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Journal of Differential Equations 453 (2026) 113835

Journal of Differential Equations

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Localized and non-localized non-blowup criteria for the 2D resistive MHD equations

Jiahong Wu [a], Jiefeng Zhao [b,*], Daoguo Zhou [c]

[a] *Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA*
[b] *School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454003, PR China*
[c] *School of Mathematics, Hangzhou Normal University, Hangzhou 311121, PR China*

Received 7 April 2025; revised 14 August 2025; accepted 3 October 2025
Available online 15 October 2025

Abstract

The global well-posedness problem on the 2D resistive MHD equations without kinematic dissipation is still open. This is a critical problem and its solution relies crucially on the control of the vorticity. Due to the ill-posedness of the vorticity formulation in the L^∞-setting, standard global type vorticity or velocity gradient criteria have no hope to be validated. By exploiting the coupling and interaction of the MHD system, this paper presents several localized non blow-up criteria that are weaker than their global counterparts and more likely to be verified. One of them is in terms of the vorticity in $L^1(-\rho, 0; BMO(B(\rho)))$ and the other in terms of the velocity gradient in $L^{\frac{1}{2}}(-\rho, 0; L^\infty(B(\rho)))$. In addition, a criterion in weighted L^∞ space is also presented to weaken the classical ones. Besides their theoretical importance, these criteria also serve as practical diagnostics in the numerical search of potential MHD singularities.

MSC: 35Q31; 35Q61

Keywords: 2D resistive MHD; Localized criterion; Non-localized criterion

* Corresponding author.
E-mail addresses: jwu29@nd.edu (J. Wu), zhaojiefeng003@hpu.edu.cn (J. Zhao), dgzhou@hznu.edu.cn (D. Zhou).

https://doi.org/10.1016/j.jde.2025.113835

1. Introduction

Whether or not classical solutions to the 2D resistive MHD equations

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + b\cdot\nabla b, \quad x\in\mathbb{R}^2,\ t>0, \\ b_t + u\cdot\nabla b - \Delta b = b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0 \end{cases} \tag{1.1}$$

are always global in time remains a prominent open problem. Here $u = u(x,t)$ is the fluid velocity, $b = b(x,t)$ is the magnetic field, $p = p(x,t)$ is the pressure. The MHD system (1.1) models various physical phenomena such as magnetic reconnection and magnetic turbulence for which the role of resistivity is important while the fluid viscosity can be ignored (see [28]). Studies on fundamental issues such as the global existence of classical solutions and the uniqueness of weak solutions help understand many spectacular events triggered by magnetic reconnection such as solar flares and northern lights. General background information on the MHD equations can be found in many references (see, e.g., [3,6,15,28]). There are substantial recent mathematical developments on (1.1) and related equations (see, e.g., [1,5,8–10,13,14,20,21,24,26,30–32,35,37]).

Global weak solutions for the 2D resistive MHD equations were first obtained in [24] in 1989. More recently, global small solutions have been established on the 2D periodic domain in [32,37]. In the 3D case, global well-posedness was proved in [20] for specially structured axially symmetric initial data. However, the global well-posedness problem on the 2D whole space remains open. Mathematically, the global regularity problem on (1.1) is a critical problem. If the Laplacian dissipation Δb in (1.1) is replaced by the hyper-dissipation $-(-\Delta)^{\beta}b$ with any $\beta > 1$, then the resulting system is globally well-posedness ([11,23]). If we keep Δb in (1.1) but add fractional dissipation $-(-\Delta)^{\alpha}u$ or even logarithmic dissipation $-\log(2-\Delta)u$, then the slightly dissipated MHD system always possesses a unique global classical solution ([19,34]). In addition, the bounds obtained in [22] on the vorticity $\omega = \nabla\times u$ also reveal the criticality. Applying the maximal regularity of the heat operator, [22] shows that any L^q-norm of ω with $1\le q<\infty$ is bounded,

$$\|\omega(t)\|_{L^q} \le C(q,t).$$

However the upper bound $C(q,t)$ depends on q, the initial data and t. In fact, $C(q,t)$ appears to grow at least exponentially in q. This fast growth rate shuts out any potential approach of bounding the L^∞-norm in terms of the L^q-norm and the logarithmic type inequalities. Whether $\|\omega(t)\|_{L^\infty}$ can blow up in a finite time remains open.

