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Global well-posedness and decay estimates for the 3D anisotropic Boussinesq system with thermal damping

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Abstract

This paper investigates the well-posedness of the 3D Boussinesq system with vertical dissipation and thermal damping in a strip domain. The present work is the first to address the case where the Boussinesq velocity equation contains dissipation in only one spatial direction. Two main challenges arise in establishing small-data global well-posedness: the lack of horizontal dissipation and the imposition of Dirichlet boundary conditions on the velocity. As a consequence, direct approach of energy estimates on the Sobolev norms of (u, θ) fails. To overcome these difficulties, we decompose the Sobolev estimates into two components: one focusing on horizontal derivatives and the other on vertical ones. The vertical derivatives are controlled using Stokes estimates in combination with bounds on the time derivatives of (u, θ). In addition, we take advantage of the structural feature of the spatial domain and make extensive use of anisotropic Sobolev inequalities.

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1. Introduction

Attention here is focused on global well-posedness and long-time behavior of solutions for the 3D incompressible Boussinesq system with only vertical dissipation and thermal damping in a strip domain $\Omega = \mathbb{R}^2 \times [0, 1]$,

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \partial_{33} u + \theta e_3, & x \in \Omega,\ t > 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \eta\theta = 0, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ u = 0, \quad \theta = 0, & x \in \partial\Omega,\ t > 0, \\ u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here, $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ denotes the fluid velocity, while $\theta(x, t)$ and $p(x, t)$ represent the buoyancy field and the total pressure, respectively. The parameters $\nu > 0$ and $\eta > 0$ denote the viscosity coefficient and damping coefficient, respectively, and $e_3 = (0, 0, 1)^T$ is the unit vector in the vertical direction.

The Boussinesq equations model large scale atmospheric and oceanic flows that are responsible for cold front and the jet stream (see, e.g., [12,22,24]). Mathematically the Boussinesq equations retain some key features of Euler and Navier-Stokes equations such as the vortex stretching mechanism. Research on the Boussinesq system has provided significant insights into its mathematical behavior under different conditions, particularly focusing on global existence, stability, and decay properties.

The Boussinesq system (1.1) models the case with vertical dissipation only, which is physically relevant in geophysical fluid dynamics. In geophysical fluid dynamics, particularly in mesoscale and large-scale regimes, vertical dissipation often dominates horizontal dissipation due to the anisotropic structure of stratified and rotating fluids in the atmosphere and oceans. These flows typically exhibit strong stable stratification in the vertical direction and relatively weak horizontal turbulence. In the Boussinesq equations, temperature evolves according to an advection-diffusion equation, where the diffusion (or damping) term originates from Fourier's law of heat conduction. Temperature damping is also physically justified in many scenarios, including subgrid-scale mixing processes in the atmosphere and ocean (see, e.g., [12,24]). Moreover, from a mathematical perspective, anisotropic or partially dissipative systems pose substantial challenges. Analyzing such equations requires the development of sophisticated mathematical tools and can also help us better understand the core difficulties of fully dissipative or inviscid systems. By selectively "turning off" certain dissipative mechanisms, we can more clearly discern the role of remaining dissipation and identify the potential pathological behaviors induced by nonlinearities in the absence of sufficient regularization. This perspective not only deepens our understanding of the anisotropic Boussinesq system but also provides important progress toward the broader goal of understanding the Boussinesq equations with minimal or no dissipation.

The goal of this paper is to investigate the global well-posedness and large-time behavior of (1.1) in the domain $\mathbb{R}^2 \times [0, 1]$. These results form part of our broader effort to understand the global dynamics of the Boussinesq system under minimal dissipation. In addition, we aim to

develop new techniques for establishing well-posedness and decay estimates for fluid equations with partial dissipation in domains with boundaries.

The global well-posedness problem on (1.1) is not trivial. There are several main difficulties. The first is the absence of velocity dissipation in two directions. The dissipation in only one direction is simply not enough to control the nonlinearity directly. This is the reason why the global well-posedness problem on the 3D Navier-Stokes equations remains open when the spatial domain is the whole space $\mathbb{R}^3$ or the periodic domain $\mathbb{T}^3$. This paper exploits the special properties of the fact that our spatial domain is bounded in the direction of dissipation together with the Dirichlet boundary condition.

The second main difficulty stems from the presence of Dirichlet boundary conditions. Standard energy estimates and integration by parts cannot be directly applied, as boundary terms would arise. To overcome this, we separate the estimates for horizontal and vertical derivatives. The horizontal derivatives of u and θ inherit the same boundary conditions as u and θ themselves, allowing us to derive uniform bounds for both the horizontal and time derivatives. In contrast, the vertical derivatives cannot be directly controlled via energy methods. Instead, we estimate them indirectly by rewriting the velocity equations in terms of the Stokes operator and applying known bounds for that operator.

The third difficulty arises from the presence of the buoyancy forcing term θe_3 in the velocity equation, which can cause the Sobolev norms of the velocity in (1.1) to grow over time. In fact, Brandolese and Schonbek [5] demonstrated that even with full viscous dissipation and thermal diffusion, the L^2-norm of the velocity in the Boussinesq system may grow in time for smooth initial data. In our setting, we overcome this difficulty by exploiting the thermal damping mechanism. Specifically, the Poincaré-type inequality (2.1) allows us to effectively control the buoyancy forcing term and prevent unbounded growth of the velocity.

It is natural to investigate the large-time behavior of these globally stable solutions. However, this problem is highly nontrivial due to the anisotropic dissipation and the geometry of the spatial domain. The dissipation acts only in the vertical direction, and the presence of Dirichlet boundary conditions leads to the formation of boundary layers. To address these challenges, we carefully combine and analyze a collection of energy functionals developed in the existence proof. This strategy enables us to derive exponential decay estimates for high-order horizontal derivatives of the velocity, all derivatives of the temperature, as well as the time derivatives of u and θ. These results are particularly remarkable given the limited dissipation in the system.

To state our main result precisely, we clarify that the constant C in inequalities of the form $A \le CB$ may vary from line to line. We also introduce the following notation and norms

$$f_h = (f_1, f_2),\ \ \nabla_h f = (\partial_1 f, \partial_2 f),\ \ \Delta_h f = \partial_1^2 f + \partial_2^2 f,\ \ \partial_t f = f_t,\ \ \partial_t f_3 = f_{3t},$$

$$x_h = (x_1, x_2),\ \ \|f\|_{L^p_{x_3} L^q_{x_h}(\Omega)} = \Big(\int_0^1 \Big(\int_{\mathbb{R}^2} |f|^q \, dx_h \Big)^{\frac{p}{q}} dx_3 \Big)^{\frac{1}{p}},$$

$$\|f\|^2_{H^{s,0}(\Omega)} = \sum_{i=1}^{2} \sum_{0 \le \alpha \le s} \|\partial_i^\alpha f\|^2_{L^2(\Omega)},\ \ \|f\|^2_{\dot{H}^{s,0}(\Omega)} = \sum_{i=1}^{2} \|\partial_i^s f\|^2_{L^2(\Omega)},$$

$$\|(f,g)\|^2_{H^s(\Omega)} = \|f\|^2_{H^s(\Omega)} + \|g\|^2_{H^s(\Omega)},\quad \|(f,g)\|^2_{H^{s,0}(\Omega)} = \|f\|^2_{H^{s,0}(\Omega)} + \|g\|^2_{H^{s,0}(\Omega)}.$$

With this notation at our disposal, we now state our main result.

Theorem 1.1. *Assume that the initial data $(u_0, \theta_0) \in H^3(\Omega)$ satisfies the incompressibility condition $\nabla \cdot u_0 = 0$ in Ω and the boundary conditions $u_0 = \theta_0 = 0$ on $\partial\Omega$. Then there exists a constant $\delta > 0$ such that if*

$$\|u_0\|_{H^3(\Omega)} + \|\theta_0\|_{H^3(\Omega)} \le \delta,$$

the 3D Boussinesq system (1.1) admits a unique global solution (u, θ) satisfying, for all $t > 0$,

$$\begin{aligned}
&\|(u(t), \theta(t))\|^2_{H^3(\Omega)} + \|(\partial_t u(t), \partial_t \theta(t))\|^2_{H^1(\Omega)} \\
&\quad + \int_0^t \left(\nu \|\partial_3 u(\tau)\|^2_{H^{3,0}(\Omega)} + \|\theta(\tau)\|^2_{H^3(\Omega)} \right) d\tau \\
&\quad + \int_0^t \left(\nu \|\partial_3 u_\tau(\tau)\|^2_{H^{1,0}(\Omega)} + \|u_{\tau\tau}(\tau)\|^2_{L^2(\Omega)} + \|\theta_\tau(\tau)\|^2_{H^3(\Omega)} \right) d\tau \le C(\delta + \delta^2),
\end{aligned}$$

for some constant $C > 0$ depending on ν and η.

Moreover, the solution satisfies the exponential decay estimate:

$$\|(u(t), \theta(t))\|_{H^3(\Omega)} + \|(\partial_t u(t), \partial_t \theta(t))\|_{H^1(\Omega)} \le C e^{-C^* t},$$

for some constants $C > 0$ and $C^* > 0$ depending on ν and η.

We now provide a more detailed outline of the main ideas involved in the proof of Theorem 1.1. Naturally, the argument is divided into two primary components: the first establishes global well-posedness, and the second derives the large-time decay estimates.

To obtain the global bounds for the well-posedness part, a key step is to isolate the horizontal derivatives by estimating $\|u\|_{H^{3,0}}$ from the full Sobolev norm $\|u\|_{H^3}$. Since the horizontal derivatives of u vanish on the boundary (due to the Dirichlet condition), this separation avoids the appearance of boundary terms during integration by parts.

Another critical observation is that certain Poincaré-type inequalities hold for functions vanishing on the boundary. Specifically, for such functions f, we have

$$\|f\|_{L^2(\Omega)} \le C \|\partial_3 f\|_{L^2(\Omega)}, \quad \|f\|_{L^\infty(\Omega)} \le C \|\partial_3 f\|_{H^{2,0}(\Omega)},$$

where the bounds involve only vertical derivatives. This property plays a crucial role throughout the analysis. In particular, the anisotropic inequalities established in Lemma 2.2 rely on this structure and help resolve difficulties posed by the buoyancy term as well as the lack of dissipation in the horizontal directions.

To illustrate this idea, consider the term

$$I_1 := \int \theta \, u_3 \, dx.$$

Using the Poincaré inequality $\|u\|_{L^2(\Omega)} \le C \|\partial_3 u\|_{L^2(\Omega)}$, together with Hölder's and Young's inequalities, we obtain

$$I_1 := \int \theta\, u_3\, dx \le C\|u\|_{L^2}\|\theta\|_{L^2} \le \frac{\nu}{2}\|\partial_3 u\|_{L^2}^2 + C_\nu\|\theta\|_{L^2}^2,$$

for some constant $C_\nu > 0$ depending on ν. The first term can be absorbed by the available vertical dissipation in the velocity equation, while the second term can be controlled via the thermal damping in the temperature equation, after multiplying the energy identity by a suitably small constant.

