_{L^2}^{\frac{1}{8}}\|\partial_2 b_1\|_{L^2}^{\frac{1}{8}}\|\partial_3 b_1\|_{L^2}^{\frac{1}{8}}\\ &\quad\times\|\partial_1\partial_2 b_1\|_{L^2}^{\frac{1}{8}}\|\partial_1\partial_3 b_1\|_{L^2}^{\frac{1}{8}}\|\partial_2\partial_3 b_1\|_{L^2}^{\frac{1}{8}}\|\partial_1\partial_2\partial_3 b_1\|_{L^2}^{\frac{1}{8}}\\ &\leq CC_0\varepsilon\,(1+t)^{-\frac{11}{8}}\,.\end{aligned} \tag{5.22}$$

With the help of (5.21), (5.22) and the anisotropic Sobolev inequalities (4.21), we infer from (5.1) and (5.3) that

$$\begin{aligned}\|\partial_1(b\cdot\nabla b)\|_{L^2} &\leq \|b_1\|_{L^\infty}\|\partial_1^2 b\|_{L^2}+\|b_\nu\|_{L^\infty}\|\partial_1\partial_\nu b\|_{L^2}+\|\partial_1 b\cdot\nabla b\|_{L^2}\\ &\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{15}{8}}+\|\partial_1 b_1\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_1 b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\ &\quad+\|\partial_1 b_\nu\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\|\partial_\nu b\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\\ &\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{15}{8}}\,,\end{aligned} \tag{5.23}$$

due to the divergence-free condition $\nabla\cdot b=0$ and the fact that $\partial_1 b_1=-\partial_\nu b_\nu$ with $\partial_\nu=(\partial_2,\partial_3)$ and $b_\nu=(b_2,b_3)$. Analogously,

$$\begin{aligned}&\|\partial_2(b\cdot\nabla b)\|_{L^2}+\|\partial_3(b\cdot\nabla b)\|_{L^2}\\ &\quad\leq\|b_1\|_{L^\infty}\|(\partial_2\partial_1 b,\partial_3\partial_1 b)\|_{L^2}+\|b_\nu\|_{L^\infty}\|(\partial_2\partial_\nu b,\partial_3\partial_\nu b)\|_{L^2}\\ &\qquad+\|(\partial_2 b_1,\partial_3 b_1)\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_1 b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\ &\qquad+\|(\partial_2 b_\nu,\partial_3 b_\nu)\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_\nu b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\ &\quad\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{19}{8}}\,.\end{aligned} \tag{5.24}$$

Based on the Hölder inequality and (4.21) that, by (5.1) and (5.3) we have

$$\begin{aligned}&\left\|\|\partial_1^2\,(b\cdot\nabla b)\,\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\quad\leq\left\|\|\partial_1^2\,(b_\nu\partial_\nu b)\,\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}+\left\|\|\partial_1^2\,(b_1\partial_1 b)\,\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\quad\leq\left\|\|\partial_1^2 b_\nu\|_{L^2_{x_2,x_3}}\|\partial_\nu b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}+2\left\|\|\partial_1 b_\nu\|_{L^2_{x_2,x_3}}\|\partial_\nu\partial_1 b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\qquad+\left\|\|b_\nu\|_{L^2_{x_2,x_3}}\|\partial_\nu\partial_1^2 b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}+\left\|\|\partial_1^2 b_1\|_{L^2_{x_2,x_3}}\|\partial_1 b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\qquad+2\left\|\|\partial_1 b_1\|_{L^2_{x_2,x_3}}\|\partial_1^2 b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}+\left\|\|b_1\|_{L^2_{x_2,x_3}}\|\partial_1^3 b\|_{L^2_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\quad\leq C\|\partial_1^2 b_\nu\|_{L^2}\|\partial_\nu b\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_\nu b\|_{L^2}^{\frac{1}{2}}+C\|\partial_\nu\partial_1 b\|_{L^2}\|\partial_1 b_\nu\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_\nu\|_{L^2}^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+C\|\partial_\nu\partial_1^2 b\|_{L^2}\|b_\nu\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_\nu\|_{L^2}^{\frac{1}{2}}+C\|\partial_1 b\|_{L^2}\|\partial_1^2 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^3 b_1\|_{L^2}^{\frac{1}{2}}\\
&+C\|\partial_1^2 b\|_{L^2}\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1^2 b_1\|_{L^2}^{\frac{1}{2}}+C\|\partial_1^3 b\|_{L^2}\|b_1\|_{L^2}^{\frac{1}{2}}\|\partial_1 b_1\|_{L^2}^{\frac{1}{2}}\\
&\leq CC_0^2\varepsilon^2(1+t)^{-\frac{5}{4}}. \qquad (5.25)
\end{aligned}
$$

Using the divergence-free condition $\nabla\cdot b=0$, (5.1) and (5.3), we deduce

$$
\begin{aligned}
\|\partial_1^2(b\cdot\nabla b)\|_{L^2}\leq{}&C\|b_1\|_{L^\infty}\|\partial_1^3 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_\nu\partial_1^2 b\|_{L^2}\\
&+C\|\partial_1^2 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_1^2 b_\nu\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_\nu b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
&+C\|\partial_1^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_1\partial_\nu b\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_1 b_\nu\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
\leq{}&CC_0^2\varepsilon^2(1+t)^{-\frac{7}{4}}. \qquad (5.26)
\end{aligned}
$$

By the same token, we have

$$
\begin{aligned}
\|\partial_l\partial_2(b\cdot\nabla b)\|_{L^2}\leq{}&C\|b_1\|_{L^\infty}\|\partial_l\partial_2\partial_1 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_l\partial_2\partial_\nu b\|_{L^2}\\
&+C\|\partial_l\partial_2 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_l\partial_2 b_\nu\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_\nu b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
&+C\|(\partial_1\partial_l b,\partial_1\partial_2 b)\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|(\partial_2 b_1,\partial_l b_1)\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|(\partial_2\partial_\nu b,\partial_l\partial_\nu b)\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|(\partial_l b_\nu,\partial_2 b_\nu)\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
\leq{}&CC_0^2\varepsilon^2(1+t)^{-2},\quad\forall\ l\in\{1,2\},\\
\|\partial_1\partial_3(b\cdot\nabla b)\|_{L^2}\leq{}&C\|b_1\|_{L^\infty}\|\partial_1^2\partial_3 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_1\partial_3\partial_\nu b\|_{L^2}\\
&+C\|\partial_1\partial_3 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_1\partial_3 b_\nu\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_\nu b\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
&+C\|\partial_1\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_1 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_3\partial_\nu b\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_1 b_\nu\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
&+C\|\partial_1^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\
&+C\|\partial_1\partial_\nu b\|_{L^\infty_{x_1}L^2_{x_2,x_3}}\|\partial_3 b_\nu\|_{L^2_{x_1}L^\infty_{x_2,x_3}}\\
\leq{}&CC_0^2\varepsilon^2(1+t)^{-\frac{19}{8}},
\end{aligned}
$$

and

$$\begin{aligned}\|\partial_i\partial_3\left(b\cdot\nabla b\right)\|_{L^2} &\le C\|b_1\|_{L^\infty}\|\partial_1\partial_i\partial_3 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_i\partial_3\partial_\nu b\|_{L^2}\\ &\quad+C\|\partial_i\partial_3 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\ &\quad+C\|\partial_i\partial_3 b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\ &\quad+C\|(\partial_1\partial_i b,\partial_1\partial_3 b)\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|(\partial_3 b_1,\partial_i b_1)\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\ &\quad+C\|(\partial_\nu\partial_i b,\partial_\nu\partial_3 b)\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|(\partial_3 b_\nu,\partial_i b_\nu)\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\ &\le CC_0^2\varepsilon^2\left(1+t\right)^{-\frac{23}{8}},\quad\forall\ i\in\{2,3\}.\end{aligned}$$

Analogously to the derivations of (5.25) and (5.26), using the fact that $\nabla\cdot b=0$, we obtain

$$\begin{aligned}&\left\|\|\partial_1^3\left(b\cdot\nabla b\right)\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\le\left\|\|\partial_1^3\left(b_\nu\partial_\nu b\right)\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}+\left\|\|\partial_1^3\left(b_1\partial_1 b\right)\|_{L^1_{x_2,x_3}}\right\|_{L^2_{x_1}}\\ &\le C\|\partial_\nu b\|_{L^2}\|\partial_1^3 b\|_{L^2}^{\frac12}\|\partial_1^4 b\|_{L^2}^{\frac12}+C\|\partial_\nu\partial_1 b\|_{L^2}\|\partial_1^2 b\|_{L^2}^{\frac12}\|\partial_1^3 b\|_{L^2}^{\frac12}\\ &\quad+C\|\partial_\nu\partial_1^2 b\|_{L^2}\|\partial_1 b\|_{L^2}^{\frac12}\|\partial_1^2 b\|_{L^2}^{\frac12}+C\|\partial_\nu\partial_1^3 b\|_{L^2}\|b_\nu\|_{L^2}^{\frac12}\|\partial_1 b_\nu\|_{L^2}^{\frac12}\\ &\quad+C\|\partial_1^4 b\|_{L^2}\|b_1\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac12}\\ &\le CC_0\varepsilon^2\left(1+t\right)^{-\frac12}\|\partial_\nu b\|_{H^3}+CC_0\varepsilon^2\left(1+t\right)^{-\frac78}\end{aligned}$$

and

$$\begin{aligned}\|\partial_1^3\left(b\cdot\nabla b\right)\|_{L^2} &\le C\|b\|_{L^\infty}\|\nabla\partial_1^3 b\|_{L^2}+C\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\nabla\partial_1^2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\ &\quad+C\|\partial_1^2 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\nabla\partial_1 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\ &\quad+C\|\partial_1^3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\nabla b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\ &\le CC_0\varepsilon^2\left(1+t\right)^{-1}.\end{aligned}\tag{5.27}$$

In a similar manner as that used for (5.27), we conclude from (5.1) and (5.3) that

$$\|\partial_l\partial_g\partial_h\left(b\cdot\nabla b\right)\|_{L^2}\le CC_0\varepsilon^2\left(1+t\right)^{-1},\quad\forall\ l,g,h\in\{1,2\}.$$