Of course, if we knew the vorticity is bounded, say

$$\int_0^T \|\omega(t)\|_{L^\infty} dt < \infty, \tag{1.2}$$

then it is a simple matter to establish the regularity of the solution on $[0,T]$. But verifying (1.2) is no simple matter. The aforementioned L^q approach fails. More importantly, the MHD system (1.1) has a very special structure that promotes the vorticity growth. The vorticity formulation of (1.1)

$$\partial_t \omega + u \cdot \nabla \omega = b_1 \Delta b_2 - b_2 \Delta b_1$$

can be rewritten as

$$\partial_t \omega + u \cdot \nabla \omega = L - b_1 b \cdot \nabla u_2 + b_2 b \cdot \nabla u_1, \tag{1.3}$$

where $L = b_1(\Delta b_2 + b \cdot \nabla u_2) - b_2(\Delta b_1 + b \cdot \nabla u_1)$ and L is globally regular (see [36]). More precisely, L is globally bounded in the following spaces

$$L \in L^\infty((0,T); L^q(\mathbb{R}^2)) \quad \text{and} \quad \nabla L \in L^r((0,T); L^s(\mathbb{R}^2))$$

for any $2 \le q \le \infty$, $2 \le r, s < \infty$ and $T > 0$. By the Biot-Savart law, ∇u_1 and ∇u_2 can be written in terms of ω by

$$\nabla u_1 = -\nabla \partial_2 \Delta^{-1} \omega, \quad \nabla u_2 = \nabla \partial_1 \Delta^{-1} \omega.$$

The operators $-\nabla \partial_2 \Delta^{-1}$ and $\nabla \partial_1 \Delta^{-1}$ are Riesz transform type singular integral operators and are thus not bounded on L^∞. By exploiting this fact, [33] established the mild ill-posedness of the vorticity formulation by constructing a suitable sequence of vorticities that have normal inflation in L^∞. The concept "mild ill-posedness" was proposed by Elgindi and Masmoudi for the 2D Euler vorticity like equations [16].

The impossibility of confirming (1.2) has prompted us to search for weaker criteria that can be potentially verified. Inspired by the work of Chae and Wolf on the 3D Euler equations [12], our paper derives localized regularity criteria relevant for the MHD system. These classes of criteria conclude the regularity by making only local in space and local in time assumptions. They are not only weaker and more likely to be verified, but also provide substantial advantage over the global criteria in the computational test of the blow-up (see, e.g. [12,17]). The localized criteria requires only computed values in a small neighborhood instead of all points in the whole domain.

Our first main result states that localized BMO-norm of the vorticity controls the regularity of the concerned MHD system (1.1). No assumption on the magnetic field is needed. In order to provide a precise statement, we recall several spaces. The notation below refers to the average of u over $\Omega \subset \mathbb{R}^n$

$$u_\Omega = \fint_\Omega u \, \mathrm{d}x.$$

We have the following standard space $BMO(\mathbb{R}^n)$ (see [29])

$$\|u\|_{BMO(\mathbb{R}^n)} = \sup_B \fint_B |u(x) - u_B| \mathrm{d}x,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

We also use local BMO space (see [12]). Denote $B(x,r) = \{y \in \mathbb{R}^n \mid |x-y| < r\}$ and $B(r) = B(0,r)$ for $r > 0$ and $x \in \mathbb{R}^n$. Let $BMO(B(r))$ denote the space of all $u \in L^1(B(r))$ such that

$$|u|_{BMO(B(r))} = \sup_{\substack{z\in B(r)\\ 0<s\le 2r B(z,s)\cap B(r)}} \fint |u - u_{B(z,s)\cap B(r)}|dy < +\infty.$$

The space $BMO(B(r))$ will be equipped with the norm

$$\|u\|_{BMO(B(r))} = |u|_{BMO(B(r))} + r^{-n}\|u\|_{L^1(B(r))}.$$

In addition, $W^{2,\kappa}(B(\rho))$ denotes the standard Sobolev space.

We remark that, for $\kappa > 2$, the Cauchy problem of (1.1) with any initial data $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2)$ and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$ always has a unique local solution (u, b) satisfying, for some $T > 0$ and $1 < q < \infty$,

$$(u, b) \in C([0, T); L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2)), \quad (u, b) \in L^\infty(0, T; H^1(\mathbb{R}^2))$$

and the corresponding pressure $p \in L^\infty(0, T; L^q(\mathbb{R}^2))$. For simplicity we assume that the possible blow-up occurs at the space-time origin $(0, 0)$, and consider equations (1.1) in the local space time domain $B(\rho) \times (-\rho, 0)$ in Theorem 1.1 and Theorem 1.2. For this shifted time interval, the corresponding local solution (u, b) is in the setting

$$(u, b) \in C([-\rho, 0); W^{2,\kappa}(B(\rho))) \cap L^\infty(-\rho, 0; L^4(B(\rho))), \quad p \in L^\infty(-\rho, 0; L^2(B(\rho))).$$

The precise statement on localized non blow-up criterion of the Beale-Kato-Majda type for (1.1) is provided in the following theorem.