We define the energy functional

$$\varepsilon(t) := \varepsilon_0(t) + \varepsilon_1(t),$$

where

$$\varepsilon_0(t) = \sup_{0\le\tau\le t}\left(\|u(\tau)\|_{H^{3,0}}^2 + \|\theta(\tau)\|_{H^3}^2\right) + \int_0^t \left(\nu\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\theta(\tau)\|_{H^3}^2\right) d\tau, \tag{1.2}$$

and

$$\varepsilon_1(t) = \sup_{0\le\tau\le t}\|(u_\tau(\tau),\theta_\tau(\tau))\|_{H^1}^2 + \int_0^t \left(\nu\|\partial_3 u_\tau(\tau)\|_{H^{1,0}}^2 + \|u_{\tau\tau}(\tau)\|_{L^2}^2 + \|\theta_\tau(\tau)\|_{H^3}^2\right) d\tau. \tag{1.3}$$

We now provide a brief explanation of the intuition behind the decomposition of the energy functional $\varepsilon(t)$. Our goal is to establish the global well-posedness of (1.1) in the Sobolev setting H^3. $\varepsilon(t)$ is divided into two parts, $\varepsilon_0(t)$ and $\varepsilon_1(t)$. The first part, $\varepsilon_0(t)$, involves the norms of the horizontal derivatives and their corresponding time-integrated contributions due to dissipation or damping, while $\varepsilon_1(t)$ captures the vertical derivatives. As aforementioned, due to the lack of boundary conditions for the vertical derivatives of (u,θ), integration by parts cannot be directly applied in the energy estimates. In particular, we can not directly establish high-order regularity estimates for the vertical derivatives of the velocity field. Therefore, our strategy is to exploit the fact that the horizontal derivative is zero on the boundary to first establish an estimate concerning $\|u\|_{H^{3,0}}$, which ultimately leads to the energy inequality (3.1) for $\varepsilon_0(t)$. For the vertical derivatives of u, we employ Stokes estimates, which require a norm estimate of the time derivative of the velocity. Furthermore, as can be seen from (3.1), the upper bound of $\varepsilon_0(t)$ depends on $\varepsilon_1(t)$. This leads us to define $\varepsilon_1(t)$ and a careful estimate of it, yielding inequality (4.1). By combining the estimates for $\varepsilon_0(t)$ and $\varepsilon_1(t)$, we obtain the inequality

$$\varepsilon(t) \le C_0^*\varepsilon(0) + C_1^*\left(\varepsilon^{3/2}(t) + \varepsilon^2(t) + \varepsilon^3(t)\right),$$

for some constants $C_0^* > 0$, $C_1^* > 0$. A standard bootstrapping argument then yields the uniform bound in (5.2), namely

$$\|u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|(u_t,\theta_t)\|_{L^2}^2 + \int_0^t (\nu\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\theta(\tau)\|_{H^3}^2)\, d\tau \le C\|(u_0,b_0)\|_{H^3}^2 \le C\delta^2,$$

provided the initial norm $\|(u_0,\theta_0)\|_{H^3}$ is sufficiently small.

Note that in $\varepsilon_0(t)$, only the horizontal regularity of u is obtained. Vertical derivatives cannot be directly estimated via standard energy methods. To address this, we employ the Stokes system estimates. Specifically, Lemma 2.4 yields

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} \le C\left(\|\partial_t u\|_{H^1} + \|\Delta_h u\|_{H^{1,0}} + \|\partial_3 \Delta_h u\|_{L^2} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{2,0}}^2\right)\\ + C\|u\|_{H^{3,0}}\left(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{2,0}} + \|\theta\|_{L^2}\right).\end{aligned}$$

With the established bound on ε_0, we can derive inequality (5.5) and subsequently control $\|\nabla u\|_{H^2}$. Note that the H^2-norm of ∇u is controlled through the time-derivative norms, highlighting the necessity of including time-derivative terms in the energy $\varepsilon_1(t)$.

Furthermore, when estimating the third-order derivatives of the temperature, especially in handling the nonlinear term J_3, control over $\|\nabla u\|_{H^2}$ becomes essential. This further justifies the introduction of $\varepsilon_1(t)$, and the careful treatment of time-derivative estimates is detailed in Lemma 4.1.

The exponential decay estimate in Theorem 1.1 relies critically on the bounds obtained for $\varepsilon_0(t)$ and $\varepsilon_1(t)$. As in the global well-posedness analysis, we cannot directly derive the decay rate for $\|\nabla u\|_{H^2}$. Our approach is to first establish exponential decay for the combined quantity

$$\|u\|_{H^{3,0}} + \|\theta\|_{H^3} + \|(u_t, \theta_t)\|_{H^1}.$$

To this end, we incorporate the estimate for $\|(u_t, \theta_t)\|_{H^1}$ from Lemma 4.1 and inequality (3.11) into equation (3.10). This leads to the following differential inequality:

$$\partial_t X(t) + C\, X(t) \le 0,$$

for some constant $C > 0$, where

$$X(t) := \|u\|_{H^{3,0}} + \|\theta\|_{H^3} + \|(u_t, \theta_t)\|_{H^1}.$$

Applying Gronwall's inequality, we deduce

$$\|u(t)\|_{H^{3,0}} + \|\theta(t)\|_{H^3} + \|(u_t(t), \theta_t(t))\|_{H^1} \le C\, e^{-Ct}. \tag{1.4}$$

With the decay estimate in (1.4), we can further deduce the exponential decay of $\|\nabla u\|_{H^2}$ by employing the Stokes estimates established in Lemma 2.4.

To place our main result in a broader context, we briefly review some related work on the Boussinesq equations. Due to their wide range of applications in geophysical fluid dynamics and their mathematical richness, Boussinesq systems for buoyancy-driven flows have attracted considerable attention. In particular, the 2D Boussinesq equations with various types of partial or fractional dissipation have been extensively studied, leading to substantial progress on the global existence and regularity problem (see, e.g., [1–4,6,8,11,13,14,16,17,19,21,29–31,34,35]). The study of stability and long-time behavior near hydrostatic balance is a more recent but equally important direction, where significant results have also been obtained (see, e.g., [5,7,9,10,18,20,27]).

In comparison with the 2D Boussinesq equations, the 3D case is substantially more difficult. Even with full dissipation, it is still unknown whether classical solutions of the 3D Boussinesq

system exist globally in time. The anisotropic setting is even more challenging: in 3D, the presence of dissipation in only one direction among three is considerably harder to handle than the corresponding 2D case, where one direction among two is dissipated. Indeed, for the 3D Navier-Stokes or Boussinesq equations with dissipation in a single direction, even small-data global well-posedness remains open, while in 2D the global existence of solutions to the Navier-Stokes equations is well established.

Because of these difficulties, relatively few results are available on the existence and long-time behavior of solutions to 3D systems with anisotropic dissipation. Some regularity criteria for global solutions have been obtained in [25,28]. Using Littlewood-Paley theory and Bony's paradifferential calculus, Qiu, Du, and Yao [25] derived a blow-up criterion for the 3D Boussinesq equations with full dissipation and thermal diffusion. In a different direction, [34] studied the 3D Boussinesq equations with fractional partial dissipation and proved global well-posedness for any initial data in H^1. Under the assumption of axisymmetric initial data without swirl, [23] established global well-posedness for the 3D Boussinesq equations with horizontal dissipation.

There have also been several recent advances concerning the stability and long-time behavior of solutions to the 3D Boussinesq equations. In particular, [15] established global stability and precisely characterized the large-time behavior of perturbations near hydrostatic balance for the 3D Boussinesq equations with only horizontal dissipation. Furthermore, under certain symmetry assumptions, [32] proved global stability for the 3D Boussinesq equations with horizontal dissipation and vertical thermal diffusion in the domain $\Omega = \mathbb{R}^2 \times \mathbb{T}$, where $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$ is a one-dimensional periodic interval. We also refer the reader to [26] for results on the stability of stationary solutions under partial dissipation.

To the best of our knowledge, the present work is the first to address the case where the velocity equation contains dissipation in only one spatial direction. This setting introduces new analytical challenges and requires the development of novel energy methods tailored to the anisotropic structure of the system.

The remainder of this paper is organized as follows. In Section 2, we present several key inequalities that will be used throughout the proof of Theorem 1.1. Section 3 is devoted to deriving estimates for $\varepsilon_0(t)$ and is further divided into two subsections. The bounds for $\varepsilon_1(t)$ are established in Section 4. Section 5 addresses the global well-posedness component of Theorem 1.1, while Section 6 is devoted to proving the exponential decay estimate stated in Theorem 1.1.

2. Preliminary

This section begins by presenting several inequalities that are essential for the subsequent analysis. These include the Poincaré-type inequalities and anisotropic inequalities, stated in Lemma 2.1 and Lemma 2.2, respectively. These tools are instrumental in fully utilizing the anisotropic dissipation and are key to establishing time-integrable upper bounds for the nonlinear terms. In addition, Lemma 2.4, which follows from Lemma 2.3, plays an equally crucial role in effectively controlling the nonlinear terms.

We now state the first lemma that involves several Poincaré-type inequalities.

Lemma 2.1. *Let $\Omega = \mathbb{R}^2 \times [0,1]$. Assume $f|_{\partial\Omega} = 0$, $f \in H^1(\Omega)$ and $\partial_3 f \in H^{2,0}(\Omega)$ and $g \in H^1(\Omega) \cap H^{2,0}(\Omega)$. Then for some constants $C > 0$, we have*

$$\|f\|_{L^2(\Omega)} \le C\|\partial_3 f\|_{L^2(\Omega)}, \tag{2.1}$$

$$\|f\|_{L^\infty(\Omega)} \le C\|\partial_3 f\|_{H^{2,0}(\Omega)}, \tag{2.2}$$

$$\|f\|_{L^4(\Omega)} \le C\|f\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_3 f\|^{\frac{1}{4}}_{L^2(\Omega)}\|\nabla_h f\|^{\frac{1}{2}}_{L^2(\Omega)} \le C\|\partial_3 f\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla_h f\|^{\frac{1}{2}}_{L^2(\Omega)}, \tag{2.3}$$

$$\|fg\|_{L^2(\Omega)} \le C\|\partial_3 f\|_{L^2(\Omega)}\|g\|_{H^{2,0}(\Omega)}, \tag{2.4}$$

$$\|fg\|_{L^2(\Omega)} \le C\|f\|_{H^{2,0}(\Omega)}(\|g\|_{L^2(\Omega)} + \|\partial_3 g\|_{L^2(\Omega)}). \tag{2.5}$$

Proof of Lemma 2.1. The first inequality is straightforward to obtain, we will focus on presenting the other four inequalities in detail. By Minkowski's inequality, interpolation inequality and the Sobolev imbedding inequality,

$$\begin{aligned}\|f\|_{L^\infty} &= \left\|\|f\|_{L^\infty_{x_3}}\right\|_{L^\infty_{x_h}} \le C\left\|\|\partial_3 f\|_{L^2_{x_3}}\right\|_{L^\infty_{x_h}} \le C\left\|\|\partial_3 f\|_{L^\infty_{x_h}}\right\|_{L^2_{x_3}}\\ &\le C\left\|\|\partial_3 f\|_{H^2_{x_h}}\right\|_{L^2_{x_3}} = C\|\partial_3 f\|_{H^{2,0}}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|f\|_{L^4} &= \left\|\|f\|_{L^4_{x_h}}\right\|_{L^4_{x_3}} \le C\left\|\|f\|^{\frac{1}{2}}_{L^2_{x_h}}\|\nabla_h f\|^{\frac{1}{2}}_{L^2_{x_h}}\right\|_{L^4_{x_3}} \le C\left\|\|f\|^2_{L^\infty_{x_3}}\right\|^{\frac{1}{4}}_{L^2_{x_h}}\|\nabla_h f\|^{\frac{1}{2}}_{L^2}\\ &\le C\left\|\|f\|_{L^2_{x_3}}\|\partial_3 f\|_{L^2_{x_3}}\right\|^{\frac{1}{4}}_{L^2_{x_h}}\|\nabla_h f\|^{\frac{1}{2}}_{L^2} \le C\|f\|^{\frac{1}{4}}_{L^2}\|\partial_3 f\|^{\frac{1}{4}}_{L^2}\|\nabla_h f\|^{\frac{1}{2}}_{L^2}\\ &\le C\|\partial_3 f\|^{\frac{1}{2}}_{L^2}\|\nabla_h f\|^{\frac{1}{2}}_{L^2}.\end{aligned}$$