For (5.16) and (5.17), by (5.1), (5.3) and direct calculations we have

$$\begin{aligned}&\|\partial_1^2\partial_3\left(b\cdot\nabla b\right)\|_{L^2}\\ &\le C\|b_1\|_{L^\infty}\|\partial_1^3\partial_3 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_1^2\partial_3\partial_\nu b\|_{L^2}\\ &\quad+C\|\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1^3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_1^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\end{aligned}$$

$$
\begin{aligned}
&+C\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_1^2\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_1\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&+C\|\partial_\nu\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_1\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_1^2 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{15}{8}},
\end{aligned}
$$

and analogously,

$$
\begin{aligned}
&\|\partial_l\partial_2\partial_3(b\cdot\nabla b)\|_{L^2}\\
&\le C\|b_1\|_{L^\infty}\|\partial_l\partial_2\partial_3\partial_1 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_l\partial_2\partial_3\partial_\nu b\|_{L^2}\\
&\quad+C\|\partial_3 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\nabla\partial_l\partial_2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_2 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\nabla\partial_l\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_l b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\nabla\partial_2\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_l\partial_2 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\nabla\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\quad+C\|\partial_l\partial_3 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\nabla\partial_2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_2\partial_3 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\nabla\partial_l b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_l\partial_2\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\nabla b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\le CC_0^2\varepsilon^2(1+t)^{-2},\quad\forall\, l\in\{1,2\}.
\end{aligned}
$$

Finally, it is easily deduced from (4.21), (5.1), (5.3) and the divergence-free condition $\nabla\cdot b=0$ that

$$
\begin{aligned}
&\|\partial_1\partial_3^2(b\cdot\nabla b)\|_{L^2}\\
&\le C\|b_1\|_{L^\infty}\|\partial_1^2\partial_3^2 b\|_{L^2}+C\|b_\nu\|_{L^\infty}\|\partial_1\partial_3^2\partial_\nu b\|_{L^2}\\
&\quad+C\|\partial_1 b_1\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_1\partial_3^2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_1 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3^2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\quad+C\|\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1^2\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_1\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_1\partial_3 b_1\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_1\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_1\partial_3 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\quad+C\|\partial_3^2 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_3^2 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_1 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_1\partial_3^2 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}+C\|\partial_1\partial_3^2 b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{21}{8}},\\
&\|\partial_2\partial_3^2(b\cdot\nabla b)\|_{L^2}\\
&\le C\|b\|_{L^\infty}\|\nabla\partial_2\partial_3^2 b\|_{L^2}+C\|\partial_\nu b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1\partial_\nu\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_\nu b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu^2\partial_3 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}+C\|\partial_2\partial_3 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_1\partial_3 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\
&\quad+C\|\partial_2\partial_3 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}+C\|\partial_3^2 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\nabla\partial_2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\quad+C\|\partial_2\partial_3^2 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\nabla b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\le CC_0^2\varepsilon^2(1+t)^{-3},
\end{aligned}
$$

and analogously,

$$\begin{aligned}
&\|\partial_3^3 (b\cdot\nabla b)\|_{L^2}\\
&\quad\le C\|b_1\|_{L^\infty}\|\partial_1\partial_3^3 b\|_{L^2} + C\|b_\nu\|_{L^\infty}\|\partial_\nu\partial_3^3 b\|_{L^2}\\
&\qquad + C\|\partial_3 b_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_1\partial_3^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}} + C\|\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_3^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\
&\qquad + C\|\partial_3^2 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1\partial_3 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}} + C\|\partial_3^2 b_\nu\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\
&\qquad + C\|\partial_3^3 b_1\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_1 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}} + C\|\partial_3^3 b_\nu\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\|\partial_\nu b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\\
&\quad\le CC_0^2\varepsilon^2(1+t)^{-\frac{27}{8}}.
\end{aligned}$$

Now, collecting the above estimates together finishes the proof of Lemma 5.1. □

5.2. The decay rates of (u,b)

In this subsection, we prove the decay estimates of (u,b) stated in Theorem 1.3. Analogously to the proof of Theorem 1.2, we only consider the large-time behavior for $t\ge 1$. Moreover, it suffices to deal with the decay estimates of u, since those of b can be achieved similarly.

Step I. The decay rates of $\|\partial_i\partial_j u\|_{L^2}$ with $i,j\in\{1,2,3\}$

We start with the decay rates of the second-order derivatives. It follows from (2.7) that for $\forall\ i,j\in\{1,2,3\}$,

$$\begin{aligned}
\|\partial_i\partial_j u\|_{L^2} &= \|\widehat{\partial_{ij}u}\|_{L^2} = \|\xi_i\xi_j\widehat{u}\|_{L^2}\\
&\le \left\|\xi_i\xi_j\widehat{K_1}(t)\widehat{u_0}\right\|_{L^2} + \left\||\xi_i\xi_j|\widehat{K_2}(t)\widehat{b_0}\right\|_{L^2}\\
&\quad + \int_0^t \left\|\xi_i\xi_j\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2}d\tau\\
&\quad + \int_0^t \left\|\xi_i\xi_j\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2}d\tau. \qquad (5.28)
\end{aligned}$$

Step I-1. The decay rates of $\|\partial_i\partial_j u\|_{L^2}$ with $i,j\in\{2,3\}$.

Similarly the treatment of Step III-1 in Section 4, we only deal with the terms associated with $b\cdot\nabla b$. By (4.23), (4.24) and Corollary 2.1, we infer from (5.5), (5.8), (5.11), (5.13) and (5.28) that for $i,j\in\{2,3\}$,

$$\|\partial_i\partial_j u\|_{L^2}\le C(1+t)^{-\frac{3}{2}}\left(\|(u_0,b_0)\|_{L^2_{x_1}L^1_{x_2,x_3}} + \|\partial_i\partial_j(u_0,b_0)\|_{L^2}\right)$$

$$
\begin{aligned}
&+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{2}}\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&+C\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\|\partial_i(b\cdot\nabla b)\|_{L^2}d\tau\\
&+C\int_0^{t}e^{-c(t-\tau)}\|\partial_i\partial_j(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-\frac{3}{2}}+CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{2}}(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&+CC_0^2\varepsilon^2\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{19}{8}}\,d\tau\\
&+CC_0^2\varepsilon^2\int_0^{t}e^{-c(t-\tau)}(1+\tau)^{-2}\,d\tau\\
&\le C\varepsilon(1+t)^{-\frac{3}{2}}+CC_0^2\varepsilon^2(1+t)^{-\frac{3}{2}}\le\frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{3}{2}},
\end{aligned}
$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step I-2. The decay rates of $\|\partial_1\partial_i u\|_{L^2}$ with $i\in\{2,3\}$

For $i\in\{2,3\}$, using (5.6), (5.11), (5.12) and Corollary 2.1 with $\alpha=1$, we deduce from (5.28) by choosing C_0 large enough and ε suitably small that

$$
\begin{aligned}
\|\partial_1\partial_i u\|_{L^2}&\le C\varepsilon(1+t)^{-1}+C\int_0^{t}(1+t-\tau)^{-1}\|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&+C\int_0^{t}e^{-c(t-\tau)}\|\partial_i\partial_1(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-1}+CC_0^2\varepsilon^2\int_0^{t}(1+t-\tau)^{-1}(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&+CC_0^2\varepsilon^2\int_0^{t}e^{-c(t-\tau)}(1+\tau)^{-2}\,d\tau\\
&\le C\varepsilon(1+t)^{-1}+CC_0^2\varepsilon^2(1+t)^{-1}\le\frac{C_0}{2}\varepsilon\,(1+t)^{-1}.
\end{aligned}
$$

Step I-3. The decay rate of $\|\partial_1^2 u\|_{L^2}$

In view of (5.9), (5.10) and Corollary 2.1 with $\alpha = 0$, we obtain by taking $i = j = 1$ in (5.28) that

$$\begin{aligned}
\|\partial_1^2 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{1}{2}} + C\int_0^t (1+t-\tau)^{-\frac{1}{2}} \|\partial_1^2(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)} \|\partial_1^2(b\cdot\nabla b)\|_{L^2} d\tau \\
&\le C\varepsilon\,(1+t)^{-\frac{1}{2}} + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-1}\,(1+\tau)^{-\frac{5}{4}}\,d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)}\,(1+\tau)^{-\frac{5}{4}}\,d\tau \\
&\le C\varepsilon(1+t)^{-\frac{1}{2}} + CC_0^2\varepsilon^2(1+t)^{-\frac{1}{2}} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{1}{2}},
\end{aligned}$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step II. The decay rates of $\|\partial_i\partial_j\partial_k u\|_{L^2}$ with $i,j,k \in \{1,2,3\}$

It follows from (2.7) that for $\forall\ i,j,k \in \{1,2,3\}$,

$$\begin{aligned}
\|\partial_{ijk} u\|_{L^2} &= \|\widehat{\partial_{ijk}u}\|_{L^2} = \|\xi_i\xi_j\xi_k\widehat{u}\|_{L^2} \\
&\le \left\|\xi_i\xi_j\xi_k\widehat{K_1}(t)\widehat{u_0}\right\|_{L^2} + \left\|\xi_i\xi_j\xi_k\widehat{K_2}(t)\widehat{b_0}\right\|_{L^2} \\
&\quad + \int_0^t \left\|\xi_i\xi_j\xi_k\widehat{K_1}(t-\tau)\widehat{N_1}(\tau)\right\|_{L^2} d\tau \\
&\quad + \int_0^t \left\|\xi_i\xi_j\xi_k\widehat{K_2}(t-\tau)\widehat{N_2}(\tau)\right\|_{L^2} d\tau. \qquad (5.29)
\end{aligned}$$

Step II-1. The decay rates of $\|\partial_1\partial_i\partial_3 u\|_{L^2}$ with $i \in \{2,3\}$

Using (5.6), (5.11), (5.12) and Corollary 2.1 with $\alpha = 2$, we deduce from (4.23), (4.24), (4.38) and (5.29) that for $i \in \{2,3\}$,

$$\|\partial_1\partial_i\partial_3 u\|_{L^2} \le C\varepsilon(1+t)^{-\frac{3}{2}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} \|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau$$

$$
\begin{aligned}
&+ C \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_1 \partial_3 (b \cdot \nabla b)\|_{L^2} d\tau \\
&+ C \int_{0}^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_i \partial_1 (b \cdot \nabla b)\|_{L^2} d\tau \\
&\leq C\varepsilon (1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2 \int_{0}^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\
&+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{19}{8}} d\tau \\
&+ CC_0^2\varepsilon^2 \int_{0}^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-2} d\tau \\
&\leq C\varepsilon (1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{3}{2}} \leq \frac{C_0}{2} \varepsilon (1+t)^{-\frac{3}{2}},
\end{aligned}
$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step II-2. The decay rates of $\|\partial_1^2 \partial_i u\|_{L^2}$ with $i \in \{2,3\}$