Theorem 1.1. *Let (u, p, b) be a solution to (1.1) such that $(u, b) \in C([-\rho, 0); W^{2,\kappa}(B(\rho))) \cap L^\infty(-\rho, 0; L^4(B(\rho)))$ for some $\kappa > 2$ and $p \in L^\infty(-\rho, 0; L^2(B(\rho)))$. If ω satisfies*

$$\int_{-\rho}^{0} |\omega(\tau)|_{BMO(B(\rho))} d\tau < +\infty, \tag{1.4}$$

then there exists no blow-up in $B(\rho) \times \{0\}$, namely

$$\limsup_{t\to 0^-} \|(u(t), b(t)\|_{W^{2,\kappa}(B(r))} < \infty \tag{1.5}$$

for any $r \in (0, \rho)$. Since $|\omega(\tau)|_{BMO(B(\rho))} \leqslant 2\|\omega(\tau)\|_{L^\infty(B(\rho))}$, a special consequence is that (1.4) can be replaced by

$$\int_{-\rho}^{0} \|\omega(\tau)\|_{L^\infty(B(\rho))} d\tau < +\infty.$$

This assumption is appropriate. In fact, one can establish local existence and uniqueness of solutions $(u, b) \in C([0, T); L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2))$ satisfying the a priori estimates $(u, b) \in L^\infty(0, T; H^1(\mathbb{R}^2))$ and $p \in L^\infty(0, T; L^q(\mathbb{R}^2))$ for some $1 < q < \infty$ and $T > 0$. This follows

from a standard local well-posedness procedure for initial data $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$.

Clearly Theorem 1.1 weakens the global criterion (1.2). It makes no assumptions on the magnetic field. Furthermore, the local criterion of Theorem 1.1 can be much more easily checked numerically and serve as a better guidance.

The proof of this theorem is highly nontrivial. We draw some ideas from [12], but the MHD system (1.1) consists of coupled equations of two different types, the hyperbolic and parabolic types. This complex situation makes our proof different from the 3D Euler case. As we can see in Section 4, in order to avoid transforming both the velocity field and magnetic field equations into new ones and localizing the resulting equations, we will exploit the special structure of (1.3). More technical details can be found in Sections 3 and 4.

When there is no magnetic diffusion in (1.1), a nonlocal Beale-Kato-Majda-type blow-up criterion was established in [8] for the ideal MHD equations. Specifically, if

$$\int_0^T (\|\omega(\tau)\|_{L^\infty} + \|j(\tau)\|_{L^\infty})\, \mathrm{d}\tau < \infty,$$

then smooth solutions do not blow up at time T, where $j = \nabla \times b$. For the non-resistive MHD equations, where viscosity is present only in the velocity equation, a Beale-Kato-Majda-type blow-up criterion for smooth solutions was proved in [13], namely

$$\sup_{k\in\mathbb{Z}} \int_0^T \|\Delta_k \omega(\tau)\|_{L^\infty}\, \mathrm{d}\tau < \infty,$$

where Δ_k denotes the Littlewood-Paley frequency localization operator. Furthermore, local versions of the Beale-Kato-Majda-type criterion for both the ideal and non-resistive MHD systems have been investigated in [12].

Our second main result establishes a localized version of Agélas' global criterion [2]. Based on [36] (or see [34]), Agélas proved that if $(u, b) \in C([0, T); H^r(\mathbb{R}^2))$ for $r > 2$ and

$$\int_0^T \|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)} \mathrm{d}\tau < +\infty, \tag{1.6}$$

then there is no blow-up for the solutions of (1.1) at T. Clearly (1.6) greatly improves the standard criterion that requires the L^1-norm in time, namely

$$\int_0^T \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} \mathrm{d}\tau < +\infty.$$

Motivated by numerical simulations on the potential finite-time singularities of the MHD equations, we prove a localized and improved version of (1.6). Our localized regularity criterion can be stated as follows.

Theorem 1.2. *Let* (u, p, b) *be a solution to* (1.1) *such that*

$$(u, b) \in C([-\rho, 0); W^{2,\kappa}(B(\rho))) \cap L^{\infty}(-\rho, 0; L^4(B(\rho)))$$

for some $\kappa > 2$ *and* $p \in L^{\infty}(-\rho, 0; L^2(B(\rho)))$. *If* u *satisfies*

$$\int_{-\rho}^{0} \|\nabla u(\tau)\|_{L^{\infty}(B(\rho))}^{\frac{1}{2}} d\tau < +\infty, \tag{1.7}$$

then there exists no blow-up in $B(\rho) \times \{0\}$, *namely*

$$\limsup_{t \to 0^-} \|(u(t), b(t)\|_{W^{2,\kappa}(B(r))} < \infty \tag{1.8}$$

for any $r \in (0, \rho)$.

This regularity criterion reveals the potential finite-time blowup difference between the 3D Euler and the 2D MHD system concerned here. Recent breakthrough results in [17] by Elgindi and [18] by Elgindi, Ghoul and Masmoudi show that the Beale-Kato-Majda criterion for the 3D incompressible Euler equations cannot be improved in the class of $C^{1,\alpha}$. That is, the L^1-time integral in the criterion condition

$$\int_0^T \|\nabla u(\tau)\|_{L^{\infty}(\mathbb{R}^3)} d\tau < +\infty$$

cannot also be weakened. Theorem 1.2 for the MHD equations requires only the $L^{\frac{1}{2}}$ time integrability.