For (2.4) and (2.5), by Minkowski's inequality, Sobolev inequality and Hölder inequality, we have

$$\begin{aligned}\|fg\|_{L^2} &\le \|f\|_{L^2_{x_h}L^\infty_{x_3}}\|g\|_{L^\infty_{x_h}L^2_{x_3}}\\ &\le C\left\|\|f\|^{\frac{1}{2}}_{L^2_{x_3}}\|\partial_3 f\|^{\frac{1}{2}}_{L^2_{x_3}}\right\|_{L^2_{x_h}}\left\|\|g\|^{\frac{1}{2}}_{L^2_{x_h}}\|\nabla_h^2 g\|^{\frac{1}{2}}_{L^2_{x_h}}\right\|_{L^2_{x_3}}\\ &\le C\|f\|^{\frac{1}{2}}_{L^2}\|\partial_3 f\|^{\frac{1}{2}}_{L^2}\|g\|^{\frac{1}{2}}_{L^2}\|\nabla_h^2 g\|^{\frac{1}{2}}_{L^2} \le C\|\partial_3 f\|_{L^2}\|g\|_{H^{2,0}}\end{aligned}$$

and

$$\begin{aligned}\|fg\|_{L^2} &\le \|f\|_{L^\infty_{x_h}L^2_{x_3}}\|g\|_{L^2_{x_h}L^\infty_{x_3}}\\ &\le C\left\|\|f\|^{\frac{1}{2}}_{L^2_{x_h}}\|\nabla_h^2 f\|^{\frac{1}{2}}_{L^2_{x_h}}\right\|_{L^2_{x_3}}\left\|\|g\|^{\frac{1}{2}}_{L^2_{x_3}}\|\partial_3 g\|^{\frac{1}{2}}_{L^2_{x_3}} + \|g\|_{L^2_{X_3}}\right\|_{L^2_{x_h}}\\ &\le C\|f\|^{\frac{1}{2}}_{L^2}\|\nabla_h^2 f\|^{\frac{1}{2}}_{L^2}(\|g\|^{\frac{1}{2}}_{L^2}\|\partial_3 g\|^{\frac{1}{2}}_{L^2} + \|g\|_{L^2})\\ &\le C\|f\|_{H^{2,0}}(\|g\|_{L^2} + \|\partial_3 g\|_{L^2}).\end{aligned}$$

This completes the proof of Lemma 2.1. □

The second lemma presents several anisotropic inequalities for triple products. When applying anisotropic inequalities to triple products, the goal is often to manage terms involving derivatives in different directions in systems with anisotropic dissipation or scaling. These inequalities help control the interactions between terms and ensure the estimates remain bounded in Sobolev spaces. The proof of this lemma follows the approach in [33].

Lemma 2.2. *Let $\Omega = \mathbb{R}^2 \times [0,1]$. Assume $f|_{\partial\Omega} = 0$. Then for some pure constants $C > 0$, we have*

$$\int |fgh|dx \le C\|\partial_3 f\|_{L^2(\Omega)}\|g\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_1 g\|^{\frac{1}{2}}_{L^2(\Omega)}\|h\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_2 h\|^{\frac{1}{2}}_{L^2(\Omega)}, \tag{2.6}$$

$$\int |fgh|dx \le C\|\partial_3 f\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_1 f\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_{13} f\|^{\frac{1}{4}}_{L^2(\Omega)}\|g\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_2 g\|^{\frac{1}{2}}_{L^2(\Omega)}\|h\|_{L^2(\Omega)}. \tag{2.7}$$

$$\begin{aligned}\int |fgh|dx &\le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)}\|\partial_3 f\|^{\frac{1}{2}}_{L^2(\Omega)}\|g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_1 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_1\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|h\|_{L^2(\Omega)}\\ &\le C\|\partial_3 f\|_{L^2(\Omega)}\|g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_1 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|\partial_1\partial_2 g\|^{\frac{1}{4}}_{L^2(\Omega)}\|h\|_{L^2(\Omega)}.\end{aligned} \tag{2.8}$$

The following lemma shows the Stokes estimates. It provides essential tools for controlling velocity, pressure, and their derivatives, especially in the construction of energy estimates. Their combination with nonlinear techniques is what enables deeper exploration of regularity. And it plays a key role in establishing the estimates in the last lemma.

Lemma 2.3 *(Stokes estimates). Let $\Omega = \mathbb{R}^2 \times [0,1]$ be the strip domain. Let $f \in H^k(\Omega)$ with $k \ge 0$ being an integer. Assume $\tilde{u} \in H^1(\Omega)$ is the weak solution of the Stokes equations*

$$\begin{cases} -\Delta\tilde{u} + \nabla\tilde{P} = f, & in\ \Omega,\\ \nabla\cdot\tilde{u} = 0, & in\ \Omega,\\ \tilde{u} = 0, & on\ \partial\Omega. \end{cases} \tag{2.9}$$

Then (2.9) has a unique strong solution $(\tilde{u}, \tilde{P}) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ and the following estimate

$$\|\nabla\tilde{u}\|_{H^{k+1}(\Omega)} + \|\tilde{P}\|_{H^k(\Omega)} \le C\|f\|_{H^k(\Omega)} \tag{2.10}$$

holds for some positive constant C.

The final lemma will use Stokes estimates to establish the regularity estimates for the velocity field. Due to the characteristics of the domain under consideration, it is challenging to directly establish the vertical regularity of the velocity field through energy estimates. Our approach is to transform the vertical derivatives into estimates involving horizontal derivatives and time derivatives by utilizing the following estimates and the established existence theory of solutions. Furthermore, this lemma plays a crucial role in nonlinear estimates, allowing us to convert higher-order vertical derivatives of the velocity field to achieve a closed energy framework.

Lemma 2.4. *Let $\Omega = \mathbb{R}^2 \times [0,1]$ be the strip domain. Assume (u,θ) is the solution of the Boussinesq system (1.1). Then we can get the following estimates*

$$\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} \le C(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + \|\theta\|_{L^2}) \tag{2.11}$$

and

$$\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} \le C(\|u\|_{H^{3,0}} + 1)(\|\partial_t u\|_{H^1} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|^2_{H^{3,0}}) \tag{2.12}$$

for some positive constant $C > 0$ depending on ν.

Proof of Lemma 2.4. Firstly, we can rewrite the velocity equation of (1.1) as

$$\begin{cases} -\nu\Delta u + \nabla p = f, \quad x \in \Omega,\ t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0, \\ \nabla \cdot u = 0, \quad x \in \Omega,\ t > 0, \end{cases}$$

with

$$f := -\partial_t u - (u\cdot\nabla)u - \nu\Delta_h u + \theta e_3.$$

By Hölder's inequality, (2.2), (2.3) and Sobolev's imbedding inequality,

$$\begin{aligned} \|u\cdot\nabla u\|_{L^2} &\le C\|u_h\|_{L^4}\|\nabla_h u\|_{L^4} + \|u_3\|_{L^\infty}\|\partial_3 u\|_{L^2} \\ &\le C\|\nabla_h u_h\|^{\frac12}_{L^2}\|\partial_3 u_h\|^{\frac12}_{L^2}\|\nabla_h^2 u\|^{\frac12}_{L^2}\|\partial_3\nabla_h u\|^{\frac12}_{L^2} \\ &\quad + C\|\partial_3 u_3\|_{H^{2,0}}\|\partial_3 u\|_{L^2} \\ &\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}}. \end{aligned}$$

Then it follows from the Stokes estimates (2.10) that

$$\begin{aligned} \nu\|\nabla u\|_{H^1} + \|\nabla p\|_{L^2} &\le C(\|\partial_t u\|_{L^2} + \|(u\cdot\nabla)u\|_{L^2} + \|\Delta_h u\|_{L^2} + \|\theta\|_{L^2}) \\ &\le C(\|\partial_t u\|_{L^2} + \|\Delta_h u\|_{L^2} + \|\theta\|_{L^2} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}}) \\ &\le C(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + \|\theta\|_{L^2}). \end{aligned}$$

Thus, we conclude the proof of (2.11).

Similar to the discussion of $\|u\cdot\nabla u\|_{L^2}$, by means of Lemma 2.1, the previously established upper bound for $\|u\cdot\nabla u\|_{L^2}$ and $\|\nabla u\|_{H^1}$, we can deduce that

$$\begin{aligned} \|(u\cdot\nabla)u\|_{H^1} &\le \|(u\cdot\nabla)u\|_{L^2} + \|\nabla_h(u\cdot\nabla)u\|_{L^2} + \|\partial_3(u\cdot\nabla)u\|_{L^2} \\ &\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + C\|\nabla_h u_h\cdot\nabla_h u\|_{L^2} + C\|\nabla_h u_3\cdot\partial_3 u\|_{L^2} \\ &\quad + C(\|u_h\cdot\nabla_h^2 u\|_{L^2} + \|u_3\|_{L^\infty}\|\partial_3\nabla_h u\|_{L^2} + \|\partial_3 u_h\cdot\nabla_h u\|_{L^2} + \|\partial_3 u_3\,\partial_3 u\|_{L^2}) \end{aligned}$$

$$
\begin{aligned}
&+ C(\|u_h\|_{L^\infty}\|\partial_3\nabla_h u\|_{L^2} + \|u_3\|_{L^\infty}\|\partial_3^2 u\|_{L^2})\\
&\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + C\|\nabla_h u_h\|_{H^{2,0}}\|\nabla_h\partial_3 u\|_{L^2} + C\|\partial_3\nabla_h u_3\|_{L^2}\|\partial_3 u\|_{H^{2,0}}\\
&\quad + C(\|u_h\|_{H^{2,0}}\|\partial_3\nabla_h^2 u\|_{L^2} + \|\partial_3 u_3\|_{H^{2,0}}\|\partial_3\nabla_h u\|_{L^2})\\
&\quad + C\|\partial_3 u_h\|_{H^{2,0}}\|\partial_3\nabla_h u\|_{L^2} + C\|\partial_3 u_3\|_{H^{2,0}}(\|\partial_3 u\|_{L^2} + \|\partial_3^2 u\|_{L^2})\\
&\quad + C\|\partial_3 u_3\|_{H^{2,0}}\|\partial_3^2 u\|_{L^2}\\
&\le C(\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{2,0}} + \|\partial_3 u\|_{H^{2,0}}^2)\\
&\quad + C\|u\|_{H^{3,0}}\big(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + \|\theta\|_{L^2}\big)\\
&\le C\|u\|_{H^{3,0}}\big(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{1,0}} + \|\theta\|_{L^2}\big) + C\|\partial_3 u\|_{H^{2,0}}^2.
\end{aligned}
$$

Also, applying the Stokes estimates (2.10) yields

$$
\begin{aligned}
&\nu\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1}\\
&\le C(\|\partial_t u\|_{H^1} + \|(u\cdot\nabla)u\|_{H^1} + \|\Delta_h u\|_{H^1} + \|\theta\|_{H^1})\\
&\le C(\|\partial_t u\|_{H^1} + \|\Delta_h u\|_{H^{1,0}} + \|\partial_3\Delta_h u\|_{L^2} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{2,0}}^2)\\
&\quad + C\|u\|_{H^{3,0}}(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{2,0}} + \|\theta\|_{L^2})\\
&\le C(\|u\|_{H^{3,0}} + 1)(\|\partial_t u\|_{H^1} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|_{H^{3,0}}^2).
\end{aligned}
\tag{2.13}
$$

Thus,

$$
\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} \le C(\|u\|_{H^{3,0}} + 1)(\|\partial_t u\|_{H^1} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|_{H^{3,0}}^2),
$$

where $C > 0$ depends on ν.

This completes the proof of Lemma 2.4. □

3. Estimates for $\varepsilon_0(t)$

This section establishes the bound for ε_0. As highlighted in the introduction, the focus is on estimating $\|u\|_{H^{3,0}}$ and $\|\theta\|_{H^3}$. For clarity, the results are organized into a proposition, which provides an inequality for ε_0, and two lemmas. Lemma 3.2 details the estimate for $\|u\|_{H^{3,0}}$, while Lemma 3.3 establishes the bound for $\|\theta\|_{H^3}$.

Proposition 3.1. *Assume $(u_0, \theta_0) \in H_0^1 \cap H^3$ and let (u, θ) be the corresponding solution to (1.1). Then $\varepsilon_0(t)$ be defined as in (1.2) satisfies*

$$
\varepsilon_0(t) \le C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t) \tag{3.1}
$$

for some constant $C > 0$ depending on ν and η.