Taking $j = k = 1$ in (5.29), using (5.9), (5.15), (5.16) and Corollary 2.1, we obtain by choosing C_0 large enough and ε suitably small that for $i \in \{2,3\}$,

$$
\begin{aligned}
\|\partial_1^2 \partial_i u\|_{L^2} &\leq C\varepsilon (1+t)^{-1} + C \int_{0}^{t} (1+t-\tau)^{-1} \|\partial_1^2 (b \cdot \nabla b)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&+ C \int_{0}^{t} e^{-c(t-\tau)} \|\partial_1^2 \partial_i (b \cdot \nabla b)\|_{L^2} d\tau \\
&\leq C\varepsilon (1+t)^{-1} + CC_0^2\varepsilon^2 \int_{0}^{t} (1+t-\tau)^{-1} (1+\tau)^{-\frac{5}{4}} d\tau \\
&+ CC_0^2\varepsilon^2 \int_{0}^{t} e^{-c(t-\tau)} (1+\tau)^{-1} d\tau \\
&\leq C\varepsilon (1+t)^{-1} + CC_0^2\varepsilon^2 (1+t)^{-1} \leq \frac{C_0}{2} \varepsilon (1+t)^{-1}.
\end{aligned}
$$

Step II-3. The decay rates of $\|\partial_l\partial_2^2 u\|_{L^2}$ with $l \in \{1,2\}$

For $i=1$ and $j=k=2$ in (5.29), by (5.6) and (5.15), we have

$$\begin{aligned}
\|\partial_1\partial_2^2 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{3}{2}} + C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|\partial_1\partial_2^2(b\cdot\nabla b)\|_{L^2} d\tau \\
&\le C\varepsilon\,(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-\frac{3}{2}}\,(1+\tau)^{-\frac{11}{8}}\,d\tau \\
&\quad + CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}\,(1+\tau)^{-1}\,d\tau \\
&\le C\varepsilon\,(1+t)^{-1} + C\left(C_0+C_0^2\right)\varepsilon^2\,(1+t)^{-1}, \qquad (5.30)
\end{aligned}$$

where we have used Corollary 2.1 with $\alpha=2$. Similarly, using (5.5), (5.15) and Corollary 2.1 with $\alpha=3$, we get

$$\begin{aligned}
\|\partial_2^3 u\|_{L^2} &\le C\varepsilon(1+t)^{-2} + C\int_0^t (1+t-\tau)^{-2} \|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_0^t e^{-c(t-\tau)}\|\partial_2^3(b\cdot\nabla b)\|_{L^2} d\tau \\
&\le C\varepsilon\,(1+t)^{-1} + CC_0^2\varepsilon^2\int_0^t (1+t-\tau)^{-2}\,(1+\tau)^{-\frac{11}{8}}\,d\tau \\
&\quad + CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}\,(1+\tau)^{-1}\,d\tau \\
&\le C\varepsilon\,(1+t)^{-1} + C\left(C_0+C_0^2\right)\varepsilon^2\,(1+t)^{-1},
\end{aligned}$$

which, together with (5.30), shows

$$\|\partial_l\partial_2^2 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-1}, \quad \forall\, l\in\{1,2\},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step II-4. The decay rates of $\|\partial_i\partial_j\partial_3 u\|_{L^2}$ with $i,j\in\{2,3\}$

Similarly to that in Step I-1, using (5.5), (5.11), (5.13), Corollary 2.1 with $\alpha=3$ and (4.38), we infer from (5.29) that for $i,j\in\{2,3\}$,

$$\begin{aligned}
\|\partial_i\partial_j\partial_3 u\|_{L^2} &\le C\varepsilon(1+t)^{-2} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-2}\,\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&\quad + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}\,\|\partial_i\partial_3(b\cdot\nabla b)\|_{L^2}d\tau\\
&\quad + C\int_0^{t} e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,\|\partial_i\partial_j(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-2} + CC_0^2\varepsilon^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-2}\,(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-\frac{23}{8}}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_0^{t} e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-2}\,d\tau\\
&\le C\varepsilon(1+t)^{-2} + CC_0^2\varepsilon^2\,(1+t)^{-2} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-2},
\end{aligned}$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step II-5. The decay rate of $\|\partial_1^3 u\|_{L^2}$

Taking $i=j=k=1$ in (5.29) and applying Corollary 2.1 with $\alpha=0$, by (5.14) and (5.15) we deduce that for any $0<\delta<1$,

$$\begin{aligned}
\|\partial_1^3 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{1}{2}} + C\int_0^{t} (1+t-\tau)^{-\frac{1}{2}}\,\|\partial_1^3(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&\quad + C\int_0^{t} e^{-c(t-\tau)}\|\partial_1^3(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-\frac{1}{2}} + CC_0\varepsilon^2\int_0^{t} (1+t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-\frac{7}{8}}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ CC_0\varepsilon \int_0^t (1+t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{1}{2}} \|\partial_\nu b\|_{H^3} d\tau \\
&+ CC_0\varepsilon^2 \int_0^t e^{-c(t-\tau)} (1+\tau)^{-1} d\tau \\
&\le C\varepsilon (1+t)^{-\frac{1}{2}} + CC_0\varepsilon^2 (1+t)^{-\frac{3}{8}} + CC_0\varepsilon^2 (1+t)^{-1} \\
&+ CC_0\varepsilon \left(\int_0^t (1+t-\tau)^{-1} (1+\tau)^{-1} d\tau \right)^{\frac{1}{2}} \left(\int_0^t \|\partial_\nu b\|_{H^3}^2 d\tau \right)^{\frac{1}{2}} \\
&\le C\varepsilon (1+t)^{-\frac{1}{2}} + CC_0\varepsilon^2 (1+t)^{-\frac{3}{8}} + CC_0\varepsilon^2 (1+t)^{-\frac{1-\delta}{2}},
\end{aligned}
$$

where we have also used (5.2). So, if we choose C_0 large enough and ε suitably small, then we have by taking $\delta \in (0, 1/4]$ that

$$
\|\partial_1^3 u(t)\|_{L^2} \le \frac{C_0}{2} \varepsilon (1+t)^{-\frac{3}{8}}, \quad \forall\, t \ge 1.
$$

Step III. The decay rates of $\|\partial_i\partial_j\partial_k\partial_l u\|_{L^2}$ with $i,j,k,l \in \{1,2,3\}$

For any $i,j,k,l \in \{1,2,3\}$, we infer from (2.7) that

$$
\begin{aligned}
\|\partial_{ijkl} u\|_{L^2} &= \|\widehat{\partial_{ijkl} u}\|_{L^2} = \|\xi_i\xi_j\xi_k\xi_l \widehat{u}\|_{L^2} \\
&\le \left\| \xi_i\xi_j\xi_k\xi_l \widehat{K_1}(t)\widehat{u_0} \right\|_{L^2} + \left\| \xi_i\xi_j\xi_k\xi_l \widehat{K_2}(t)\widehat{b_0} \right\|_{L^2} \\
&\quad + \int_0^t \left\| \xi_i\xi_j\xi_k\xi_l \widehat{K_1}(t-\tau)\widehat{N_1}(\tau) \right\|_{L^2} d\tau \\
&\quad + \int_0^t \left\| \xi_i\xi_j\xi_k\xi_l \widehat{K_2}(t-\tau)\widehat{N_2}(\tau) \right\|_{L^2} d\tau. \qquad (5.31)
\end{aligned}
$$

Step III-1. The decay rate of $\|\partial_1^3\partial_3 u\|_{L^2}$

Taking $i=j=k=1$ and $l=3$ in (5.31), using (4.38) and Corollary 2.1 with $\alpha = 1$, by (5.14) and (5.15) we see that for any $0 < \delta < 1$,

$$
\|\partial_1^3\partial_3 u\|_{L^2} \le C\varepsilon(1+t)^{-1} + C \int_0^t (1+t-\tau)^{-1} \|\partial_1^3 (b\cdot\nabla b)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau
$$

$$
\begin{aligned}
&+C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_1^3(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon(1+t)^{-1}+CC_0\varepsilon\int_0^t(1+t-\tau)^{-1}(1+\tau)^{-\frac{1}{2}}\|\partial_\nu b\|_{H^3}d\tau\\
&+CC_0\varepsilon^2\int_0^t(1+t-\tau)^{-1}(1+\tau)^{-\frac{7}{8}}d\tau\\
&+CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}d\tau\\
&\le C\varepsilon(1+t)^{-1}+CC_0\varepsilon^2(1+t)^{-\frac{7}{8}+\delta}+CC_0\varepsilon^2(1+t)^{-1}\\
&+CC_0\varepsilon\left(\int_0^t(1+t-\tau)^{-2}(1+\tau)^{-1}d\tau\right)^{\frac{1}{2}}\left(\int_0^t\|\partial_\nu b\|_{H^3}^2d\tau\right)^{\frac{1}{2}}\\
&\le C\varepsilon(1+t)^{-\frac{1}{2}}+CC_0\varepsilon^2(1+t)^{-\frac{1}{2}}\le\frac{C_0}{2}\varepsilon(1+t)^{-\frac{1}{2}},
\end{aligned}
$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III-2. The decay rates of $\|\partial_l\partial_h\partial_2\partial_3 u\|_{L^2}$ with $l,h\in\{1,2\}$

Based upon (5.6), (5.15), (4.38) and Corollary 2.1, it follows from (5.31) that

$$
\begin{aligned}
\|\partial_1\partial_2^2\partial_3 u\|_{L^2}&\le C\varepsilon(1+t)^{-2}+C\int_0^t(1+t-\tau)^{-2}\|\partial_1(b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&+C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_1\partial_2^2(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon(1+t)^{-2}+CC_0^2\varepsilon^2\int_0^t(1+t-\tau)^{-2}(1+\tau)^{-\frac{11}{8}}d\tau\\
&+CC_0\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-1}d\tau\\
&\le C\varepsilon(1+t)^{-2}+C\left(C_0+C_0^2\right)\varepsilon^2(1+t)^{-1}.
\end{aligned}
\tag{5.32}
$$