This localized regularity criterion offers a convenient and easily checkable diagnostics in numerical search of MHD singularities. One crucial index used in determining the singularities of numerical solutions at $t = T$ is the growth rate of $\|\nabla u(t)\|_{L^{\infty}}$ near T. Numerical simulations on the ideal MHD equations suggest that $\|\nabla u(t)\|_{L^{\infty}}$ blow up at the rate of the form $(T - t)^{-\beta}$ for $1 < \beta \leq 2$ (see, e.g., [7]). If we use the standard criterion $\int_0^T \|\nabla u(t)\|_{L^{\infty}} dt$, we would falsely conclude the finite-time singularity. But our new criterion in Theorem 1.2 would rule out the blowup at $t = T$. Furthermore, the localized criterion only requires computed values of ∇u in a small space-time neighborhood.

The proof of Theorem 1.2 is very elaborated and much more complex than its global counterpart. A detailed proof is given in Section 5.

Based on Theorem 1.2, our next result provides a pointwise critical growth rate of $\|\nabla u(t)\|_{L^{\infty}}$ for which the solutions of the MHD equations remain regular. It states that $\|\nabla u(t)\|_{L^{\infty}}$ has to grow at least like t^{-2} in order for the solution to blow up at $t = 0$.

Theorem 1.3. *Let* (u, p, b) *be a solution to* (1.1) *such that* $(u, b) \in C([-\rho, 0); W^{2,\kappa}(B(\rho)))$ *for some* $\kappa > 2$ *and*

$$\|(u, b)\|_{L^{\infty}(-\rho, 0; L^4(B(\rho)))} < \tilde{c}$$

and

$$\|p\|_{L^\infty(-\rho,0;L^2(B(\rho)))} < \tilde{c}.$$

Then there exists a small positive constant ε depending on $\tilde{c}$, ρ, $\|(u(-\rho),b(-\rho)\|_{H^1(B(\rho))}$ and $\|b(-\rho)\|_{L^\infty(B(\rho))}$ such that, if u satisfies

$$\limsup_{t\to 0^-}(-t)^2\|\nabla u(t)\|_{L^\infty(B(\rho))} < \varepsilon, \tag{1.9}$$

then there exists no blow-up in $B(\frac{1}{2}\rho)\times\{0\}$, namely

$$\limsup_{t\to 0^-}\|(u(t),b(t)\|_{W^{2,\kappa}(B(r))} < \infty$$

for any $r\in(0,\frac{1}{2}\rho)$.

As aforementioned, due to the norm inflation of ω in the L^∞ space, directly checking the global criteria such as (1.2) is impossible. The next theorem presents another approach and proves a non-local regularity criterion for ∇u in weighted L^∞ space. It weakens the integrability in space variables and makes it potentially checkable.

Theorem 1.4. *Let $h(x)$ be defined on $\mathbb{R}^2$ such that $|\nabla^i h(x)|\leqslant c|h(x)|$ with $|h(x)|$ having a positive bound from below for $i=1,2$. Let initial data $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap W^{2,\kappa}(\mathbb{R}^2)$ satisfying $\nabla\cdot u_0=\nabla\cdot b_0=0$ for $\kappa>2$. Let $T>0$ such that there exists a unique solution $(u,b)\in L^\infty(0,T;L^2(\mathbb{R}^2))\cap C([0,T);W^{2,\kappa}(\mathbb{R}^2))$ of equations (1.1). Set $0\leqslant\epsilon\leqslant\frac{1}{2}$. Assume $\|b_0h\|_{L^\infty(\mathbb{R}^2)}<+\infty$. If*

$$\int_0^T\|\nabla u(\tau)\|^{\epsilon}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau\int_0^T\|\frac{1}{h^2}\nabla u(\tau)\|^{1-\epsilon}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau<+\infty, \tag{1.10}$$

then there exists no blow-up for the solution (u,b) in $W^{2,\kappa}(\mathbb{R}^2)$ at time T, that is, $(u,b)\in L^\infty(0,T;W^{2,\kappa}(\mathbb{R}^2))$. In particular, assume $b_0\in C_c^\infty(\mathbb{R}^2)$, if

$$\int_0^T\|\nabla u(\tau)\|^{\epsilon}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau\int_0^T\inf_{y\in\overline{B(R)}}\|e^{-c(1+|\bullet-y|^2)^{\frac{1}{2}}}\nabla u(\tau)\|^{1-\epsilon}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau<+\infty \tag{1.11}$$

for some $0\leqslant R<\infty$, then there exists no blow-up for the solution (u,b) in $W^{2,\kappa}(\mathbb{R}^2)$ at time T.

Some special examples of $h(x)$ are $h(x)=(1+|x|^2)^k$ for $k>0$ or $h(x)=e^{c(1+|x|^2)^{\frac{1}{2}}}$ for $c>0$. In addition, if $\epsilon=\frac{1}{2}$ in (1.10), namely $\int_0^T\|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau<+\infty$, we obtain the non blow-up criterion in [2].

Finally, we prove the following non-local regularity criterion for solutions in the Hölder space setting $C^{1,\alpha}(\mathbb{R}^2)$. We remark that, for initial data $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap C^{1,\alpha}(\mathbb{R}^2)$ with

$\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, we can actually obtain the local existence and uniqueness of solution $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap C([0, T); C^{1,\alpha}(\mathbb{R}^2))$ for some $T > 0$ via a standard procedure.