Proof of Proposition 3.1. Multiplying (3.3) by a suitably small constant δ_1 with $\delta_1 < \frac{2\eta\nu}{C^*}$ and adding it to (3.11), we obtain

$$\delta_1\|u\|^2_{H^{3,0}} + \|\theta\|^2_{H^3} + \delta_1\nu\int_0^t \|\partial_3 u(\tau)\|^2_{H^{3,0}}\,d\tau + (2\eta - \frac{C^*\delta_1}{\nu})\int_0^t \|\theta(\tau)\|^2_{H^3}\,d\tau$$
$$\leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t).$$

Thus,

$$\|u\|^2_{H^{3,0}} + \|\theta\|^2_{H^3} + \nu\int_0^t \|\partial_3 u(\tau)\|^2_{H^{3,0}}\,d\tau + \int_0^t \|\theta(\tau)\|^2_{H^3}\,d\tau$$
$$\leq \frac{1}{\tilde{C}}(C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t))$$
$$\leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t),$$

where $\tilde{C} = \min\{\delta_1, 1, (2\eta - \frac{C^*\delta_1}{\nu})\}$.

This completes the proof of Proposition 3.1. □

3.1. Estimates for $\|u\|_{H^{3,0}}$

To make the following proof clear, we first emphasize the following boundary conditions resulting from $u|_{\partial\Omega} = \theta|_{\partial\Omega} = 0$,

$$\partial_i^k u|_{\partial\Omega} = \partial_i^k \theta|_{\partial\Omega} = 0$$

for $i = 1, 2$ and $k = 1, 2, 3$. Furthermore, from (2.1) we can derive the following inequalities which will be used frequently in the subsequent proof.

$$\|\partial_i^k u\|_{L^2(\Omega)} \leq C\|\partial_3\partial_i^k u\|_{L^2(\Omega)},\ \ \|\partial_i^k\theta\|_{L^2(\Omega)} \leq C\|\partial_3\partial_i^k\theta\|_{L^2(\Omega)},\ i = 1, 2; k = 1, 2, 3. \tag{3.2}$$

Lemma 3.2. *Assume $(u_0, \theta_0) \in H_0^1 \cap H^{3,0}$ and let (u, θ) be the corresponding solution to (1.1). Then we have*

$$\|u\|^2_{H^{3,0}} + \nu\int_0^t \|\partial_3 u(\tau)\|^2_{H^{3,0}}\,d\tau \leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + \frac{C^*}{\nu}\int_0^t \|\theta(\tau)\|^2_{H^{3,0}}\,d\tau \tag{3.3}$$

for some constants $C > 0$ and $C^* > 0$.

Proof of Lemma 3.2. Taking the L^2-inner product of $(1.1)_1$ with u, integrating by parts and applying $\nabla \cdot u = 0$ and the boundary conditions, we find

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \nu\|\partial_3 u\|^2_{L^2} = \int \theta\, u_3\,dx := I_1. \tag{3.4}$$

Applying ∂_i^3 $(i = 1, 2)$ to $(1.1)_1$ and taking L^2-inner product of the resulting equations with $\partial_i^3 u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{2}\|\partial_i^3 u\|_{L^2}^2 + \sum_{i=1}^{2}\nu\|\partial_i^3\partial_3 u\|_{L^2}^2 := I_2 + I_3, \tag{3.5}$$

where

$$I_2 = -\sum_{i=1}^{2}\int \partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx,$$

$$I_3 = \sum_{i=1}^{2}\int \partial_i^3\theta\,\partial_i^3 u_3\,dx.$$

Here we have used the fact

$$\int \partial_i^3 u\cdot\partial_i^3\nabla p\,dx = 0,$$

which can be obtained by integration by parts, divergence-free condition for u and the boundary conditions $\partial_i^3 u|_{\partial\Omega} = 0$.

The estimates for I_1 and I_3 are straightforward. Using Hölder's inequality, Young's inequality and (3.2), we get

$$I_1 + I_3 \le \frac{\nu}{2}(\|\partial_3 u\|_{L^2}^2 + \|\partial_i^3\partial_3 u\|_{L^2}^2) + \frac{C_1}{\nu}(\|\theta\|_{L^2}^2 + \|\partial_i^3\theta\|_{L^2}^2). \tag{3.6}$$

To fully utilize the limited dissipation of the equations, we handle I_2 by decomposing $u\cdot\nabla u$ into $u_h\cdot\nabla_h u$ and $u_3\,\partial_3 u$, allowing I_2 to be expanded into two terms for separate analysis,

$$\begin{aligned} I_2 &= -\sum_{i=1}^{2}\int \partial_i^3(u_h\cdot\nabla_h u)\cdot\partial_i^3 u\,dx - \sum_{i=1}^{2}\int \partial_i^3(u_3\,\partial_3 u)\cdot\partial_i^3 u\,dx \\ &:= I_{2,1} + I_{2,2}. \end{aligned}$$

Further, we use Leibniz's formula and integration by parts to decompose $I_{2,1}$ into three terms based on the order k of the derivative,

$$\begin{aligned} I_{2,1} &= -\sum_{i=1}^{2}\sum_{k=1}^{3}C_3^k\int \partial_i^k u_h\cdot\partial_i^{3-k}\nabla_h u\cdot\partial_i^3 u\,dx \\ &= -3\sum_{i=1}^{2}\int \partial_i u_h\cdot\partial_i^2\nabla_h u\cdot\partial_i^3 u\,dx - 3\sum_{i=1}^{2}\int \partial_i^2 u_h\cdot\nabla_h\partial_i u\cdot\partial_i^3 u dx \\ &\quad -\sum_{i=1}^{2}\int \partial_i^3 u_h\cdot\nabla_h u\cdot\partial_i^3 u\,dx. \end{aligned}$$

By Hölder's inequality, (2.2), (2.3) and (3.2) with the Dirichlet boundary condition, we get

$$\begin{aligned} I_{2,1} &\le C\sum_{i=1}^{2}\|\nabla_h u_h\|_{L^\infty}\|\nabla_h^3 u\|_{L^2}\|\partial_i^3 u\|_{L^2} + C\sum_{i=1}^{2}\|\partial_i^2 u_h\|_{L^4}\|\nabla_h\partial_i u\|_{L^4}\|\partial_i^3 u\|_{L^2} \\ &\le C\|\partial_3\nabla_h u_h\|_{H^{2,0}}\|\partial_3 u\|_{H^{3,0}}\|u\|_{H^{3,0}} \\ &\quad + C\sum_{i=1}^{2}\|\partial_i^2\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_i^2\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_i\nabla_h^2 u\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_i\nabla_h u\|_{L^2}^{\frac{1}{2}}\|\partial_i^3\partial_3 u\|_{L^2} \\ &\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \end{aligned} \tag{3.7}$$

For $I_{2,2}$, we firstly split it into two parts. Applying the anisotropic inequalities (2.6) and (2.8), we have

$$\begin{aligned} I_{2,2} &= -\sum_{i=1}^{2}\sum_{k=1}^{2}C_3^k\int \partial_i^k u_3\,\partial_i^{3-k}\partial_3 u\cdot\partial_i^3 u\,dx - \sum_{i=1}^{2}\int \partial_i^3 u_3\,\partial_3 u\cdot\partial_i^3 u\,dx \\ &\le C\sum_{i=1}^{2}\sum_{k=1}^{2}\|\partial_i^k u_3\|_{L^2}^{\frac{1}{2}}\|\partial_i^k\partial_1 u_3\|_{L^2}^{\frac{1}{2}}\|\partial_i^{3-k}\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_i^{3-k}\partial_2\partial_3 u\|_{L^2}^{\frac{1}{2}}\|\partial_i^3\partial_3 u\|_{L^2} \\ &\quad + C\sum_{i=1}^{2}\|\partial_i^3 u_3\|_{L^2}\|\partial_3 u\|_{L^2}^{\frac{1}{4}}\|\partial_3\partial_1 u\|_{L^2}^{\frac{1}{4}}\|\partial_3\partial_2 u\|_{L^2}^{\frac{1}{4}}\|\partial_3\partial_{12} u\|_{L^2}^{\frac{1}{4}}\|\partial_i^3\partial_3 u\|_{L^2} \\ &\le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$I_2 \le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2. \tag{3.9}$$

By substituting (3.9) and (3.6) into the equation obtained from the summation of (3.4) and (3.5), we derive

$$\frac{d}{dt}(\|u\|_{L^2}^2 + \sum_{i=1}^{2}\|\partial_i^3 u\|_{L^2}^2) + \nu(\|\partial_3 u\|_{L^2}^2 + \sum_{i=1}^{2}\|\partial_i^3\partial_3 u\|_{L^2}^2) \le C\|u\|_{H^{3,0}}\|\partial_3 u\|_{H^{3,0}}^2 + \frac{C_1}{\nu}\|\theta\|_{H^{3,0}}^2. \tag{3.10}$$

Taking the time integral of equation (3.10) yields

$$\begin{aligned} \|u\|_{H^{3,0}}^2 + \nu\int_0^t \|\partial_3 u(\tau)\|_{H^{3,0}}^2\,d\tau &\le C\|u_0\|_{H^{3,0}}^2 + C\sup_{0\le\tau\le t}\|u\|_{H^{3,0}}\int_0^t \|\partial_3 u(\tau)\|_{H^{3,0}}^2\,d\tau \\ &\quad + \frac{C^*}{\nu}\int_0^t \|\theta(\tau)\|_{H^{3,0}}^2\,d\tau \end{aligned}$$

$$\leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + \frac{C^*}{\nu}\int_0^t \|\theta(\tau)\|_{H^{3,0}}^2\, d\tau,$$

where we also use the fact that

$$\begin{aligned}\|\partial_i u\|_{L^2}^2 + \|\partial_i^2 u\|_{L^2}^2 &\leq C(\|u\|_{L^2}^{\frac{4}{3}}\|\partial_i^3 u\|_{L^2}^{\frac{2}{3}} + \|u\|_{L^2}^{\frac{2}{3}}\|\partial_i^3 u\|_{L^2}^{\frac{4}{3}})\\ &\leq C(\|u\|_{L^2}^2 + \|\partial_i^3 u\|_{L^2}^2).\end{aligned}$$

This completes the proof of Lemma 3.2. □

3.2. Estimates for $\|\theta\|_{H^3}$

This subsection presents the estimates for $\|\theta\|_{H^3}$. Unlike the velocity field, due to the structural characteristics of the equation for θ, we will establish its regularity in all directions. Furthermore, the upcoming estimates for $\|\theta\|_{L^2}^2 + \|\partial_i^3\theta\|_{L^2}^2$ will play a crucial role in addressing the challenges posed by the buoyancy term.