In a similar manner, we have by (5.6), (5.9) and (5.15) that

$$\begin{aligned}\|\partial_1^2\partial_2\partial_3 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{3}{2}} + C\int_0^t (1+t-\tau)^{-\frac{3}{2}} \|\partial_1^2 (b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\ &\quad + C\int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_1\partial_2^2 (b\cdot\nabla b)\|_{L^2} d\tau \\ &\le C\varepsilon(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{5}{4}} d\tau \\ &\quad + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\ &\quad + CC_0\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-1} d\tau \\ &\le C\varepsilon(1+t)^{-\frac{3}{2}} + C\left(C_0 + C_0^2\right)\varepsilon^2 (1+t)^{-1}\end{aligned}$$

and

$$\begin{aligned}\|\partial_2^3\partial_3 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{5}{2}} + C\int_0^t (1+t-\tau)^{-\frac{5}{2}} \|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\ &\quad + C\int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_2^3 (b\cdot\nabla b)\|_{L^2} d\tau \\ &\le C\varepsilon(1+t)^{-\frac{5}{2}} + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{5}{4}} d\tau \\ &\quad + CC_0^2\varepsilon^2 \int_0^t (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\ &\quad + CC_0\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-1} d\tau \\ &\le C\varepsilon(1+t)^{-\frac{5}{2}} + C\left(C_0 + C_0^2\right)\varepsilon^2 (1+t)^{-1},\end{aligned}$$

which, together with (5.32), give rise to

$$\|\partial_l\partial_h\partial_2\partial_3 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-1}, \quad \forall\, l,h\in\{1,2\},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III-3. The decay rate of $\|\partial_1^2\partial_3^2 u\|_{L^2}$

Using (5.9), (5.16) and (4.38), from (5.31) we obtain after choosing C_0 large enough and ε suitably small that

$$\begin{aligned}
\|\partial_1^2\partial_3^2 u\|_{L^2} \le{}& C\varepsilon(1+t)^{-\frac{3}{2}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} \|\partial_1^2 (b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_1^2\partial_3 (b\cdot\nabla b)\|_{L^2} d\tau \\
&+ C\int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_1^2\partial_3 (b\cdot\nabla b)\|_{L^2} d\tau \\
\le{}& C\varepsilon\,(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{3}{2}} (1+\tau)^{-\frac{5}{4}}\, d\tau \\
&+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{15}{8}}\, d\tau \\
&+ CC_0^2\varepsilon^2 \int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{15}{8}}\, d\tau \\
\le{}& C\varepsilon(1+t)^{-\frac{3}{2}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{8}} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{11}{8}}.
\end{aligned}$$

Step III-4. The decay rates of $\|\partial_2^2\partial_3^2 u\|_{L^2}, \|\partial_1\partial_i\partial_3^2 u\|_{L^2}$ with $i\in\{2,3\}$

Using (2.10), (2.12)–(2.14) of Proposition 2.1 and Corollary 2.1 with $\alpha = 3$, we infer from (1.6), (5.6), (5.17) and (5.18) that for $i\in\{2,3\}$,

$$\begin{aligned}
\|\partial_1\partial_i\partial_3^2 u\|_{L^2} \le{}& C\varepsilon(1+t)^{-2} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|\partial_1 (b\cdot\nabla b)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_1\partial_3^2 (b\cdot\nabla b)\|_{L^2} d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^t e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,\|\partial_1\partial_i\partial_3(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-2} + CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-2}\,(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-\frac{21}{8}}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-2}\,d\tau\\
&\le C\varepsilon(1+t)^{-2} + CC_0^2\varepsilon^2\,(1+t)^{-2}\,,
\end{aligned}
\tag{5.33}
$$

where we have also used (4.38). Analogously,

$$
\begin{aligned}
\|\partial_2^2\partial_3^2 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{5}{2}} + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{2}}\,\|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}\,\|\partial_2\partial_3^2(b\cdot\nabla b)\|_{L^2}d\tau\\
&\quad + C\int_0^t e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,\|\partial_2^2\partial_3(b\cdot\nabla b)\|_{L^2}d\tau\\
&\le C\varepsilon\,(1+t)^{-\frac{5}{2}} + CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{2}}\,(1+\tau)^{-\frac{11}{8}}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-3}\,d\tau\\
&\quad + CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}\,(t-\tau)^{-\frac{1}{2}}\,(1+\tau)^{-2}\,d\tau\\
&\le C\varepsilon(1+t)^{-\frac{5}{2}} + CC_0^2\varepsilon^2\,(1+t)^{-2}\,,
\end{aligned}
$$

which, combined with (5.33), yields

$$
\|\partial_2^2\partial_3^2 u(t)\|_{L^2} + \|\partial_1\partial_i\partial_3^2 u(t)\|_{L^2} \le \frac{C_0}{2}\varepsilon(1+t)^{-2},\quad \forall\ i\in\{2,3\},
$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Step III-5. The decay rates of $\|\partial_i\partial_3^3 u\|_{L^2}$ with $i \in \{2,3\}$

In terms of Proposition 2.1 and Corollary 2.1 with $\alpha = 4$, by choosing C_0 large enough and ε suitably small we infer from (5.19) and (5.20) that for $i \in \{2,3\}$,

$$\begin{aligned}
\|\partial_i\partial_3^3 u\|_{L^2} &\le C\varepsilon(1+t)^{-\frac{5}{2}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} \|b\cdot\nabla b\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|\partial_3^3(b\cdot\nabla b)\|_{L^2} d\tau \\
&\quad + C\int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_i\partial_3^2(b\cdot\nabla b)\|_{L^2} d\tau \\
&\le C\varepsilon\,(1+t)^{-\frac{5}{2}} + CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{11}{8}}\, d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{27}{8}}\, d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_0^{t} e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-3}\, d\tau \\
&\le C\varepsilon(1+t)^{-\frac{5}{2}} + CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{2}} \le \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{5}{2}}.
\end{aligned}$$

5.3. Enhanced decay rates of (u_1, b_1)

In this subsection, we aim to improve the decay rates of (u_1, b_1), based on (4.3), (4.60) and Corollary 2.2. Note that the decay estimates of (u_1, b_1), $\partial_i(u_1, b_1)$, $\partial_i\partial_3(u_1, b_1)$ with $i \in \{2,3\}$ and $\partial_3^3(u_1, b_1)$ stated in (5.1) have been achieved in Theorem 1.2.

Part I. Improved decay rate of $\|\partial_2^2 u_1\|_{L^2}$

It follows from (2.7) and Plancherel's theorem that

$$\begin{aligned}
\|\partial_2^2 u_1\|_{L^2} &= \|\widehat{\partial_2^2 u_1}\|_{L^2} = \|\xi_2^2 \widehat{u_1}\|_{L^2} \\
&\le \left\|\xi_2^2 \widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2} + \left\|\xi_2^2 \widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2}
\end{aligned}$$

$$+\int_0^t \left\|\xi_2^2 \widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2} d\tau$$

$$+\int_0^t \left\|\xi_2^2 \widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^{4} \Omega_{1,m}. \tag{5.34}$$

It is easily derived from (4.23), (4.24) and Corollary 2.2 with $\beta = 2$ that

$$\begin{aligned}\Omega_{1,1} + \Omega_{1,2} &\le C(1+t)^{-\frac{7}{4}} \|(u_0, b_0)\|_{L^1} + Ce^{-ct} \|\partial_2^2 (u_{10}, b_{10})\|_{L^2} \\ &\le C\varepsilon (1+t)^{-\frac{7}{4}}.\end{aligned} \tag{5.35}$$

Analogously to the treatment in Subsection 4.3, for simplicity we only deal with the terms associated with $\mathbb{P}_1\widehat{(b\cdot\nabla b)}$ in $\widehat{N_{11}}$ and $\widehat{b_2 u_1}, \widehat{b_3 u_1}$ in $\widehat{N_{21}}$, which are still denoted by $\Omega_{1,3}$ and $\Omega_{1,4}$. First, by (4.3) and Proposition 2.1 we have

$$\begin{aligned}\Omega_{1,3} \le\ & C\int_0^t \left\| \xi_2^2 e^{-c\xi_\nu^2 (t-\tau)} \sum_{k=1}^{3} \xi_\nu^2 |\xi|^{-2} \xi_k \widehat{b_k b_1} \right\|_{L^2} d\tau \\ &+ C\int_0^t \left\| \xi_2^2 e^{-c\xi_\nu^2 (t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1 |\xi|^{-2} \xi_k \xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \\ &+ C\int_0^t \left\| \xi_2^2 e^{-c(1+\xi_3^2)(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2 |\xi|^{-2} \xi_k \widehat{b_k b_1} \right\|_{L^2} d\tau \\ &+ C\int_0^t \left\| \xi_2^2 e^{-c(1+\xi_3^2)(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1 |\xi|^{-2} \xi_k \xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \\ \triangleq\ & \Omega_{1,31} + \Omega_{1,32} + \Omega_{1,33} + \Omega_{1,34}.\end{aligned} \tag{5.36}$$

In terms of (4.54) and Corollary 2.1 with $\alpha = 3$, we have

$$\begin{aligned}\Omega_{1,31} &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|bb_1\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\ &\quad + C\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \|\partial_2 (b\cdot\nabla b_1)\|_{L^2} d\tau \\ &\le CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} (1+\tau)^{-\frac{11}{8}} d\tau\end{aligned}$$

$$+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{11}{4}}\, d\tau$$
$$\leq CC_0^2\varepsilon^2 (1+t)^{-2}, \tag{5.37}$$

where we have used (5.1), (5.3), (5.4), (5.21), (5.22) and the divergence-free condition $\nabla \cdot b = 0$ to get that ($\partial_\nu = (\partial_2, \partial_3)$ and $b_\nu = (b_2, b_3)$)

$$\begin{aligned}
&\|\partial_2(b\cdot\nabla b_1)\|_{L^2} \\
&\leq C\|\partial_2 b_\nu\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 b_\nu\|_{L^2}^{\frac{1}{2}} \|\partial_\nu b_1\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_\nu b_1\|_{L^2}^{\frac{1}{4}} \|\partial_3\partial_\nu b_1\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_3\partial_\nu b_1\|_{L^2}^{\frac{1}{4}} \\
&\quad + C\|\partial_2 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 b_1\|_{L^2}^{\frac{1}{2}} \|\partial_1 b_1\|_{L^2}^{\frac{1}{4}} \|\partial_1^2 b_1\|_{L^2}^{\frac{1}{4}} \|\partial_1\partial_3 b_1\|_{L^2}^{\frac{1}{4}} \|\partial_1^2\partial_3 b_1\|_{L^2}^{\frac{1}{4}} \\
&\quad + C\|b_1\|_{L^\infty} \|\partial_2\partial_1 b_1\|_{L^2} + C\|b_\nu\|_{L^\infty} \|\partial_2\partial_\nu b_1\|_{L^2} \\
&\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{4}}.
\end{aligned}$$