Theorem 1.5. *Let initial data $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap C^{1,\alpha}(\mathbb{R}^2)$ for $0 < \alpha < 1$ satisfying $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Let $T > 0$ such that there exists a unique solution $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap C([0, T); C^{1,\alpha}(\mathbb{R}^2))$ of equations (1.1). If*

$$\int_0^T \frac{\|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{\frac{1}{2}}}{\log(e + \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)})} \mathrm{d}\tau < +\infty, \tag{1.12}$$

then there exists no blow-up for the solution (u, b) in $C^{1,\alpha}(\mathbb{R}^2)$ at time T, that is, $(u, b) \in L^\infty(0, T; C^{1,\alpha}(\mathbb{R}^2))$.

The rest of this paper is divided into seven sections. Section 2 recalls a list of facts, proves some multiplicative inequalities on BMO space and local BMO space used in the proofs of the theorems. In Section 3 we prove local estimates of equations (1.1) under the assumption of Theorem 1.1 by using the structure of MHD equations and standard iteration inequalities. Here we also obtain local estimates of a special combined quantities by exploiting the special structure of 2D resistive MHD equations and obtain good high order local estimates for magnetic field. In Section 4 we use the special structure of (1.1) to prove Theorem 1.1 by using the arguments in [12]. Section 5 proves Theorem 1.2 by using the special structure of (1.1). Here we obtain new iteration inequalities to lead to the conclusion. Section 6, Section 7 and Section 8 prove Theorem 1.3, Theorem 1.4 and Theorem 1.5, respectively.

2. Preparations

In this section, we review some facts about the inhomogeneous Littlewood-Paley decomposition, Bernstein-type lemma, maximal $L^p(L^q)$ estimates for the heat kernel, local version of logarithmic Sobolev's inequality and iteration lemma. In addition, we prove some multiplicative inequalities on BMO space and local BMO space.

2.1. Littlewood-Paley decomposition

The definition of the inhomogeneous Littlewood-Paley decomposition relies on the following dyadic partition of unity.

Lemma 2.1. ([4,27]) *There exist two radial functions $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$ supported in $\{\xi \in \mathbb{R}^n,\ |\xi| \leqslant \frac{4}{3}\}$ and $\{\xi \in \mathbb{R}^n,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$, respectively, satisfying*

$$\chi(\xi) + \sum_{j \geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

We use $\widehat{f}$ or $\mathcal{F}(f)$ to denotes the Fourier transform of f, and $\mathcal{F}^{-1}(f)$ to denote the inverse Fourier transform of f. We set

$$h_1(x) = \mathcal{F}^{-1}\big(\chi(\xi)\big), \quad h_2(x) = \mathcal{F}^{-1}\big(\varphi(\xi)\big)$$

and define the dyadic blocks as follows

$$\Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^n} h_1(y)u(x-y)\mathrm{d}y,$$

$$\Delta_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^n} h_2(2^j y)u(x-y)\mathrm{d}y, \ \forall j \geqslant 0.$$

Then the inhomogeneous Littlewood-Paley decomposition is defined as

$$u = \sum_{j\geqslant -1} \Delta_j u \qquad \text{for} \quad u \in \mathcal{S}'. \tag{2.1}$$

Let us recall the definition of inhomogeneous Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov space $B^s_{p,r}$ is defined as

$$B^s_{p,r} \triangleq \{u \in \mathcal{S}', \ \|u\|_{B^s_{p,r}} < \infty\},$$

where the inhomogeneous Besov norm is given by

$$\|u\|_{B^s_{p,r}} \triangleq \Big(\sum_{j=-1}^{\infty} 2^{jsr}\|\Delta_j u\|^r_{L^p}\Big)^{\frac{1}{r}}.$$

The following is Bernstein-type lemma.

Lemma 2.2. ([4,27]) *Let $k \geqslant 0$. Let $1 \leqslant p \leqslant q \leqslant +\infty$.*

(1) *Let $j \in \mathbb{Z}$ and $m > 0$. If f satisfies*

$$\mathrm{supp}\widehat{f} \subseteq \{\xi \in \mathbb{R}^n, \ |\xi| \leqslant m\, 2^j\},$$

then, for some constant C independent of f and j,

$$\|\nabla^k f\|_{L^q(\mathbb{R}^n)} \leqslant C\, 2^{jk+jn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^n)}.$$

(2) *Let $j \in \mathbb{Z}$ and $n_1, n_2 > 0$. If f satisfies*

$$\mathrm{supp}\widehat{f} \subseteq \{\xi \in \mathbb{R}^n, \ n_1 2^j \leqslant |\xi| \leqslant n_2 2^j\},$$

then, for two constants C_1 and C_2 independent of f and j,

$$C_1 2^{kj}\|f\|_{L^q(\mathbb{R}^n)} \leqslant \sup_{|\gamma|=k} \|\partial^\gamma f\|_{L^q(\mathbb{R}^n)} \leqslant C_2 2^{kj+jn(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^n)}.$$

Especially, Lemma 2.2 holds for the dyadic blocks, namely for $f = \Delta_j u$.