Lemma 3.3. *Let (u, θ) be the solution to the system (1.1). Then we have*

$$\|\theta\|_{H^3}^2 + 2\eta\int_0^t \|\theta(\tau)\|_{H^3}^2\, d\tau \leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t). \tag{3.11}$$

Proof of Lemma 3.3. Taking the L^2-inner product of $(1.1)_2$ with θ, and then applying ∂_i^3 $(i = 1, 2)$ to $(1.1)_2$, followed by taking L^2-inner product of the resulting equations with $\partial_i^3\theta$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\theta\|_{L^2}^2 + \sum_{i=1}^2\|\partial_i^3\theta\|_{L^2}^2) + \eta(\|\theta\|_{L^2}^2 + \sum_{i=1}^2\|\partial_i^3\theta\|_{L^2}^2) = -\sum_{i=1}^2\int \partial_i^3(u\cdot\nabla\theta)\,\partial_i^3\theta\, dx := J_1.$$

The discussion regarding J_1 is similar to that of I_2. We begin by expanding it using Leibniz's formula, and then, by utilizing Hölder's inequality, (2.2), (2.7) and (2.8), we can derive the following estimate

$$\begin{aligned}J_1 &= -\sum_{i=1}^2\sum_{k=1}^3 C_3^k\int \partial_i^k u\cdot\partial_i^{3-k}\nabla\theta\,\partial_i^3\theta\, dx\\ &= -3\sum_{i=1}^2\Big(\int \partial_i u\cdot\partial_i^2\nabla\theta\,\partial_i^3\theta\, dx + \int \partial_i^2 u\cdot\partial_i\nabla\theta\,\partial_i^3\theta dx\Big)\\ &\quad -\sum_{i=1}^2\int \partial_i^3 u\cdot\nabla\theta\,\partial_i^3\theta\, dx\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{2}\|\partial_i u\|_{L^\infty}\|\partial_i^2\nabla\theta\|_{L^2}\|\partial_i^3\theta\|_{L^2}\\
&\quad + C\sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{\frac14}\|\partial_i^2\partial_1 u\|_{L^2}^{\frac14}\|\partial_i^2\partial_3 u\|_{L^2}^{\frac14}\|\partial_i^3\partial_3 u\|_{L^2}^{\frac14}\|\partial_i\nabla\theta\|_{L^2}^{\frac12}\|\partial_i\nabla\partial_2\theta\|_{L^2}^{\frac12}\|\partial_i^3\theta\|_{L^2}\\
&\quad + C\sum_{i=1}^{2}\|\nabla\theta\|_{L^2}^{\frac14}\|\nabla\partial_1\theta\|_{L^2}^{\frac14}\|\nabla\partial_2\theta\|_{L^2}^{\frac14}\|\nabla\partial_{12}\theta\|_{L^2}^{\frac14}\|\partial_i^3 u\|_{L^2}^{\frac12}\|\partial_i^3\partial_3 u\|_{L^2}^{\frac12}\|\partial_i^3\theta\|_{L^2}\\
&\le C\sum_{i=1}^{2}\|\partial_3 u\|_{H^{3,0}}\|\nabla\theta\|_{H^{2,0}}\|\partial_i^3\theta\|_{L^2} \le C\|\theta\|_{H^3}\big(\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^{3,0}}^2\big).
\end{aligned}
$$

From the above estimates for J_1, it is sufficient to obtain

$$
\frac{d}{dt}(\|\theta\|_{L^2}^2+\sum_{i=1}^{2}\|\partial_i^3\theta\|_{L^2}^2)+2\eta(\|\theta\|_{L^2}^2+\sum_{i=1}^{2}\|\partial_i^3\theta\|_{L^2}^2)\le C\|\theta\|_{H^3}(\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^{3,0}}^2). \tag{3.12}
$$

Applying $\partial_i\partial_3^2$ and ∂_3^3 to $(1.1)_2$, respectively, and then taking the L^2-inner product of the resulting equations with $\partial_3^2\partial_i\theta$ and $\partial_3^3\theta$, respectively, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}(\sum_{i=1}^{2}\|\partial_3^2\partial_i\theta\|_2^2+\|\partial_3^3\theta\|_{L^2}^2)+\eta(\sum_{i=1}^{2}\|\partial_3^2\partial_i\theta\|_{L^2}+\|\partial_3^3\theta\|_{L^2}^2)\\
&=-\sum_{i=1}^{2}\int\partial_3^2\partial_i(u\cdot\nabla\theta)\,\partial_3^2\partial_i\theta\,dx-\int\partial_3^3(u\cdot\nabla\theta)\,\partial_3^3\theta\,dx\\
&:=J_2+J_3.
\end{aligned}
$$

Now, we proceed to estimate J_2. To do this, we decompose it into five terms. By applying (2.2), (2.8), and utilizing the divergence-free condition and Young's inequality, we deduce that

$$
\begin{aligned}
J_2&=-\sum_{i=1}^{2}\int\partial_3^2(\partial_i u\cdot\nabla\theta+u\,\partial_i\nabla\theta)\,\partial_3^2\partial_i\theta\,dx\\
&=-\sum_{i=1}^{2}\Big(\int\partial_i u\cdot\nabla\partial_3^2\theta\,\partial_3^2\partial_i\theta\,dx+2\int\partial_3\partial_i u\cdot\nabla\partial_3\theta\,\partial_3^2\partial_i\theta\,dx+\int\partial_3^2\partial_i u\cdot\nabla\theta\,\partial_3^2\partial_i\theta\,dx\Big)\\
&\quad-\sum_{i=1}^{2}\Big(2\int\partial_3 u\cdot\partial_3\partial_i\nabla\theta\,\partial_3^2\partial_i\theta\,dx+\int\partial_3^2 u\cdot\partial_i\nabla\theta\,\partial_3^2\partial_i\theta\,dx\Big)\\
&\le C(\|\nabla u\|_{L^\infty}\|\nabla^2\partial_3\theta\|_{L^2}+\|\partial_3\nabla u\|_{L^4}\|\nabla^2\theta\|_{L^4})\|\partial_3^2\partial_i\theta\|_{L^2}\\
&\quad+C\|\partial_3^2\partial_i u\|_{L^2}\|\nabla\theta\|_{L^\infty}\|\partial_3^2\partial_i\theta\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\nabla u\|_{H^2}\|\theta\|_{H^3}^2\\
&\le C\|\theta\|_{H^3}^2(\|u\|_{H^{3,0}}+1)(\|\partial_t u\|_{H^1}+\|\partial_3 u\|_{H^{3,0}}+\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^1})\\
&\le C\|\theta\|_{H^3}^2\|u\|_{H^{3,0}}(\|\partial_t u\|_{H^1}+\|\partial_3 u\|_{H^{3,0}}+\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^1})\\
&\quad + C\|\theta\|_{H^3}^2(\|\partial_t u\|_{H^1}+\|\partial_3 u\|_{H^{3,0}}+\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^1})\\
&\le C\|\theta\|_{H^3}^2(\|u\|_{H^{3,0}}^2+\|\partial_t u\|_{H^1}^2)+C(\|\theta\|_{H^3}^4+\|u\|_{H^{3,0}}^2\|\partial_3 u\|_{H^{3,0}}^2)\\
&\quad + C\|\partial_3 u\|_{H^{3,0}}^2(\|\theta\|_{H^3}^4+\|u\|_{H^{3,0}}^2)+C(\|\theta\|_{H^3}^4+\|u\|_{H^{3,0}}^2\|\theta\|_{H^1}^2)\\
&\quad + C(\|\partial_t u\|_{H^1}+\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^1})\|\theta\|_{H^3}^2+C\|\theta\|_{H^3}(\|\theta\|_{H^3}^2+\|\partial_3 u\|_{H^{3,0}}^2)\\
&\le C(\|\partial_t u\|_{H^1}+\|\theta\|_{H^3}+\|\partial_t u\|_{H^1}^2+\|u\|_{H^{3,0}}^2+\|\theta\|_{H^3}^2+\|\theta\|_{H^3}^4)(\|\theta\|_{H^3}^2+\|\partial_3 u\|_{H^{3,0}}^2).
\end{aligned}
$$

A similar argument to J_2, we can get the following bounds for J_3,

$$
\begin{aligned}
J_3 &= -\int (3\partial_3 u\cdot\nabla\partial_3^2\theta+3\partial_3^2 u\cdot\nabla\partial_3\theta+\partial_3^3 u\cdot\nabla\theta)\,\partial_3^3\theta\,dx\\
&\le C(\|\partial_3 u\|_{L^\infty}\|\nabla\partial_3^2\theta\|_{L^2}+\|\partial_3^2 u\|_{L^4}\|\nabla\partial_3\theta\|_{L^4}+\|\partial_3^3 u\|_{L^2}\|\nabla\theta\|_{L^\infty})\|\partial_3^3\theta\|_{L^2}\\
&\le C\|\nabla u\|_{H^2}\|\theta\|_{H^3}^2\\
&\le C\|\theta\|_{H^3}^2(\|u\|_{H^{3,0}}+1)(\|\partial_t u\|_{H^1}+\|\partial_3 u\|_{H^{3,0}}+\|\partial_3 u\|_{H^{3,0}}^2+\|\theta\|_{H^1})\\
&\le C(\|\partial_t u\|_{H^1}+\|\theta\|_{H^3}+\|\partial_t u\|_{H^1}^2+\|u\|_{H^{3,0}}^2+\|\theta\|_{H^3}^2+\|\theta\|_{H^3}^4)(\|\theta\|_{H^3}^2+\|\partial_3 u\|_{H^{3,0}}^2).
\end{aligned}
$$

From the above, we can sufficiently conclude that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{i=1}^{2}\|\partial_3^2\partial_i\theta\|_{L^2}^2+\|\partial_3^3\theta\|_{L^2}^2\Big)+2\eta\Big(\sum_{i=1}^{2}\|\partial_3^2\partial_i\theta\|_{L^2}+\|\partial_3^3\theta\|_{L^2}^2\Big)\\
&\le C(\|\partial_t u\|_{H^1}+\|\theta\|_{H^3}+\|\partial_t u\|_{H^1}^2+\|u\|_{H^{3,0}}^2+\|\theta\|_{H^3}^2+\|\theta\|_{H^3}^4)(\|\theta\|_{H^3}^2+\|\partial_3 u\|_{H^{3,0}}^2).
\end{aligned}
\tag{3.13}
$$

Finally, summing (3.12) and (3.13), we conclude

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\theta\|_{L^2}^2+\sum_{i=1}^{2}\|(\partial_i^3\theta,\partial_3^2\partial_i\theta)\|_{L^2}^2+\|\partial_3^3\theta\|_{L^2}^2\Big)+2\eta\Big(\|\theta\|_{L^2}^2+\sum_{i=1}^{2}\|(\partial_i^3\theta,\partial_3^2\partial_i\theta)\|_{L^2}+\|\partial_3^3\theta\|_{L^2}^2\Big)\\
&\le C(\|\partial_t u\|_{H^1}+\|\theta\|_{H^3}+\|\partial_t u\|_{H^1}^2+\|u\|_{H^{3,0}}^2+\|\theta\|_{H^3}^2+\|\theta\|_{H^3}^4)(\|\theta\|_{H^3}^2+\|\partial_3 u\|_{H^{3,0}}^2),
\end{aligned}
\tag{3.14}
$$

and then integrating (3.14) over $[0,t]$ for any $t>0$,

$$
\begin{aligned}
&\|\theta\|_{H^3}^2+2\eta\int_0^t\|\theta(\tau)\|_{H^3}^2\,d\tau\\
&\le C\|\theta_0\|_{H^3}^2+C\sup_{0\le\tau\le t}(\|\partial_\tau u(\tau)\|_{H^1}+\|\theta(\tau)\|_{H^3}+\|\partial_\tau u(\tau)\|_{H^1}^2+\|u(\tau)\|_{H^{3,0}}^2+\|\theta(\tau)\|_{H^3}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\theta(\tau)\|_{H^3}^4) \\
&\times \int_0^t (\|\theta(\tau)\|_{H^3}^2 + \|\partial_3 u(\tau)\|_{H^{3,0}}^2)\, d\tau \\
&\leq C\varepsilon_0(0) + C\varepsilon_1^{\frac{1}{2}}(t)\varepsilon_0(t) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1(t)\varepsilon_0(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t) \\
&\leq C\varepsilon_0(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_1^2(t) + C\varepsilon_0^2(t) + C\varepsilon_0^3(t),
\end{aligned}
$$

where we use the fact that

$$
\|\partial_3 \partial_i^2 \theta\|_{L^2}^2 \leq \frac{1}{2}(\|\partial_3^2 \partial_i \theta\|_{L^2}^2 + \|\partial_i^3 \theta\|_{L^2}^2)
$$

and

$$
\begin{aligned}
\|\theta\|_{H^3}^2 &\leq C(\|\theta\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2 + \|\nabla^2\theta\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2) \\
&\leq C(\|\theta\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2) \\
&\quad + C(\|\theta\|_{L^2}^{\frac{2}{3}} \|\nabla^3\theta\|_{L^2}^{\frac{1}{3}} + \|\theta\|_{L^2})^2 + C(\|\theta\|_{L^2}^{\frac{1}{3}} \|\nabla^3\theta\|_{L^2}^{\frac{2}{3}} + \|\theta\|_{L^2})^2 \\
&\leq C(\|\theta\|_{L^2}^2 + \|\nabla^3\theta\|_{L^2}^2) \leq C(\|\theta\|_{L^2}^2 + \sum_{i=1}^2 \|\partial_i^3\theta\|_{L^2}^2 + \sum_{i=1}^2 \|\partial_3^2\partial_i\theta\|_{L^2}^2 + \|\partial_3^3\theta\|_{L^2}^2).
\end{aligned} \tag{3.15}
$$

This completes the proof of Lemma 3.3. □

4. Estimates for ε_1

This section focuses on proving the bound for ε_1. In the third section, we established an energy inequality for $\|u\|_{H^{3,0}} + \|\theta\|_{H^3}$. Notably, the velocity estimates from the previous section involve only horizontal derivatives. To establish the well-posedness in H^3, we also need to construct bounds for $\|\nabla u\|_{H^2}$. Unfortunately, $\|\nabla u\|_{H^2}$ cannot be directly estimated due to the absence of boundary conditions for the vertical derivatives of (u, θ). Moreover, in the process of estimating $\|\theta\|_{H^3}$, the upper bounds of J_2 and J_3 depend on $\|u_t\|_{H^1}$. Thus, it becomes necessary to establish an upper bound for $\|(u_t, b_t)\|_{H^1}$, which is achieved in Proposition 4.1. For brevity, we sometimes denote f_t as $\partial_t f$ in the following sections.