Similarly, by choosing $\delta \in (0, 1/4]$ we deduce from (4.55) that

$$\begin{aligned}
\Omega_{1,32} &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} \|(bb_2, bb_3)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} \left\| \xi_\nu e^{-c\xi_\nu^2 (t-\tau)} \right\|_{L^\infty} \|(\partial_2^2(bb_2), \partial_2\partial_3(bb_3))\|_{L^2} d\tau \\
&\leq CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-2} (1+\tau)^{-1}\, d\tau \\
&\quad + CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{5}{2}}\, d\tau \\
&\leq CC_0^2\varepsilon^2 \left((1+t)^{-2+\delta} + (1+t)^{-2}\right) \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{7}{4}},
\end{aligned} \tag{5.38}$$

since it follows from (5.3), (5.21) and (5.22) that

$$\begin{aligned}
&\|\partial_2^2 (bb_2)\|_{L^2} + \|\partial_2\partial_3 (bb_3)\|_{L^2} \\
&\leq C\|b\|_{L^\infty} \|\partial_2\partial_\nu b\|_{L^2} + C\|\partial_2 b\|_{L^2}^{\frac{1}{2}} \|\partial_1\partial_2 b\|_{L^2}^{\frac{1}{2}} \\
&\qquad \times \|\partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_2\partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_3\partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_2\partial_3\partial_\nu b\|_{L^2}^{\frac{1}{4}} \\
&\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{2}}.
\end{aligned}$$

Thanks to (5.11), it is easily seen from Lemma 2.5 that

$$\Omega_{1,33}+\Omega_{1,34}\leq C\int_0^t e^{-c(t-\tau)}\left\|\partial_2^2\left(b\cdot\nabla b\right)\right\|_{L^2}d\tau$$
$$\leq CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}\left(1+\tau\right)^{-2}d\tau\leq CC_0^2\varepsilon^2\left(1+t\right)^{-2}. \tag{5.39}$$

Plugging (5.37), (5.38) and (5.39) into (5.36), we obtain

$$\Omega_{1,3}\leq CC_0^2\varepsilon^2\left(1+t\right)^{-\frac{7}{4}}. \tag{5.40}$$

Recalling the definition of $\widehat{N_{21}}$ in (4.60), by (2.10) and (2.12)–(2.14) of Proposition 2.1 we have

$$\Omega_{1,4}=\int_0^t\left\|\xi_2^3e^{-c\xi_\nu^2(t-\tau)}\widehat{b_2u_1}\right\|_{L^2}d\tau+\int_0^t\left\|\xi_2^2\xi_3e^{-c\xi_\nu^2(t-\tau)}\widehat{b_3u_1}\right\|_{L^2}d\tau$$
$$+\int_0^t\left\|\xi_2^3e^{-c(1+\xi_3^2)(t-\tau)}\widehat{b_2u_1}\right\|_{L^2}d\tau+\int_0^t\left\|\xi_2^2\xi_3e^{-c(1+\xi_3^2)(t-\tau)}\widehat{b_3u_1}\right\|_{L^2}d\tau,$$

and hence, similarly the derivation of (4.83), we deduce from (4.38), (4.54) and Corollary 2.1 with $\alpha=3$ that

$$\Omega_{1,4}\leq C\int_0^{\frac{t}{2}}\left(1+t-\tau\right)^{-2}\left\|\left(b_2u_1,b_3u_1\right)\right\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau$$
$$+C\int_{\frac{t}{2}}^t\left(t-\tau\right)^{-\frac{1}{2}}\left\|\left(\partial_2^2(b_2u_1),\partial_2\partial_3(b_3u_1)\right)\right\|_{L^2}d\tau$$
$$+C\int_0^t e^{-c(t-\tau)}\left(\left\|\partial_2^3(b_2u_1)\right\|_{L^2}+\left(t-\tau\right)^{-\frac{1}{2}}\left\|\partial_2^2(b_3u_1)\right\|_{L^2}\right)d\tau$$
$$\leq CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}\left(1+t-\tau\right)^{-2}\left(1+\tau\right)^{-\frac{11}{8}}d\tau$$
$$+CC_0^2\varepsilon^2\int_{\frac{t}{2}}^t\left(t-\tau\right)^{-\frac{1}{2}}\left(1+\tau\right)^{-\frac{11}{4}}d\tau$$

$$+ CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} \left[(1+\tau)^{-2} + (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{11}{4}}\right] d\tau$$
$$\leq CC_0^2\varepsilon^2 (1+t)^{-2}. \tag{5.41}$$

Here, we have also used the following estimates due to (5.3), (5.21) and (5.22),

$$\begin{aligned}\|\partial_2\partial_\nu (bu_1)\|_{L^2} &\leq C\|b\|_{L^\infty}\|\partial_2\partial_\nu u_1\|_{L^2} + C\|u_1\|_{L^\infty}\|\partial_2\partial_\nu b\|_{L^2}\\ &\quad + C\|\partial_\nu b\|_{L^\infty_{x_1,x_3}L^2_{x_3}}\|\partial_\nu u_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\ &\leq CC_0^2\varepsilon^2(1+t)^{-\frac{11}{4}},\end{aligned}$$

and

$$\begin{aligned}\|\partial_2^3 (bu_1)\|_{L^2} &\leq C\|b\|_{L^\infty}\|\partial_2^3 u_1\|_{L^2} + C\|\partial_2 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2^2 u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &\quad + C\|u_1\|_{L^\infty}\|\partial_2^3 b\|_{L^2} + C\|\partial_2 u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &\leq CC_0^2\varepsilon^2(1+t)^{-2}.\end{aligned}$$

Thus, combining (5.35), (5.40) and (5.41) with (5.34), we have

$$\|\partial_2^2 u_1(t)\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{7}{4}} + CC_0^2\varepsilon^2(1+t)^{-\frac{7}{4}} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{7}{4}}, \tag{5.42}$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Part II. Improved decay rate of $\|\partial_2\partial_3^2 u_1\|_{L^2}$

Clearly, from (2.7) and Plancherel's theorem it follows that

$$\begin{aligned}\|\partial_2\partial_3^2 u_1\|_{L^2} &= \|\widehat{\partial_2\partial_3^2 u_1}\|_{L^2} = \|\xi_2\xi_3^2\widehat{u_1}\|_{L^2}\\ &\leq \left\|\xi_2\xi_3^2\widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2} + \left\|\xi_2\xi_3^2\widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2}\\ &\quad + \int_0^t \left\|\xi_2\xi_3^2\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2} d\tau\\ &\quad + \int_0^t \left\|\xi_2\xi_3^2\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2} d\tau \triangleq \sum_{m=1}^4 \Omega_{2,m}.\end{aligned} \tag{5.43}$$

Using (4.23), (4.24) and Corollary 2.2 with $\beta = 3$, we see that

$$\Omega_{2,1} + \Omega_{2,2} \leq C(1+t)^{-\frac{9}{4}}\left(\|u_0\|_{L^1} + \|\partial_2\partial_3^2 u_0\|_{L^2}\right) \leq C\varepsilon(1+t)^{-\frac{9}{4}}. \tag{5.44}$$

Analogously to that in (5.36), by (4.3), (2.10) and (2.12)–(2.14) one has

$$\begin{aligned}\Omega_{2,3} \le & C\int_0^t \left\|\xi_2\xi_3^2 e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3 \xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2} d\tau \\ & + C\int_0^t \left\|\xi_2\xi_3^2 e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3 \xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\right\|_{L^2} d\tau \\ & + C\int_0^t \left\|\xi_2\xi_3^2 e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3 \xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2} d\tau \\ & + C\int_0^t \left\|\xi_2\xi_3^2 e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3 \xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_k b_l}\right\|_{L^2} d\tau \\ \triangleq & \ \Omega_{2,31}+\Omega_{2,32}+\Omega_{2,33}+\Omega_{2,34}. \end{aligned} \tag{5.45}$$

To deal with $\Omega_{2,31}$, we first utilize (5.1), (5.3), (5.4), (5.21), (5.22) and the divergence-free condition $\nabla\cdot b=0$ to deduce

$$\begin{aligned}&\|\partial_3^2(b\cdot\nabla b_1)\|_{L^2}\\ &\le C\|\partial_3^2 b_1\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 b_1\|_{L^2}^{\frac12}\|\partial_1 b_1\|_{L^2}^{\frac14}\|\partial_1^2 b_1\|_{L^2}^{\frac14}\|\partial_3\partial_1 b_1\|_{L^2}^{\frac14}\|\partial_1^2\partial_3 b_1\|_{L^2}^{\frac14}\\ &\quad + C\|\partial_3^2 b_\nu\|_{L^2}^{\frac12}\|\partial_2\partial_3^2 b_\nu\|_{L^2}^{\frac12}\|\partial_\nu b_1\|_{L^2}^{\frac14}\|\partial_1\partial_\nu b_1\|_{L^2}^{\frac14}\|\partial_3\partial_\nu b_1\|_{L^2}^{\frac14}\|\partial_1\partial_3\partial_\nu b_1\|_{L^2}^{\frac14}\\ &\quad + C\|\partial_3 b_1\|_{L^2}^{\frac14}\|\partial_1\partial_3 b_1\|_{L^2}^{\frac14}\|\partial_3^2 b_1\|_{L^2}^{\frac14}\|\partial_1\partial_3^2 b_1\|_{L^2}^{\frac14}\|\partial_1\partial_3 b_1\|_{L^2}^{\frac12}\|\partial_1\partial_2\partial_3 b_1\|_{L^2}^{\frac12}\\ &\quad + C\|\partial_3 b_\nu\|_{L^2}^{\frac14}\|\partial_1\partial_3 b_\nu\|_{L^2}^{\frac14}\|\partial_2\partial_3 b_\nu\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3 b_\nu\|_{L^2}^{\frac14}\|\partial_3\partial_\nu b_1\|_{L^2}^{\frac12}\|\partial_3^2\partial_\nu b_1\|_{L^2}^{\frac12}\\ &\quad + C\|b_1\|_{L^\infty}\|\partial_1\partial_3^2 b_1\|_{L^2} + C\|b_\nu\|_{L^\infty}\|\partial_\nu\partial_3^2 b_1\|_{L^2}\\ &\le CC_0^2\varepsilon^2(1+t)^{-\frac{13}{4}}, \end{aligned}$$

which, combined with (4.54) and Corollary 2.1 with $\alpha=4$, yields

$$\begin{aligned}\Omega_{2,31} &\le C\int_0^{\frac t2}(1+t-\tau)^{-\frac52}\,\|\|bb_1\|_{L^1_{x_2,x_3}}\|_{L^2_{x_1}}\,d\tau\\ &\quad + C\int_{\frac t2}^t (t-\tau)^{-\frac12}\,\|\partial_3^2(b\cdot\nabla b_1)\|_{L^2}d\tau\\ &\le CC_0^2\varepsilon^2\int_0^{\frac t2}(1+t-\tau)^{-\frac52}(1+\tau)^{-\frac{11}{8}}\,d\tau \end{aligned}$$

$$+ CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{13}{4}}\, d\tau$$
$$\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{5}{2}}. \tag{5.46}$$