We also will use the following lemma which describe the action of the semigroup of the heat kernel on distributions with Fourier transforms supported in an annulus.

Lemma 2.3. ([4,27]) *Let $\mathcal{C}$ be annulus, then there exist c and C such that for any $p \in [1, \infty]$ and any positive constants t and λ we have*

$$\operatorname{supp} \widehat{f} \subset \lambda \mathcal{C} \Rightarrow \|e^{t\Delta} f\|_{L^p(\mathbb{R}^n)} \leqslant C e^{-ct\lambda^2} \|f\|_{L^p(\mathbb{R}^n)}.$$

Let S_k denote the identity approximation operator,

$$S_k = \sum_{-1 \leqslant k_1 \leqslant k-1} \Delta_{k_1}.$$

In addition, we write

$$R_k = (S_{k-1} v \cdot \nabla) \Delta_k u - \Delta_k (v \cdot \nabla u).$$

Lemma 2.4. (see [27]) *Let u, v be two vector functions with $\nabla \cdot v = 0$. Assume $1 \leqslant p \leqslant p_1 \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $p' = (1 - \frac{1}{p})^{-1}$ and $s > -1 - \min(\frac{d}{p_1}, \frac{d}{p'})$. Then,*

$$\begin{cases} 2^{ks} \|R_k\|_{L^p(\mathbb{R}^d)} \leqslant c h_k \|\nabla v\|_{B^{\frac{d}{p_1}}_{p_1,\infty}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)} \|u\|_{B^s_{p,q}(\mathbb{R}^d)}, & \text{if } s < 1 + \frac{d}{p_1}, \\ 2^{ks} \|R_k\|_{L^p(\mathbb{R}^d)} \leqslant c h_k \|\nabla v\|_{B^{s-1}_{p_1,q}(\mathbb{R}^d)} \|u\|_{B^s_{p,q}(\mathbb{R}^d)}, & \text{if } s > 1 + \frac{d}{p_1}, \text{ or } s = 1 + \frac{d}{p_1}, q = 1, \end{cases}$$

where $c = c(d, p, p_1, s, q)$, and $\sum (h_k^q)^{\frac{1}{q}} = 1$.

2.2. Maximal $L^p(L^q)$ estimates

Recalled the following maximal $L^p(L^q)$ regularity theorem for the heat kernel.

Lemma 2.5. ([25]) *Assume $f \in L^p((0,T), L^q(\mathbb{R}^n))(1 < p, q < \infty)$. Let*

$$A : v \mapsto Af(x,t) = \int_0^t e^{(t-s)\Delta} \Delta f(\cdot, s) \mathrm{d}s,$$

then

$$\|Af\|_{L^p((0,T), L^q(\mathbb{R}^n))} \leqslant C \|f\|_{L^p((0,T), L^q(\mathbb{R}^n))}$$

for every $T \in (0, \infty]$ and some positive constants C (independent of T).

2.3. Properties of local BMO space

We recall the following local version of logarithmic Sobolev inequality (see [12]).

Lemma 2.6. *Let $B(r)$ be a ball in $\mathbb{R}^n$ with the radius $r > 0$. For every $u \in W^{1,q}(B(r))$, $n < q < +\infty$, the following inequality holds true*

$$\|u\|_{L^\infty(B(r))} \leqslant c(1 + \|u\|_{BMO(B(r))}) \log\left(e + c\|\nabla u\|_{L^q(B(r))} + cr^{-1+\frac{n}{q}-\frac{n}{2}}\|u\|_{L^2(B(r))}\right)$$

with a constant $c > 0$ depending only on n and q.

The following result of embedding of $BMO(B(r))$ into $L^p(B(r))$ is well known (see [12]).

Lemma 2.7. *Let $u \in BMO(B(r))$. Then $u \in \cap_{1\leqslant q<\infty} L^q(B(r))$, and it holds*

$$\|u\|_{L^q(B(r))} \leqslant cr^{\frac{n}{q}}|u|_{BMO(B(r))} + cr^{\frac{n}{q}-n}\|u\|_{L^1(B(r))} = cr^{\frac{n}{q}}\|u\|_{BMO(B(r))}.$$

From the definition of local BMO space, we obtain the following two lemmas.