Proposition 4.1. *Assume $(u_0, \theta_0) \in H_0^1 \cap H^3$ and let (u, θ) be the corresponding solution to (1.1). Then $\varepsilon_1(t)$ be defined as in (1.3) satisfies,*

$$
\varepsilon_1(t) \leq C\varepsilon_1(0) + C\varepsilon_0^{\frac{3}{2}}(t) + C\varepsilon_1^{\frac{3}{2}}(t) + C\varepsilon_0^2(t) + C\varepsilon_1^2(t) + C\varepsilon_1^3(t) \tag{4.1}
$$

for some constant $C > 0$ depending on ν and η.

Proof of Proposition 4.1. Applying ∂_t to the system (1.1) yields

$$\begin{cases} \partial_{tt}u + ((u\cdot\nabla)u)_t + \nabla p_t = \nu\partial_{33}u_t + (\theta e_3)_t, \\ \partial_{tt}\theta + \eta\theta_t + ((u\cdot\nabla)\theta)_t = 0, \\ \nabla\cdot u_t = 0, \\ u_t|_{\partial\Omega} = 0. \end{cases} \tag{4.2}$$

Taking the L^2-inner product of $(4.2)_1$ with u_t, u_{tt}, $\Delta_h u_t$, respectively, by integration by parts and applying the boundary conditions $u_t|_{\partial\Omega} = 0$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u_t\|_{L^2}^2 + \nu\|\partial_3 u_t\|_{L^2}^2 + \|\nabla_h u_t\|_{L^2}^2\Big) + \Big(\nu\|\partial_3 u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 + \nu\|\partial_3\nabla_h u_t\|_{L^2}^2\Big) \\ &= -\int (u\cdot\nabla u)_t\cdot u_t dx + \int \theta_t u_{3t} dx - \int (u\cdot\nabla u)_t\cdot u_{tt}\,dx + \int (\theta e_3)_t\cdot u_{tt}\,dx \\ &\quad + \int (u\cdot\nabla u)_t\cdot\Delta_h u_t\,dx - \int (\theta e_3)_t\cdot\Delta_h u_t\,dx \\ &:= K_1 + \cdots + K_6, \end{aligned} \tag{4.3}$$

where we have used the fact that

$$\int \nabla p_t\cdot u_t\,dx = 0, \quad \int \nabla p_t\cdot u_{tt}\,dx = 0 \quad \text{and} \quad \int \nabla p_t\cdot\Delta_h u_t\,dx = 0.$$

By Hölder's inequality and the inequalities (2.2) and (2.8),

$$\begin{aligned} K_1 &= -\int \partial_t u_h\cdot\nabla_h u\cdot u_t dx - \int \partial_t u_3\,\partial_3 u\cdot u_t dx \\ &\le \|\nabla_h u\|_{L^\infty}\|u_t\|_{L^2}^2 + C\|\partial_3 u_t\|_{L^2}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac14}\|u_t\|_{L^2} \\ &\le C\|\partial_3\nabla_h u\|_{H^{2,0}}\|u_t\|_{L^2}\|\partial_3 u_t\|_{L^2} + C\|\partial_3 u_t\|_{L^2}\|\partial_3 u\|_{H^{2,0}}\|u_t\|_{L^2} \\ &\le C\|\partial_3 u\|_{H^{3,0}}\|u_t\|_{L^2}\|\partial_3 u_t\|_{L^2} \le C\|u_t\|_{L^2}(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 u_t\|_{L^2}^2). \end{aligned} \tag{4.4}$$

With a minor modification of the estimate for K_1, we get

$$\begin{aligned} K_3 &= -\int \partial_t u_h\cdot\nabla_h u\cdot u_{tt} dx - \int \partial_t u_3\,\partial_3 u\cdot u_{tt} dx - \int u\cdot\nabla u_t\cdot u_{tt} dx \\ &\le \|\nabla_h u\|_{L^\infty}\|u_t\|_{L^2}\|u_{tt}\|_{L^2} + C\|\partial_3 u_t\|_{L^2}\|\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_3 u\|_{L^2}^{\frac14}\|\partial_2\partial_3 u\|_{L^2}^{\frac14}\|\partial_1\partial_2\partial_3 u\|_{L^2}^{\frac14}\|u_{tt}\|_{L^2} \\ &\quad + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2} \\ &\le C(\|\partial_3\nabla_h u\|_{H^{2,0}}\|u_t\|_{L^2} + \|\partial_3 u_t\|_{L^2}\|\partial_3 u\|_{H^{2,0}} + \|\partial_3 u\|_{H^{2,0}}\|\nabla u_t\|_{L^2})\|u_{tt}\|_{L^2} \\ &\le C\|\partial_3 u\|_{H^{3,0}}\|\nabla u_t\|_{L^2}\|u_{tt}\|_{L^2} \le C\|\nabla u_t\|_{L^2}(\|\partial_3 u\|_{H^{3,0}}^2 + \|u_{tt}\|_{L^2}^2). \end{aligned} \tag{4.5}$$

For the term K_5, we firstly split it into three parts. Then using Hölder's inequality and (2.2), (2.6) and (3.2) yields

$$\begin{aligned}
K_5 &= -\int u_t \cdot \nabla u \cdot \Delta_h u_t dx - \int u \cdot \nabla u_t \cdot \Delta_h u_t dx \\
&= \int \nabla_h u_t \cdot \nabla u \cdot \nabla_h u_t dx + \int u_t \cdot \nabla \nabla_h u \cdot \nabla_h u_t dx + \int \nabla_h u \cdot \nabla u_t \cdot \nabla_h u_t dx \\
&\le \Big(\|\nabla_h u_t\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{4}} \|\nabla \partial_1 u\|_{L^2}^{\frac{1}{4}} \|\nabla \partial_2 u\|_{L^2}^{\frac{1}{4}} \|\nabla \partial_{12} u\|_{L^2}^{\frac{1}{4}} \\
&\quad + \|u_t\|_{L^2}^{\frac{1}{2}} \|\partial_1 u_t\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla \nabla_h \partial_2 u\|_{L^2}^{\frac{1}{2}} \Big) \|\nabla_h \partial_3 u_t\|_{L^2} \\
&\quad + \|\nabla_h u\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\nabla_h u_t\|_{L^2} \\
&\le C(\|\nabla_h u_t\|_{L^2} \|\partial_3 u\|_{H^{3,0}} + \|u_t\|_{H^1} \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|_{H^{3,0}} \|\nabla u_t\|_{L^2}) \|\partial_3 \nabla_h u_t\|_{L^2} \\
&\le C \|\partial_3 u\|_{H^{3,0}} \|u_t\|_{H^1} \|\partial_3 \nabla_h u_t\|_{L^2} \le C \|u_t\|_{H^1} (\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 \nabla_h u_t\|_{L^2}^2).
\end{aligned}$$

A similar argument to $I_1 + I_3$, the following inequalities for K_2, K_4 and K_6 can be showed as

$$K_2 = \int \theta_t \, u_{3t} dx \le \frac{\nu}{2} \|\partial_3 u_t\|_{L^2}^2 + \frac{\bar{C}}{\nu} \|\theta_t\|_{L^2}^2. \tag{4.6}$$

$$K_4 = \int \theta_t \, u_{3tt} dx \le \frac{1}{2} \|u_{tt}\|_{L^2}^2 + \bar{C} \|\theta_t\|_{L^2}^2. \tag{4.7}$$

$$K_6 = \int \theta_t \, \Delta_h u_{3t} dx \le \frac{\nu}{2} \|\partial_3 \nabla_h u_t\|_{L^2}^2 + \frac{\bar{C}}{\nu} \|\nabla_h \theta_t\|_{L^2}^2. \tag{4.8}$$

Substituting (4.4) through (4.8) into (4.3), it holds that

$$\begin{aligned}
&\frac{d}{dt} \Big(\|u_t\|_{L^2}^2 + \nu \|\partial_3 u_t\|_{L^2}^2 + \|\nabla_h u_t\|_{L^2}^2 \Big) + \Big(\nu \|\partial_3 u_t\|_{L^2}^2 + \|u_{tt}\|_{L^2}^2 + \nu \|\partial_3 \nabla_h u_t\|_{L^2}^2 \Big) \\
&\le C \|u_t\|_{H^1} (\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 u_t\|_{H^{1,0}}^2 + \|u_{tt}\|_{L^2}^2) + C_\nu \|(\theta_t, \nabla_h \theta_t)\|_{L^2}^2,
\end{aligned} \tag{4.9}$$

where $C_\nu = \max\{\frac{2\bar{C}}{\nu}, 2\bar{C}\}$.

Now, taking the L^2-inner product of $(4.2)_2$ with θ_t, $\partial_3^2 \theta_t$ and $\Delta_h \theta_t$, respectively, and applying the boundary conditions for θ, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \big(\|\theta_t\|_{L^2}^2 + \|\partial_3 \theta_t\|_{L^2}^2 + \|\nabla_h \theta_t\|_{L^2}^2 \big) + \eta \Big(\|\theta_t\|_{L^2}^2 + \|\partial_3 \theta_t\|_{L^2}^2 + \|\nabla_h \theta_t\|_{L^2}^2 \Big) \\
&= -\int (u \cdot \nabla \theta)_t \, \theta_t \, dx + \int (u \cdot \nabla \theta)_t \, \partial_3^2 \theta_t \, dx + \int (u \cdot \nabla \theta)_t \, \Delta_h \theta_t \, dx \\
&:= K_7 + K_8 + K_9.
\end{aligned} \tag{4.10}$$

Firstly, due to (3.2), K_7 can be estimated as

$$\begin{aligned} K_7 &= -\int u_t \cdot \nabla\theta\, \theta_t\, dx \\ &\le C\|\partial_3 u_t\|_{L^2}\|\nabla\theta\|_{L^\infty}\|\theta_t\|_{L^2} \le C\|\theta\|_{H^3}(\|\partial_3 u_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2). \end{aligned} \tag{4.11}$$

Using the fact

$$(u\cdot\nabla\theta)_t|_{\partial\Omega} = (u_t\cdot\nabla\theta + u\cdot\nabla\theta_t)|_{\partial\Omega} = 0$$

and integration by parts, we obtain

$$\begin{aligned} K_8 &= -\int \partial_3(u\cdot\nabla\theta)_t\, \partial_3\theta_t\, dx \\ &= -\int \partial_3 u_t\cdot\nabla\theta\, \partial_3\theta_t\, dx - \int u_t\cdot\nabla\partial_3\theta\, \partial_3\theta_t\, dx - \int \partial_3 u\cdot\nabla\theta_t\, \partial_3\theta_t\, dx \\ &\le C\|\partial_3 u_t\|_{L^2}\|\nabla\theta\|_{L^\infty}\|\partial_3\theta_t\|_{L^2} + C\|\partial_3 u\|_{L^\infty}\|\nabla\theta_t\|_{L^2}\|\partial_3\theta_t\|_{L^2} \\ &\quad + C\|u_t\|_{L^2}^{\frac14}\|\partial_1 u_t\|_{L^2}^{\frac14}\|\partial_3 u_t\|_{L^2}^{\frac14}\|\partial_{13} u_t\|_{L^2}^{\frac14}\|\nabla\partial_3\theta\|_{L^2}^{\frac12}\|\nabla\partial_{23}\theta\|_{L^2}^{\frac12}\|\partial_3\theta_t\|_{L^2} \\ &\le C(\|\partial_3 u_t\|_{L^2} + \|\partial_3 u_t\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_t\|_{L^2}^{\frac12})\|\theta\|_{H^3}\|\partial_3\theta_t\|_{L^2} + C\|\nabla u\|_{H^2}\|\nabla\theta_t\|_{L^2}\|\partial_3\theta_t\|_{L^2} \\ &\le C\|\theta\|_{H^3}\|\partial_3 u_t\|_{H^{1,0}}\|\partial_3\theta_t\|_{L^2} \\ &\quad + C(\|u\|_{H^{3,0}}+1)\big(\|\partial_t u\|_{H^1} + \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|_{H^{3,0}}^2 + \|\theta\|_{H^1}\big)\|\nabla\theta_t\|_{L^2}\|\partial_3\theta_t\|_{L^2} \\ &\le C(\|u_t\|_{H^1} + \|\theta\|_{H^3} + \|u_t\|_{H^1}^2 + \|\theta\|_{H^1}^2)(\|\partial_3 u_t\|_{H^{1,0}}^2 + \|\nabla\theta_t\|_{L^2}^2) \\ &\quad + (\|\nabla\theta_t\|_{L^2} + \|u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^4)(\|\partial_3 u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2), \end{aligned} \tag{4.12}$$

where we also used Hölder inequality, (2.7) and (2.12).