Analogously, since

$$\begin{aligned}\|\partial_\nu \partial_3^2 (b b_\nu)\|_{L^2} &\leq C\|b\|_{L^\infty}\|\partial_\nu \partial_3^2 b\|_{L^2} + C\|\partial_\nu \partial_3 b\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_\nu \partial_3 b\|_{L^2}^{\frac{1}{2}} \\ &\quad \times \|\partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_3 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \|\partial_1 \partial_3 \partial_\nu b\|_{L^2}^{\frac{1}{4}} \\ &\leq CC_0^2\varepsilon^2 (1+t)^{-3},\end{aligned}$$

due to (5.3) and (5.21), by choosing $0 < \delta \leq 1/4$ we infer from (4.55) that

$$\begin{aligned}\Omega_{2,32} &\leq C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} \|(bb_2, bb_3)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\ &\quad + C \int_{\frac{t}{2}}^{t} \left\| \xi_2 e^{-c\xi_\nu^2 (t-\tau)} \right\|_{L^\infty} \|(\partial_2 \partial_3^2 (bb_2), \partial_3^3 (bb_3))\|_{L^2} d\tau \\ &\leq CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-1}\, d\tau \\ &\quad + CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-3}\, d\tau \\ &\leq CC_0^2\varepsilon^2 \left((1+t)^{-\frac{5}{2}+\delta} + (1+t)^{-\frac{5}{2}} \right) \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{9}{4}}.\end{aligned} \tag{5.47}$$

Similarly to the treatment of $\Omega_{1,4}$, by (5.13) and (4.38) we obtain

$$\begin{aligned}\Omega_{2,33} + \Omega_{2,34} &\leq C \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|\partial_2 \partial_3 (b \cdot \nabla b)\|_{L^2}\, d\tau \\ &\leq CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{23}{8}}\, d\tau \\ &\leq CC_0^2\varepsilon^2 (1+t)^{-\frac{23}{8}}.\end{aligned} \tag{5.48}$$

Thus, inserting (5.46), (5.47) and (5.48) into (5.45) gives

$$\Omega_{2,3} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{9}{4}}. \tag{5.49}$$

In view of (4.60), we observe that

$$\Omega_{2,4} = \int_0^t \left\| \xi_2^2 \xi_3^2 e^{-c\xi_\nu^2 (t-\tau)} \widehat{b_2 u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_2 \xi_3^3 e^{-c\xi_\nu^2 (t-\tau)} \widehat{b_3 u_1} \right\|_{L^2} d\tau$$
$$+ \int_0^t \left\| \xi_2^2 \xi_3^2 e^{-c(1+\xi_3^2)(t-\tau)} \widehat{b_2 u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_2 \xi_3^3 e^{-c(1+\xi_3^2)(t-\tau)} \widehat{b_3 u_1} \right\|_{L^2} d\tau,$$

so that, similarly to the treatment of $\Omega_{1,4}$ in (5.41), we find

$$\begin{aligned}
\Omega_{2,4} &\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} \|(b_2 u_1, b_3 u_1)\|_{L^2_{x_1} L^1_{x_2,x_3}} d\tau \\
&\quad + C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \|(\partial_2 \partial_3^2 (b_2 u_1), \partial_3^3 (b_3 u_1))\|_{L^2} d\tau \\
&\quad + C \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|(\partial_2^2 \partial_3 (b_2 u_1), \partial_2 \partial_3^2 (b_3 u_1))\|_{L^2} d\tau \\
&\le C C_0^2 \varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{2}} (1+\tau)^{-\frac{11}{8}} d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{13}{4}} d\tau \\
&\quad + C C_0^2 \varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \left((1+\tau)^{-3} + (1+\tau)^{-\frac{13}{4}} \right) d\tau \\
&\le C C_0^2 \varepsilon^2 (1+t)^{-\frac{5}{2}},
\end{aligned} \tag{5.50}$$

where we have used (4.54), (4.38) and the following estimates due to (5.1), (5.3), (5.4), (5.21), (5.22) and the divergence-free condition $\nabla \cdot u = 0$,

$$\begin{aligned}
\|\partial_\nu \partial_3^2 (b u_1)\|_{L^2} &\le C \|u_1\|_{L^\infty} \|\partial_\nu \partial_3^2 b\|_{L^2} + C \|b\|_{L^\infty} \|\partial_\nu \partial_3^2 u_1\|_{L^2} \\
&\quad + C \|\partial_\nu b\|_{L^\infty_{x_1,x_3} L^2_{x_2}} \|\partial_3^2 u_1\|_{L^2_{x_1,x_3} L^\infty_{x_2}} \\
&\quad + C \|\partial_3 b\|_{L^\infty_{x_1,x_2} L^2_{x_3}} \|\partial_\nu \partial_3 u_1\|_{L^2_{x_1,x_2} L^\infty_{x_3}} \\
&\quad + C \|\partial_\nu \partial_3 b\|_{L^2_{x_1,x_3} L^\infty_{x_2}} \|\partial_\nu u_1\|_{L^\infty_{x_1,x_3} L^2_{x_2}}
\end{aligned}$$

$$\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{13}{4}}\,,$$

and

$$\begin{aligned}\|\partial_2^2\partial_3(bu_1)\|_{L^2} \leq\ & C\|b\|_{L^\infty}\|\partial_2^2\partial_3 u_1\|_{L^2} + C\|u_1\|_{L^\infty}\|\partial_2^2\partial_3 b_2\|_{L^2}\\ &+ C\|\partial_\nu b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2\partial_\nu u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &+ C\|\partial_3\partial_2 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_2 u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\ &+ C\|\partial_2^2 b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\ \leq\ & CC_0^2\varepsilon^2\,(1+t)^{-3}\,.\end{aligned}$$

Thus, inserting (5.44), (5.49) and (5.50) into (5.43), we obtain after choosing C_0 large enough and ε suitably small that

$$\|\partial_2\partial_3^2 u_1(t)\|_{L^2} \leq C\varepsilon(1+t)^{-\frac{9}{4}} + CC_0^2\varepsilon^2\,(1+t)^{-\frac{9}{4}} \leq \frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{9}{4}}\,.$$

Part III. Improved decay rate of $\|\partial_2\partial_3^3 u_1\|_{L^2}$

Based on (2.7) and Plancherel's theorem, one has

$$\begin{aligned}\|\partial_2\partial_3^3 u_1\|_{L^2} &= \|\widehat{\partial_2\partial_3^3 u_1}\|_{L^2} = \||\xi_2\xi_3^3|\widehat{u_1}\|_{L^2}\\ &\leq \left\|\xi_2\xi_3^3\widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2} + \left\|\xi_2\xi_3^3\widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2}\\ &\quad + \int_0^t \left\|\xi_2\xi_3^3\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2}d\tau\\ &\quad + \int_0^t \left\|\xi_2\xi_3^3\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2}d\tau \triangleq \sum_{m=1}^4 \Omega_{3,m}.\end{aligned} \tag{5.51}$$

It follows from (4.23), (4.24) and Corollary 2.2 with $\beta = 4$ that

$$\Omega_{3,1} + \Omega_{3,2} \leq C(1+t)^{-\frac{11}{4}}\left(\|u_0\|_{L^1} + \|\partial_2\partial_3^3 u_0\|_{L^2}\right) \leq C\varepsilon(1+t)^{-\frac{11}{4}}. \tag{5.52}$$

Next, we deal with the term $\Omega_{3,3}$, which can be written in terms of (2.10), (2.12)–(2.14) and (4.3) as follows,

$$\Omega_{3,3} \leq C\int_0^t \left\|\xi_2\xi_3^3 e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3 \xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_k b_1}\right\|_{L^2}d\tau$$

$$
\begin{aligned}
&+C\int_0^t \left\| \xi_2\xi_3^3 e^{-c\xi_\nu^2(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \\
&+C\int_0^t \left\| \xi_2\xi_3^3 e^{-c(1+\xi_3^2)(t-\tau)} \sum_{k=1}^{3} \xi_\nu^2|\xi|^{-2}\xi_k \widehat{b_k b_1} \right\|_{L^2} d\tau \\
&+C\int_0^t \left\| \xi_2\xi_3^3 e^{-c(1+\xi_3^2)(t-\tau)} \sum_{k=1}^{3}\sum_{l=2}^{3} \xi_1|\xi|^{-2}\xi_k\xi_l \widehat{b_k b_l} \right\|_{L^2} d\tau \\
&\triangleq \Omega_{3,31}+\Omega_{3,32}+\Omega_{3,33}+\Omega_{3,34}. \qquad (5.53)
\end{aligned}
$$