Lemma 2.8. *For $r_1 < r_2$, we have $|u|_{BMO(B(r_1))} \leqslant c|u|_{BMO(B(r_2))}$.*

Proof. Using the definition of $BMO(B(r))$, we obtain

$$\begin{aligned}
&|u|_{BMO(B(r_1))}\\
&= \sup_{\substack{z\in B(r_1)\\ 0<s\leqslant 2r_1}} \fint_{B(z,s)\cap B(r_1)} |u - u_{B(z,s)\cap B(r_1)}|\mathrm{d}y\\
&\leqslant 2 \sup_{\substack{z\in B(r_1)\\ 0<s\leqslant 2r_1}} \fint_{B(z,s)\cap B(r_1)} |u - u_{B(z,s)\cap B(r_2)}|\mathrm{d}y\\
&\leqslant 2 \sup_{\substack{z\in B(r_1)\\ 0<s\leqslant 2r_1}} \frac{|B(z,s)\cap B(r_2)|}{|B(z,s)\cap B(r_1)|} \fint_{B(z,s)\cap B(r_2)} |u - u_{B(z,s)\cap B(r_2)}|\mathrm{d}y\\
&\leqslant c \sup_{\substack{z\in B(r_1)\\ 0<s\leqslant 2r_1}} \fint_{B(z,s)\cap B(r_2)} |u - u_{B(z,s)\cap B(r_2)}|\mathrm{d}y\\
&\leqslant c|u|_{BMO(B(r_2))}. \quad \square
\end{aligned}$$

Lemma 2.9. *Assume $u \in BMO(B(lr))$. If $u(\cdot) = v(\frac{\cdot}{l})$ for $l > 0$, we have $|v|_{BMO(B(r))} = |u|_{BMO(B(lr))}$.*

Proof. From the definition of $BMO(B(r))$, we have

$$
\begin{aligned}
&|v|_{BMO(B(r))}\\
&= \sup_{\substack{z\in B(r)\\ 0<s\leqslant 2r}} \fint_{B(z,s)\cap B(r)} |v - v_{B(z,s)\cap B(r)}|\mathrm{d}y\\
&= \sup_{\substack{z\in B(lr)\\ 0<s\leqslant 2lr}} \fint_{B(z,s)\cap B(lr)} |u - u_{B(z,s)\cap B(lr)}|\mathrm{d}y. \quad \square
\end{aligned}
$$

Similar to [12], we can prove the following three multiplicative inequalities on BMO space and local BMO space in $\mathbb{R}^2$.

Lemma 2.10. *Let $\psi \in C_c^\infty(B(r))$ with $0 \leqslant \psi \leqslant 1$. For every $u \in W^{1,1}(B(r))$ such that $\nabla \cdot u = 0$ it holds*

$$\|\nabla u\psi\|_{BMO(\mathbb{R}^2)} \leqslant c\|(\nabla \times u)\psi\|_{BMO(\mathbb{R}^2)} + c\|\nabla\psi\|_{L^\infty}\|u\|_{L^\infty(B(r))}.$$

Proof. Thanks to $\nabla \cdot u = 0$, it holds

$$-\Delta\nabla(u\psi) = \nabla\nabla\times(\nabla\times(u\psi)) - \nabla\nabla(u\cdot\nabla\psi).$$

Using Calderón-Zygmund's inequality [29], we find that

$$
\begin{aligned}
\|\nabla u\psi\|_{BMO((\mathbb{R}^2))} &\leqslant c\|\nabla\times(\psi u)\|_{BMO(\mathbb{R}^2)} + c\|u\nabla\psi\|_{BMO(\mathbb{R}^2)}\\
&\leqslant c\|(\nabla\times u)\psi\|_{BMO(\mathbb{R}^2)} + c\|\nabla\psi\|_{L^\infty}\|u\|_{L^\infty(B(r))}. \quad \square
\end{aligned}
$$

Lemma 2.11. *Let $u \in W^{1,1}(B(r))$ with $\nabla\times u \in BMO(B(r))$. Then for all $\psi \in C_c^\infty(B(\frac{(k-1)r}{k}))$ with k is a large positive integer and $0 \leqslant \psi \leqslant 1$ we get*

$$
\begin{aligned}
\|(\nabla\times u)\psi\|_{BMO(\mathbb{R}^2)} \leqslant c(k)\Big(1 + r\|\nabla\psi\|_{L^\infty}\Big)\Big((|\nabla\times u|_{BMO(B(r))}\\
+r^{-1}\|\nabla\times u\|_{L^2(B(r))} + r^{-2}\|u\|_{L^2(B(r))})\Big)
\end{aligned}
$$

where $c(k)$ depends on k.

Proof. Assume $r = 1$. Let $\eta \in C_c^\infty(B(1))$ such that $|\nabla\eta| \leqslant c$ and $\int_{B(1)}\eta\mathrm{d}x \geqslant c$, where $c > 0$ stands for a constant depending only on n. For $f \in L^1(B(1))$ we define the mean

$$\widetilde{f}_{B(1)} = \frac{1}{\int_{B(1)}\eta dx}\fint_{B(1)} f\eta\mathrm{d}x.$$

First, using triangle inequality and applying integration by parts, we see that

$$
\begin{aligned}
\|\nabla\times u\|_{L^1(B(1))} &= \|\nabla\times u - \widetilde{\nabla\times u}_{B(1)}\|_{L^1(B(1))} + |\widetilde{\nabla\times u}_{B(1)}|\\
&\leqslant c|\nabla\times u|_{BMO(B(1))} + c\|u\|_{L^1(B(1))}.
\end{aligned}
$$