Similar to K_8, we have

$$\begin{aligned} K_9 &= \int u_t\cdot\nabla\theta\, \Delta_h\theta_t\, dx + \int u\cdot\nabla\theta_t\, \Delta_h\theta_t\, dx \\ &= -\int \nabla_h u_t\cdot\nabla\theta\cdot\nabla_h\theta_t dx - \int u_t\cdot\nabla\nabla_h\theta\cdot\nabla_h\theta_t dx - \int \nabla_h u\cdot\nabla\theta_t\cdot\nabla_h\theta_t dx \\ &\le C\|\partial_3\nabla_h u_t\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac14}\|\nabla\partial_1\theta\|_{L^2}^{\frac14}\|\nabla\partial_2\theta\|_{L^2}^{\frac14}\|\nabla\partial_{12}\theta\|_{L^2}^{\frac14}\|\nabla_h\theta_t\|_{L^2} \\ &\quad + C\|u_t\|_{L^2}^{\frac14}\|\partial_1 u_t\|_{L^2}^{\frac14}\|\partial_3 u_t\|_{L^2}^{\frac14}\|\partial_{13} u_t\|_{L^2}^{\frac14}\|\nabla\nabla_h\theta\|_{L^2}^{\frac12}\|\nabla\nabla_h\partial_2\theta\|_{L^2}^{\frac12}\|\nabla_h\theta_t\|_{L^2} \\ &\quad + C\|\nabla u\|_{L^\infty}\|\nabla\theta_t\|_{L^2}\|\nabla_h\theta_t\|_{L^2} \\ &\le C\|\theta\|_{H^3}\|\partial_3 u_t\|_{H^{1,0}}|\nabla_h\theta_t\|_{L^2} \\ &\quad + C(\|u\|_{H^{3,0}}+1)\big(\|\partial_t u\|_{H^1} + \|\partial_3 u\|_{H^{3,0}} + \|\partial_3 u\|_{H^{3,0}}^2 + \|\theta\|_{H^1}\big)\|\nabla\theta_t\|_{L^2}\|\nabla_h\theta_t\|_{L^2} \\ &\le C(\|u_t\|_{H^1} + \|\theta\|_{H^3} + \|u_t\|_{H^1}^2 + \|\theta\|_{H^1}^2)(\|\partial_3 u_t\|_{H^{1,0}}^2 + \|\nabla\theta_t\|_{L^2}^2) \\ &\quad + (\|\nabla\theta_t\|_{L^2} + \|u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^4)(\|\partial_3 u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2). \end{aligned} \tag{4.13}$$

Substituting (4.11) through (4.13) into (4.10), we infer

$$\begin{aligned}
&\frac{d}{dt}\big(\|\theta_t\|_{L^2}^2+\|\partial_3\theta_t\|_{L^2}^2+\|\nabla_h\theta_t\|_{L^2}^2\big)+2\eta\Big(\|\theta_t\|_{L^2}^2+\|\partial_3\theta_t\|_{L^2}^2+\|\nabla_h\theta_t\|_{L^2}^2\Big)\\
&\le C(\|u_t\|_{H^1}+\|\theta\|_{H^3}+\|u_t\|_{H^1}^2+\|\theta\|_{H^1}^2)(\|\partial_3 u_t\|_{H^{1,0}}^2+\|\theta_t\|_{H^1}^2)\\
&\quad+(\|\nabla\theta_t\|_{L^2}+\|u\|_{H^{3,0}}^2+\|\nabla\theta_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^4)(\|\partial_3 u\|_{H^{3,0}}^2+\|\nabla\theta_t\|_{L^2}^2). \qquad (4.14)
\end{aligned}$$

Multiplying (4.9) by a suitably small constant $\delta_2>0$, and combining with (4.14) yield

$$\begin{aligned}
&\frac{d}{dt}\Big(\delta_2\|u_t\|_{H^{1,0}}^2+\nu\delta_2\|\partial_3 u_t\|_{L^2}^2+\|\theta_t\|_{H^1}^2\Big)\\
&\quad+\Big(\nu\delta_2\|(\partial_3 u_t,\partial_3\nabla_h u_t)\|_{L^2}^2+\delta_2\|u_{tt}\|_{L^2}^2+(2\eta-4\delta_2 C_\nu)\|\theta_t\|_{H^1}^2\Big)\\
&\le C(\|u_t\|_{H^1}+\|\theta\|_{H^3}+\|u_t\|_{H^1}^2+\|\theta\|_{H^1}^2)(\|\partial_3 u_t\|_{H^{1,0}}^2+\|\theta_t\|_{H^1}^2+\|u_{tt}\|_{L^2})\\
&\quad+C(\|\nabla\theta_t\|_{L^2}+\|u_t\|_{H^1}+\|u\|_{H^{3,0}}^2+\|\nabla\theta_t\|_{L^2}^2+\|\nabla\theta_t\|_{L^2}^4)(\|\partial_3 u\|_{H^{3,0}}^2+\|\nabla\theta_t\|_{L^2}^2). \qquad (4.15)
\end{aligned}$$

By taking the time integral of (4.15) and performing simplifications, we readily obtain

$$\begin{aligned}
&\|u_t\|_{H^1}^2+\|\theta_t\|_{H^1}^2+\int_0^t\Big(\nu\|(\partial_3 u_\tau(\tau),\partial_3\nabla_h u_\tau(\tau))\|_{L^2}^2+\|u_{\tau\tau}(\tau)\|_{L^2}^2+\|\theta_\tau(\tau)\|_{H^1}^2\Big)\,d\tau\\
&\le C\|(u_t(0),\theta_t(0))\|_{H^1}^2+C\sup_{0\le\tau\le t}(\|u_\tau(\tau)\|_{H^1}+\|\theta(\tau)\|_{H^3}+\|u_\tau(\tau)\|_{H^1}^2+\|\theta(\tau)\|_{H^1}^2)\\
&\quad\times\int_0^t(\|\partial_3 u_\tau(\tau)\|_{H^{1,0}}^2+\|\theta_\tau(\tau)\|_{H^1}^2+\|u_{\tau\tau}(\tau)\|_{L^2}^2)\,d\tau\\
&\quad+\sup_{0\le\tau\le t}(\|\nabla\theta_\tau(\tau)\|_{L^2}+\|u_\tau(\tau)\|_{H^1}+\|u(\tau)\|_{H^{3,0}}^2+\|\nabla\theta_\tau(\tau)\|_{L^2}^2+\|\nabla\theta_\tau(\tau)\|_{L^2}^4)\\
&\quad\times\int_0^t(\|\partial_3 u(\tau)\|_{H^{3,0}}^2+\|\nabla\theta_\tau(\tau)\|_{L^2}^2)\,d\tau\\
&\le C\varepsilon_1(0)+C\varepsilon_1^{\frac12}(t)\varepsilon_0(t)+C\varepsilon_0^{\frac12}(t)\varepsilon_1(t)+C\varepsilon_0(t)\varepsilon_1(t)+C\varepsilon_0^{\frac32}(t)+C\varepsilon_1^{\frac32}(t)\\
&\quad+C\varepsilon_0^2(t)+C\varepsilon_1^2(t)+C\varepsilon_1^3(t)\\
&\le C\varepsilon_1(0)+C\varepsilon_0^{\frac32}(t)+C\varepsilon_1^{\frac32}(t)+C\varepsilon_0^2(t)+C\varepsilon_1^2(t)+C\varepsilon_1^3(t)
\end{aligned}$$

for some constant $C>0$ depending on ν and η.

This completes the proof of Proposition 4.1. □

5. The global well-posedness

5.1. Proof of the global well-posedness part of Theorem 1.1

This subsection completes the proof of the global well-posedness part of Theorem 1.1 by combining the energy estimates obtained in the previous two subsections.

Proof of the global well-posedness. First, we apply the bootstrapping argument to establish a global bound for $\varepsilon(t) := \varepsilon_0(t) + \varepsilon_1(t)$, assuming a sufficiently small initial solution.

Recalling the inequalities (3.1) and (4.1) in Proposition 3.1 and Proposition 4.1, we can show

$$\varepsilon(t) \le C_0^* \varepsilon(0) + C_1^*(\varepsilon^{\frac{3}{2}}(t) + \varepsilon^2(t) + \varepsilon^3(t)).$$

Making the ansatz

$$\varepsilon(t) \le C_2^* := \min\{\frac{1}{(6C_1^*)^2}, \frac{1}{\sqrt{6C_1^*}}\}. \tag{5.1}$$

Then it implies

$$\varepsilon(t) \le 2C_0^* \varepsilon(0).$$

Therefore, if we take $\|u_0\|_{H^{3,0}} + \|\theta_0\|_{H^3} + \|(u_{t0}, \theta_{t0})\|_{H^1}$ sufficiently small such that

$$\varepsilon(0) \le \frac{C_2^*}{2C_0^*},$$

then we will get a smaller bound of $\varepsilon(t)$ than the ansatz (5.1). By bootstrapping argument, for any $t > 0$ and some constant $C > 0$ depending on ν and η, it holds that

$$\|u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|(u_t, \theta_t)\|_{L^2}^2 + \int_0^t (\nu\|\partial_3 u(\tau)\|_{H^{3,0}}^2 + \|\theta(\tau)\|_{H^3}^2)\, d\tau \le C\|(u_0, b_0)\|_{H^3}^2 \le C\delta^2, \tag{5.2}$$

where we have used the fact that

$$\|(u_t(0), \theta_t(0)\|_{H^1} \le C\|(u_0, \theta_0)\|_{H^3},$$

which follows from

$$\begin{aligned} \partial_t u(0) &= -u_0 \cdot \nabla u_0 - \nabla p_0 + \nu\partial_3^2 u_0 + \theta_0 e_3, \\ \partial_t \theta(0) &= -u_0 \cdot \nabla \theta_0 + \theta_0, \end{aligned}$$

with the pressure determined by the elliptic equations

$$-\Delta p_0 = \nabla \cdot (u_0 \cdot \nabla u_0 + \theta_0 e_3),\ x \in \Omega, \quad \nabla p_0 \cdot n = \partial_3^2 u_0 \cdot n,\ x \in \partial\Omega.$$