Using (5.1), (5.3), (5.4), (5.21), (5.22) and the divergence-free condition $\nabla\cdot b=0$, we have

$$
\begin{aligned}
\|\partial_3^3(b\cdot\nabla b_1)\|_{L^2} \le{}& C\|b_1\|_{L^\infty}\|\partial_1\partial_3^3 b_1\|_{L^2} + C\|b_\nu\|_{L^\infty}\|\partial_\nu\partial_3^3 b_1\|_{L^2} \\
&+C\|\partial_3^3 b_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_1 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \\
&+C\|\partial_3^3 b_\nu\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_\nu b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \\
&+C\|\partial_3^2 b_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_1\partial_3 b_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \\
&+C\|\partial_3^2 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_3 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}} \\
&+C\|\partial_3 b_1\|_{L^\infty_{x_1x_2}L^2_{x_3}}\|\partial_1\partial_3^2 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}} \\
&+C\|\partial_3 b_\nu\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_3^2 b_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}} \\
\le{}& CC_0^2\varepsilon^2(1+t)^{-\frac{15}{4}}, \qquad (5.54)
\end{aligned}
$$

which, together with (4.54) and Corollary 2.1 with $\alpha=5$, yields

$$
\begin{aligned}
\Omega_{3,31} &\le C\int_0^{\frac{t}{2}} (1+t-\tau)^{-3}\,\|\|bb_1\|_{L^1_{x_2x_3}}\|_{L^2_{x_1}}\,d\tau \\
&\quad + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}\,\|\partial_3^3(b\cdot\nabla b_1)\|_{L^2}d\tau \\
&\le CC_0^2\varepsilon^2\int_0^{\frac{t}{2}} (1+t-\tau)^{-3}(1+\tau)^{-\frac{11}{8}}\,d\tau \\
&\quad + CC_0^2\varepsilon^2\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{15}{4}}\,d\tau
\end{aligned}
$$

$$\leq CC_0^2\varepsilon^2\left(1+t\right)^{-3}. \tag{5.55}$$

Analogously, noting that

$$\begin{aligned}\|\partial_\nu\partial_3^3(bb_\nu)\|_{L^2} &\leq C\|b\|_{L^\infty}\|\partial_\nu\partial_3^3 b\|_{L^2}+C\|\partial_3^3 b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_\nu b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\\ &\quad+C\|\partial_3^2\partial_\nu b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3 b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\ &\quad+C\|\partial_3^2 b\|_{L^\infty_{x_2,x_3}L^2_{x_1}}\|\partial_\nu\partial_3 b\|_{L^2_{x_2,x_3}L^\infty_{x_1}}\\ &\leq CC_0^2\varepsilon^2\left(1+t\right)^{-\frac{7}{2}},\end{aligned} \tag{5.56}$$

from which and (4.55), we obtain by choosing $\delta\in(0,1/4]$ that

$$\begin{aligned}\Omega_{3,32} &\leq C\int_0^{\frac{t}{2}}\left(1+t-\tau\right)^{-3}\|(bb_2,bb_3)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\ &\quad+C\int_{\frac{t}{2}}^t\left\|\xi_2 e^{-c\xi_\nu^2(t-\tau)}\right\|_{L^\infty}\|(\partial_2\partial_3^3(bb_2),\partial_3^4(bb_3))\|_{L^2}d\tau\\ &\leq CC_0^2\varepsilon^2\int_0^{\frac{t}{2}}\left(1+t-\tau\right)^{-3}\left(1+\tau\right)^{-1}d\tau\\ &\quad+CC_0^2\varepsilon^2\int_{\frac{t}{2}}^t\left(t-\tau\right)^{-\frac{1}{2}}\left(1+\tau\right)^{-\frac{7}{2}}d\tau\\ &\leq CC_0^2\varepsilon^2\left(1+t\right)^{-3+\delta}+CC_0^2\varepsilon^2\left(1+\tau\right)^{-3}\\ &\leq CC_0^2\varepsilon^2\left(1+t\right)^{-\frac{11}{4}}.\end{aligned} \tag{5.57}$$

It is easily deduced from (5.19) and (4.38) that

$$\begin{aligned}\Omega_{3,33}+\Omega_{3,34} &\leq C\int_0^t e^{-c(t-\tau)}\left(t-\tau\right)^{-\frac{1}{2}}\left\|\partial_2\partial_3^2\left(b\cdot\nabla b\right)\right\|_{L^2}d\tau\\ &\leq CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}\left(t-\tau\right)^{-\frac{1}{2}}\left(1+\tau\right)^{-3}d\tau\\ &\leq CC_0^2\varepsilon^2\left(1+t\right)^{-3}.\end{aligned} \tag{5.58}$$

Thus, by virtue of (5.55), (5.57) and (5.58), we infer from (5.53) that

$$\Omega_{3,3}\leq CC_0^2\varepsilon^2\left(1+t\right)^{-\frac{11}{4}}. \tag{5.59}$$

Similarly to that for $\Omega_{2,4}$, we have from (2.10), (2.12)–(2.14) and (4.60) that

$$\begin{aligned}\Omega_{3,4} \le & \int_0^t \left\| \xi_2^2\xi_3^3 e^{-c\xi_\nu^2(t-\tau)} \widehat{b_2u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_2\xi_3^4 e^{-c\xi_\nu^2(t-\tau)} \widehat{b_3u_1} \right\|_{L^2} d\tau \\ & + \int_0^t \left\| \xi_2^2\xi_3^3 e^{-c(1+\xi_3^2)(t-\tau)} \widehat{b_2u_1} \right\|_{L^2} d\tau + \int_0^t \left\| \xi_2\xi_3^4 e^{-c(1+\xi_3^2)(t-\tau)} \widehat{b_3u_1} \right\|_{L^2} d\tau\end{aligned}$$

and thus,

$$\begin{aligned}\Omega_{3,4} \le & C \int_0^{\frac{t}{2}} (1+t-\tau)^{-3} \|(b_2u_1, b_3u_1)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\ & + C \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \|(\partial_2\partial_3^3(b_2u_1), \partial_3^4(b_3u_1))\|_{L^2} d\tau \\ & + C \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|(\partial_2^2\partial_3^2(b_2u_1), \partial_2\partial_3^3(b_3u_1))\|_{L^2} d\tau \\ \le & CC_0^2\varepsilon^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-3} (1+\tau)^{-\frac{11}{8}} d\tau \\ & + CC_0^2\varepsilon^2 \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} (1+\tau)^{-\frac{15}{4}} d\tau \\ & + CC_0^2\varepsilon^2 \int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \left((1+\tau)^{-3} + (1+\tau)^{-\frac{15}{4}} \right) d\tau \\ \le & CC_0^2\varepsilon^2 (1+t)^{-3}, \end{aligned} \tag{5.60}$$

where we have used (5.1), (5.3), (5.4), (5.21), (5.22) and the divergence-free condition $\nabla \cdot u = 0$ to get that

$$\begin{aligned}& \|\partial_2\partial_3^3(bu_1)\|_{L^2} + \|\partial_3^4(bu_1)\|_{L^2} + \|\partial_2\partial_3^3(bu_1)\|_{L^2} \\ & \le C\|b\|_{L^\infty}\|\partial_\nu\partial_3^3u_1\|_{L^2} + C\|u_1\|_{L^\infty}\|\partial_\nu\partial_3^3b\|_{L^2} \\ & \quad + C\|\partial_\nu\partial_3^2b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_3u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}} + C\|\partial_3^3b\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\|\partial_\nu u_1\|_{L^\infty_{x_1,x_3}L^2_{x_2}} \\ & \quad + C\|\partial_3^2b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_\nu\partial_3u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}} + \|\partial_\nu\partial_3b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3^2u_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}} \\ & \quad + C\|\partial_3b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2\partial_3^2u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}} + C\|\partial_2b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3^3u_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\end{aligned}$$

$$\leq CC_0^2\varepsilon^2\,(1+t)^{-\frac{15}{4}}\,, \tag{5.61}$$

and

$$\begin{aligned}\|\partial_2^2\partial_3^2\,(b_2u_1)\,\|_{L^2} &\leq C\|b\|_{L^\infty}\|\partial_2^2\partial_3^2u_1\|_{L^2}+C\|u_1\|_{L^\infty}\|\partial_2^2\partial_3^2b\|_{L^2}\\ &\quad+C\|\partial_i\partial_j\partial_3b\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\|\partial_\nu u_1\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\\ &\quad+C\|\partial_\nu\partial_3b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_2\partial_\nu u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &\quad+C\|\partial_2^2b\|_{L^\infty_{x_1,x_3}L^2_{x_2}}\|\partial_3^2u_1\|_{L^2_{x_1,x_3}L^\infty_{x_2}}\\ &\quad+\|\partial_\nu b\|_{L^\infty_{x_1,x_2}L^2_{x_3}}\|\partial_i\partial_j\partial_3u_1\|_{L^2_{x_1,x_2}L^\infty_{x_3}}\\ &\leq CC_0^2\varepsilon^2\,(1+t)^{-3}\end{aligned}$$

with $i,j\in\{2,3\}$. Thus, inserting (5.52), (5.59) and (5.60) into (5.51) gives rise to

$$\|\partial_2\partial_3^2u_1(t)\|_{L^2}\leq\left(C\varepsilon+CC_0^2\varepsilon^2\right)(1+t)^{-\frac{11}{4}}\leq\frac{C_0}{2}\varepsilon\,(1+t)^{-\frac{11}{4}}\,, \tag{5.62}$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Part IV. Improved decay rate of $\|\partial_3^4u_1\|_{L^2}$

In terms of (2.7), we have

$$\begin{aligned}\|\partial_3^4u_1\|_{L^2}&=\|\widehat{\partial_3^4u_1}\|_{L^2}=\||\xi_3^4|\widehat{u_1}\|_{L^2}\\ &\leq\left\|\xi_3^4\widehat{K_1}(t)\widehat{u_{10}}\right\|_{L^2}+\left\|\xi_3^4\widehat{K_2}(t)\widehat{b_{10}}\right\|_{L^2}\\ &\quad+\int_0^t\left\|\xi_3^4\widehat{K_1}(t-\tau)\widehat{N_{11}}(\tau)\right\|_{L^2}d\tau\\ &\quad+\int_0^t\left\|\xi_3^4\widehat{K_2}(t-\tau)\widehat{N_{21}}(\tau)\right\|_{L^2}d\tau\triangleq\sum_{m=1}^4\Omega_{4,m}.\end{aligned} \tag{5.63}$$

It follows from Corollary 2.2 with $\beta=4$ that

$$\Omega_{4,1}+\Omega_{4,2}\leq C(1+t)^{-\frac{11}{4}}\left(\|u_0\|_{L^1}+\|\partial_3^4u_0\|_{L^2}\right)\leq C\varepsilon(1+t)^{-\frac{11}{4}}. \tag{5.64}$$

Due to (2.10), (2.12)–(2.14) and (4.3), we have

$$\Omega_{4,3}\leq C\int_0^t\left\|\xi_3^4e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_kb_1}\right\|_{L^2}d\tau$$

$$
\begin{aligned}
&+C\int_0^t\left\||\xi_3|^4e^{-c\xi_\nu^2(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_kb_l}\right\|_{L^2}d\tau\\
&+C\int_0^t\left\||\xi_3|^4e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\xi_\nu^2|\xi|^{-2}\xi_k\widehat{b_kb_1}\right\|_{L^2}d\tau\\
&+C\int_0^t\left\||\xi_3|^4e^{-c(1+\xi_3^2)(t-\tau)}\sum_{k=1}^3\sum_{l=2}^3\xi_1|\xi|^{-2}\xi_k\xi_l\widehat{b_kb_l}\right\|_{L^2}d\tau\\
&\triangleq\Omega_{4,31}+\Omega_{4,32}+\Omega_{4,33}+\Omega_{4,34}
\end{aligned}\tag{5.65}
$$

Similarly to the derivations of (5.55) and (5.57), by (4.54) and (5.54) we find

$$
\begin{aligned}
\Omega_{4,31}&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-3}\|bb_1\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&\quad+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac12}\|\partial_3^3(b\cdot\nabla b_1)\|_{L^2}d\tau\\
&\le CC_0^2\varepsilon^2(1+t)^{-3},
\end{aligned}\tag{5.66}
$$

and by (4.55) and (5.56) we obtain

$$
\begin{aligned}
\Omega_{4,32}&\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-3}\|(bb_2,bb_3)\|_{L^2_{x_1}L^1_{x_2,x_3}}d\tau\\
&\quad+C\int_{\frac{t}{2}}^t\left\||\xi_\nu|e^{-c\xi_\nu^2(t-\tau)}\right\|_{L^\infty}\|\partial_3^4(bb_\nu)\|_{L^2}d\tau\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{11}{4}}.
\end{aligned}\tag{5.67}
$$

Due to (5.20) and (4.38), we deduce

$$
\begin{aligned}
\Omega_{4,33}+\Omega_{4,34}&\le C\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac12}\left\|\partial_3^3(b\cdot\nabla b)\right\|_{L^2}d\tau\\
&\le CC_0^2\varepsilon^2\int_0^t e^{-c(t-\tau)}(t-\tau)^{-\frac12}(1+\tau)^{-\frac{27}{8}}d\tau\\
&\le CC_0^2\varepsilon^2(1+t)^{-\frac{27}{8}}.
\end{aligned}\tag{5.68}
$$

By (5.66), (5.67) and (5.68), we conclude from (5.65) that

$$\Omega_{4,3} \leq CC_0^2\varepsilon^2 (1+t)^{-\frac{11}{4}}. \tag{5.69}$$

Finally, analogously to the treatment of $\Omega_{3,4}$ in (5.60), using (4.54), (5.56) and (5.61), we obtain

$$\begin{aligned}\Omega_{4,4} \leq & C\int_0^{\frac{t}{2}} (1+t-\tau)^{-3} \|(b_2u_1, b_3u_1)\|_{L^2_{x_1}L^1_{x_2,x_3}} d\tau \\ & + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}} \|(\partial_3^4(b_2u_1), \partial_3^4(b_3u_1)\|_{L^2} d\tau \\ & + C\int_0^t e^{-c(t-\tau)} (t-\tau)^{-\frac{1}{2}} \|(\partial_2\partial_3^3(b_2u_1), \partial_3^4(b_3u_1)\|_{L^2} d\tau \\ \leq & CC_0^2\varepsilon^2 (1+t)^{-3}. \end{aligned} \tag{5.70}$$

Now, plugging (5.64), (5.69) and (5.70) into (5.63) gives rise to

$$\|\partial_3^4 u_1\|_{L^2} \leq \left(C\varepsilon + CC_0^2\varepsilon^2\right)(1+t)^{-\frac{11}{4}} \leq \frac{C_0}{2}\varepsilon(1+t)^{-\frac{11}{4}},$$

provided C_0 is chosen to be large enough and ε is chosen to be suitably small.

Proof of Theorem 1.3. With all the estimates established in Subsections 5.2 and 5.3 at hand, we immediately obtain the desired decay rates of the solutions stated in Theorem 1.3, based on (5.1), (5.3), (5.4) and the bootstrapping arguments. □

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References

[1] H. Abidi, P. Zhang, On the global solution of 3D MHD system with initial data near equilibrium, Commun. Pure Appl. Math. 70 (2017) 1509–1561.
[2] A. Alemany, R. Moreau, P.-L. Sulem, U. Frisch, Influence of an external magnetic field on homogeneous MHD turbulence, J. Méc. 18 (1979) 277–313.
[3] A. Alexakis, Two-dimensional behavior of three-dimensional magnetohydrodynamic flow with a strong guiding field, Phys. Rev. E 84 (2011) 056330.
[4] H. Alfvén, Existence of electromagnetic-hydrodynamic waves, Nature 150 (1942) 405–406.
[5] C. Bardos, C. Sulem, P.L. Sulem, Longtime dynamics of a conductive fluid in the presence of a strong magnetic field, Trans. Am. Math. Soc. 305 (1988) 175–191.
[6] D. Biskamp, Nonlinear Magnetohydrodynamics, Cambridge University Press, Cambridge, 1993.
[7] N. Boardman, H. Lin, J. Wu, Stabilization of a background magnetic field on a 2 dimensional magnetohydrodynamic flow, SIAM J. Math. Anal. 52 (2020) 5001–5035.
[8] Y. Cai, Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, Arch. Ration. Mech. Anal. 228 (2018) 969–993.
[9] Q. Chen, X. Ren, Global well-posedness for the 2D non-resistive MHD equations in two kinds of periodic domains, Z. Angew. Math. Phys. 70 (2019) 18.
[10] P.A. Davidson, An Introduction to Magnetohydrodynamics, Cambridge University Press, Cambridge, England, 2001.
[11] W. Deng, P. Zhang, Large time behavior of solutions to 3-D MHD system with initial data near equilibrium, Arch. Ration. Mech. Anal. 230 (2018) 1017–1102.
[12] Y. Du, W. Yang, Y. Zhou, On the exponential stability of a stratified flow to the 2D ideal MHD equations with damping, SIAM J. Math. Anal. 51 (2019) 5077–5102.
[13] W. Feng, F. Hafeez, J. Wu, Influence of a background magnetic field on a 2D magnetohydrodynamic flow, Nonlinearity 234 (2021) 2527–2562.
[14] B. Gallet, M. Berhanu, N. Mordant, Influence of an external magnetic field on forced turbulence in a swirling flow of liquid metal, Phys. Fluids 21 (2009) 085107.
[15] L. He, L. Xu, P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, Ann. PDE 4 (2018) 5.
[16] C. Ivory, W. Gobie, J. Beckwith, R. Hergenrother, M. Malec, Electromagnetic stabilization of weakly conducting fluids, Science 238 (1987) 58–61.
[17] F. Jiang, S. Jiang, Asymptotic behaviors of global solutions to the two-dimensional non-resistive MHD equations with large initial perturbations, Adv. Math. 393 (2021) 108084.
[18] S. Lai, J. Wu, J. Zhang, Stabilizing phenomenon for 2D anisotropic magnetohydrodynamic system near a background magnetic field, SIAM J. Math. Anal. 53 (2021) 6073–6093.
[19] S. Lai, J. Wu, J. Zhang, Stabilizing effect of magnetic field on the 2D ideal magnetohydrodynamic flow with mixed partial damping, Calc. Var. Partial Differ. Equ. 53 (2022) 126.
[20] C. Li, J. Wu, X. Xu, Smoothing and stabilization effects of magnetic field on electrically conducting fluids, J. Differ. Equ. 276 (2021) 368–403.
[21] F. Lin, L. Xu, P. Zhang, Global small solutions to 2-D incompressible MHD system, J. Differ. Equ. 259 (2015) 5440–5485.
[22] H. Lin, R. Ji, J. Wu, L. Yan, Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation, J. Funct. Anal. 279 (2020) 108519.
[23] H. Lin, J. Wu, X. Suo, The global well-posedness and decay estimates for the 3D incompressible MHD equations with vertical dissipation in a strip, Int. Math. Res. Not. (2023) 1–41.
[24] H. Lin, J. Wu, Y. Zhu, Global solutions to 3D incompressible MHD system with dissipation in only one direction, SIAM J. Math. Anal. 55 (2023) 4570–4598.
[25] H. Lin, J. Wu, Y. Zhu, Stability and large-time behavior on 3D incompressible MHD equations with partial dissipation near a background magnetic field, Arch. Ration. Mech. Anal. 249 (2025) 26.
[26] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, 2002.
[27] H.K. Moffatt, On the suppression of turbulence by a uniform magnetic field, J. Fluid Mech. 28 (1967) 571–592.
[28] M. Paicu, P. Zhang, Global strong solutions to 3-D Navier-Stokes system with strong dissipation in one direction, Sci. China, Math. 62 (2019) 1175–1204.
[29] M. Paicu, N. Zhu, Global regularity for the 2D MHD and tropical climate model with horizontal dissipation, J. Nonlinear Sci. 31 (2021) 99.
[30] R. Pan, Y. Zhou, Y. Zhu, Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes, Arch. Ration. Mech. Anal. 227 (2018) 637–662.

[31] E. Priest, T. Forbes, Magnetic Reconnection, MHD Theory and Applications, Cambridge University Press, Cambridge, 2000.
[32] X. Ren, J. Wu, Z. Xiang, Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, J. Funct. Anal. 267 (2014) 503–541.
[33] X. Ren, Z. Xiang, Z. Zhang, Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain, Nonlinearity 29 (2016) 1257–1291.
[34] M.E. Schonbek, T. Schonbek, Moments and lower bounds in the far-field of solutions to quasi-geostrophic flows, Discrete Contin. Dyn. Syst. 13 (2005) 1277–1304.
[35] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, New Jersey, 1970.
[36] Z. Tan, Y. Wang, Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems, SIAM J. Math. Anal. 50 (2018) 1432–1470.
[37] T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics, vol. 106, American Mathematical Society, Providence, RI, 2006.
[38] R.H. Wan, Global existence and long time behavior of smooth solution for the 2D Boussinesq-Navier-Stokes equations, Preprint.
[39] D. Wei, Z. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, Anal. PDE 10 (2017) 1361–1406.
[40] J. Wu, Dissipative quasi-geostrophic equations with L^p data, Electron. J. Differ. Equ. 2001 (2001) 1–13.
[41] J. Wu, Y. Wu, X. Xu, Global small solution to the 2D MHD system with a velocity damping term, SIAM J. Math. Anal. 47 (2015) 2630–2656.
[42] J. Wu, The 2D magnetohydrodynamic equations with partial or fractional dissipation, in: Lectures on the Analysis of Nonlinear Partial Differential Equations, Morningside Lectures on Mathematics, Part 5, MLM5, International Press, Somerville, MA, 2018, pp. 283–332.
[43] J. Wu, Y. Zhu, Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium, Adv. Math. 377 (2021) 107466.
[44] T. Zhang, Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field, J. Differ. Equ. 260 (2016) 5450–5480.
[45] Y. Zhou, Y. Zhu, Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain, J. Math. Phys. 59 (2018) 081505.