Next we combine (5.2) with the energy estimate in Lemma 2.4 to establish a global bound for $\|u\|_{H^3}$. Firstly, by integration by parts, Hölder's inequality and Young's inequality,

$$\begin{aligned}\|\partial_3 \Delta_h u\|_{L^2} &= \Big(\int \partial_3^2 \nabla_h u \cdot \nabla_h \Delta_h u \, dx \Big)^{\frac{1}{2}} \le C \|\partial_3^2 \nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h^3 u\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{1}{2} \|\partial_3^2 \nabla_h u\|_{L^2} + C \|\nabla_h^3 u\|_{L^2} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\|\partial_3 u\|_{H^{2,0}}^2 &\le C(\|\partial_3 u\|_{L^2}^2 + \|\partial_3 \nabla_h u\|_{L^2}^2 + \|\partial_3 \nabla_h^2 u\|_{L^2}^2) \\ &\le C(\|u\|_{L^2} \|\partial_3^2 u\|_{L^2} + \|\nabla_h u\|_{L^2} \|\partial_3^2 \nabla_h u\|_{L^2} + \|\nabla_h^3 u\|_{L^2} \|\partial_3^2 \nabla_h u\|_{L^2}) \\ &\le C \|u\|_{H^{3,0}} \|\nabla u\|_{H^2}. \end{aligned} \tag{5.4}$$

Then it follows from (5.3), (5.4) and (2.13) that

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} &\le C\big(\|\partial_t u\|_{H^1} + \|\Delta_h u\|_{H^{1,0}} + \|\partial_3 \Delta_h u\|_{L^2} + \|\theta\|_{H^1} + \|\partial_3 u\|_{H^{2,0}}^2\big) \\ &\quad + C\|u\|_{H^{3,0}}\big(\|\partial_t u\|_{L^2} + \|\partial_3 u\|_{H^{2,0}} + \|u\|_{H^{3,0}} \|\partial_3 u\|_{H^{2,0}} + \|\theta\|_{L^2}\big) \\ &\le C\big(\|\partial_t u\|_{H^1} + \|u\|_{H^{3,0}} + \|\theta\|_{H^1} + \|u\|_{H^{3,0}} \|\partial_t u\|_{L^2} + \|u\|_{H^{3,0}} \|\theta\|_{L^2}\big) \\ &\quad + C(\|u\|_{H^{3,0}} + \|u\|_{H^{3,0}}^2)(\|\nabla u\|_{H^2} + \|\partial_3 u\|_{H^{2,0}}) + \frac{1}{2}\|\partial_3^2 \nabla_h u\|_{L^2} \\ &\le C\big(\|\partial_t u\|_{H^1} + \|u\|_{H^{3,0}} + \|\theta\|_{H^1} + \|u\|_{H^{3,0}} \|\partial_t u\|_{L^2} + \|u\|_{H^{3,0}} \|\theta\|_{L^2}\big) \\ &\quad + \Big(C(\delta + \delta^2) + \frac{1}{2}\Big)\|\nabla u\|_{H^2} \end{aligned}$$

for a uniform constant C (independent of δ). Taking δ sufficiently small, we have

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} &\le C\big(\|\partial_t u\|_{H^1} + \|u\|_{H^{3,0}} + \|\theta\|_{H^1} + \|u\|_{H^{3,0}} \|\partial_t u\|_{L^2} + \|u\|_{H^{3,0}} \|\theta\|_{L^2}\big) \\ &\le C\big(\delta + \delta^2\big). \end{aligned} \tag{5.5}$$

This completes the proof of the global well-posedness part in Theorem 1.1. □

6. The decay estimates

This section is devoted to proving precise exponential decay estimate in Theorem 1.1. To derive the decay estimate for $\|(u, \theta)\|_{H^3}$, the three key inequalities (3.10), (3.14) and (4.15) established earlier are closely interconnected and indispensable. Through straightforward calculations, we will combine these inequalities to form a refined inequality. Next, leveraging the results from the existence part and applying the Poincaré inequality, we simplify this into an inequality of the form

$$\partial_t X(t) + C\,X(t) \le 0,$$

from which exponential decay follows.

Our proof is divided into two steps. In the first step, the combined inequality does not directly yield estimates for the vertical derivatives of the velocity field. Therefore, in the final step, we must derive the H^2-estimate for ∇u based on the inequality in (5.5).

Next we prove the decay rate in Theorem 1.1.

Proof of the decay estimate in Theorem 1.1. As mentioned earlier, the proof of the decay estimates is divided into two main steps. The first step establishes the exponential decay of $\|u\|_{H^{3,0}} + \|\theta\|_{H^3}$ and $\|(u_t, \theta_t)\|_{H^1}$ by making use of the estimates provided in Lemma 3.2, Lemma 3.3 and Proposition 4.1.

Multiplying (3.10) by a suitably small constant δ_3 with $\delta_3 < \frac{2\eta\nu}{C_1}$ and adding it to (3.14) and (4.15). Invoking the global bound in (5.2), we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\delta_3\|u\|_{L^2}^2 + \|(\theta, \partial_3^3\theta)\|_{L^2}^2 + \sum_{i=1}^{2}(\delta_3\|\partial_i^3 u\|_{L^2}^2 + \|(\partial_i^3\theta, \partial_3^2\partial_i\theta)\|_{L^2}^2) + \delta_2\|u_t\|_{H^{1,0}}^2 \\
&\qquad + \|\theta_t\|_{H^1}^2 + \nu\delta_2\|\partial_3 u_t\|_{L^2}^2\Big) \\
&\quad + \Big(\delta_3\nu\|\partial_3 u\|_{L^2}^2 + \sum_{i=1}^{2}\delta_3\nu\|\partial_i^3\partial_3 u\|_{L^2}^2 + (2\eta - \frac{C_1\delta_3}{\nu})(\|\theta\|_{L^2}^2 + \sum_{i=1}^{2}\|\partial_i^3\theta\|_{L^2}^2)\Big) \\
&\quad + \Big(\sum_{i=1}^{2}2\eta\|\partial_3^2\partial_i\theta\|_{L^2}^2 + 2\eta\|\partial_3^3\theta\|_{L^2}^2 + \delta_2\|u_{tt}\|_{L^2}^2 + \nu\delta_2\|\partial_3 u_t\|_{H^{1,0}}^2 + (2\eta - 4\delta_2 C_\nu)\|\theta_t\|_{H^1}^2\Big) \\
&\le C(\|u\|_{H^{3,0}} + \|\theta\|_{H^3} + \|\partial_t u\|_{H^1}^2 + \|u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|\theta\|_{H^3}^4)(\|\theta\|_{H^3}^2 + \|\partial_3 u\|_{H^{3,0}}^2) \\
&\quad + C(\|u_t\|_{H^1} + \|\theta\|_{H^3} + \|u_t\|_{H^1}^2 + \|\theta\|_{H^1}^2)(\|\partial_3 u_t\|_{H^{1,0}}^2 + \|\theta_t\|_{H^1}^2 + \|u_{tt}\|_{L^2}^2) \\
&\quad + C(\|\nabla\theta_t\|_{L^2} + \|u_t\|_{H^1} + \|u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2 + \|\nabla\theta_t\|_{L^2}^4)(\|\partial_3 u\|_{H^{3,0}}^2 + \|\nabla\theta_t\|_{L^2}^2) \\
&\le C(\delta + \delta^2 + \delta^4)(\|\partial_3 u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|\theta_t\|_{H^1}^2 + \|\partial_3 u_t\|_{H^{1,0}}^2 + \|u_{tt}\|_{L^2}).
\end{aligned}$$

Based on the equivalence relationship between $\|f\|_{H^{3,0}}^2$ and $\|f\|_{L^2}^2 + \sum_{i=1}^{2}\|\partial_i^3 f\|_{L^2}^2$, together with (3.15), we can directly obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\delta_3\|u\|_{L^2}^2 + \|(\theta, \partial_3^3\theta)\|_{L^2}^2 + \sum_{i=1}^{2}(\delta_3\|\partial_i^3 u\|_{L^2}^2 + \|(\partial_i^3\theta, \partial_3^2\partial_i\theta)\|_{L^2}^2) + \delta_2\|u_t\|_{H^{1,0}}^2 \\
&\qquad + \|\theta_t\|_{H^1}^2 + \nu\delta_2\|\partial_3 u_t\|_{L^2}^2\Big) \\
&\quad + \Big(C_2\delta_3\nu\|\partial_3 u\|_{H^{3,0}}^2 + C_2(2\eta - \frac{C_1\delta_3}{\nu})\|\theta\|_{H^3}^2 + \delta_2\|u_{tt}\|_{L^2}^2 + \nu\delta_2\|\partial_3 u_t\|_{H^{1,0}}^2 \\
&\qquad + (2\eta - 4\delta_2 C_\nu)\|\theta_t\|_{H^1}^2\Big) \\
&\le C(\delta + \delta^2 + \delta^4)(\|\partial_3 u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|\theta_t\|_{H^1}^2 + \|\partial_3 u_t\|_{H^{1,0}}^2 + \|u_{tt}\|_{L^2}).
\end{aligned}$$

Now we select δ to be sufficiently small such that $C(\delta+\delta^2+\delta^4) < \min\Big\{C_2\delta_3\nu, C_2\big(2\eta - \frac{C_1\delta_3}{\nu}\big), \delta_2, \delta_2\nu, (2\eta - 4\delta_2 C_\nu)\Big\}$, then

$$\begin{aligned}\frac{d}{dt}\Big(&\delta_3\|u\|_{L^2}^2 + \|(\theta, \partial_3^3\theta)\|_{L^2}^2 + \sum_{i=1}^{2}(\delta_3\|\partial_i^3 u\|_{L^2}^2 + \|(\partial_i^3\theta, \partial_3^2\partial_i\theta)\|_{L^2}^2) + \delta_2\|u_t\|_{H^{1,0}}^2\\ &+ \|\theta_t\|_{H^1}^2 + \nu\delta_2\|\partial_3 u_t\|_{L^2}^2\Big)\\ &+ C_3\big(\|\partial_3 u\|_{H^{3,0}}^2 + \|\theta\|_{H^3}^2 + \|\partial_3 u_t\|_{H^{1,0}}^2 + \|\theta_t\|_{H^1}^2\big) \le 0,\end{aligned}$$

where the constant C_3 depends on δ_2, δ_3, ν and η.

Due to $u|_{\partial\Omega} = u_t|_{\partial\Omega} = 0$, by virtues of (3.2), we have

$$\|u\|_{L^2}^2 + \sum_{i=1}^{2}\|\partial_i^3 u\|_{L^2}^2 + \|u_t\|_{H^{1,0}}^2 \le C(\|\partial_3 u\|_{H^{3,0}}^2 + \|\partial_3 u_t\|_{H^{1,0}}^2).$$

Then, for some constant $C_4 > 0$ related to δ_2, δ_3, ν and η, we have

$$\frac{d}{dt}X(t) + C_4 X(t) \le 0,$$

where

$$\begin{aligned}X(t) = &\|u\|_{L^2}^2 + \|(\theta, \partial_3^3\theta)\|_{L^2}^2 + \sum_{i=1}^{2}(\|\partial_i^3 u\|_{L^2}^2 + \|(\partial_i^3\theta, \partial_3^2\partial_i\theta)\|_{L^2}^2) + \|u_t\|_{H^{1,0}}^2 + \|\theta_t\|_{H^1}^2\\ &+ \|\partial_3 u_t\|_{L^2}^2.\end{aligned}$$

Therefore,

$$X(t) \le e^{-C_4 t}X(0)$$

or

$$\|u\|_{H^{3,0}} + \|\theta\|_{H^3} + \|(u_t, \theta_t)\|_{H^1} \le Ce^{-C_4 t}\|(u_0, \theta_0)\|_{H^3}.$$

The last step of the proof is to derive the exponential decay rate for $\|\nabla u\|_{H^2}$ via the decay rates above. Invoking the estimate (5.5) yields

$$\begin{aligned}\|\nabla u\|_{H^2} + \|\nabla p\|_{H^1} &\le C\big(\|\partial_t u\|_{H^1} + \|u\|_{H^{3,0}} + \|\theta\|_{H^1} + \|u\|_{H^{3,0}}\|\partial_t u\|_{L^2} + \|u\|_{H^{3,0}}\|\theta\|_{L^2}\big)\\ &\le Ce^{-Ct}\end{aligned}$$

for some constant C depending on ν and η.

This completes the proof of the decay estimate in Theorem 1.1. □

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No data was used for the research described in the article.

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