For $\rho \geqslant \frac{1}{2k}$ and $x_0 \in \mathbb{R}^2$, we obtain

$$\begin{aligned}
&\fint_{B(x_0,\rho)} |(\nabla \times u)\psi - ((\nabla \times u)\psi)_{B(x_0,\rho)}|\mathrm{d}x \\
&\quad \leqslant c\|\nabla \times u\|_{L^1(B(1))} \\
&\quad \leqslant c|\nabla \times u|_{BMO(B(1))} + c\|u\|_{L^1(B(1))}.
\end{aligned}$$

In case $\rho \leqslant \frac{1}{2k}$ and $B(x_0, \rho) \cap B(\frac{k-1}{k}) \neq \emptyset$, we see that $B(x_0, \rho) \subset B(1)$ and

$$\begin{aligned}
&\fint_{B(x_0,\rho)} |(\nabla \times u)\psi - ((\nabla \times u)\psi)_{B(x_0,\rho)}|\mathrm{d}x \\
&\leqslant c|\nabla \times u|_{BMO(B(1))} + c \fint_{B(x_0,\rho)} \fint_{B(x_0,\rho)} |(\nabla \times u)(x)||\psi(x) - \psi(y)|\mathrm{d}x\mathrm{d}y. \\
&\leqslant c|\nabla \times u|_{BMO(B(1))} + \|\nabla\psi\|_{L^\infty} \fint_{B(x_0,\rho)} \fint_{B(x_0,\rho)} |(\nabla \times u)(x)|\,|x - y|\mathrm{d}x\mathrm{d}y \\
&\leqslant c|\nabla \times u|_{BMO(B(1))} + c\|\nabla\psi\|_{L^\infty}\|\nabla \times u\|_{L^2(B(r))}.
\end{aligned}$$

Therefore, we get

$$\|(\nabla \times u)\psi\|_{BMO(\mathbb{R}^2)} \leqslant c(1 + \|\nabla\psi\|_{L^\infty})(|\nabla \times u|_{BMO(B(1))} + \|\nabla \times u\|_{L^2(B(r))} + \|u\|_{L^2(B(r))}).$$

This completes the proof of the lemma by using a standard scaling argument. □

Combining Lemma 2.10 and Lemma 2.11, we have

Corollary 2.12. *Let $u \in W^{1,1}(B(r))$ with $\nabla \times u \in BMO(B(r))$. Then for all $\psi \in C_c^\infty(B(\frac{(k-1)r}{k}))$ with k is a large positive integer and $0 \leqslant \psi \leqslant 1$, we get*

$$\begin{aligned}
\|(\nabla u)\psi\|_{BMO(\mathbb{R}^2)} \leqslant c(k)\Big(1 + r\|\nabla\psi\|_{L^\infty}\Big)\Big((|\nabla \times u|_{BMO(B(r))} + r^{-1}\|\nabla \times u\|_{L^2(B(r))} \\
+ r^{-2}\|u\|_{L^2(B(r))})\Big) + c\|\nabla\psi\|_{L^\infty}\|u\|_{L^\infty(B(r))},
\end{aligned}$$

where $c(k)$ depends on k.

2.4. Iteration lemma

Finally, we review the following lemma (see [12]).

Lemma 2.13. *Let $Y_m : [t_0, t_1] \to \mathbb{R}$, $m \in \mathbb{N} \cup \{0\}$ be a sequences of bounded functions. Assume there exists $0 < M = M(t) < +\infty$ for each $t \in [t_0, t_1)$ such that*

$$|Y_m(t)| < M(t)^m \qquad \forall t \in [t_0, t_1), \forall m \in \mathbb{N}. \tag{2.2}$$

Suppose that the following recursive integral inequality holds true

$$Y_m(t) \leqslant Cm + \int_{t_0}^{t} a(s) Y_{m+1}(s) \mathrm{d}s, \quad m \in \mathbb{N} \cup \{0\},$$

where C is a positive constant and $a \in L^1(t_0, t_1)$ with $a(t) \geqslant 0$ for almost every $t \in [t_0, t_1]$. Then for all $t \in [t_0, t_1]$, we have the following inequality

$$Y_0(t) \leqslant C \int_{t_0}^{t} a(s) \mathrm{d}s \, e^{\int_{t_0}^{t} a(s) \mathrm{d}s}.$$

3. Local estimates for MHD equations

Lemma 3.1. *Assume $(u, b) \in L^\infty(-\rho, 0; L^4(B(\rho)))$ and $p \in L^\infty(-\rho, 0; L^2(B(\rho)))$, then for any $0 < r < \rho$, we have $\nabla b \in L^2(-\rho, 0; L^2(B(r)))$.*

Proof. Let $\varphi \in C_c^\infty(B(\rho))$ be a cut off function such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B(r)$. Multiplying equations (1.1) by φ, we have

$$(u\varphi)_t + u \cdot \nabla u\varphi = -\nabla p\varphi + b \cdot \nabla b\varphi, \tag{3.1}$$

and

$$(b\varphi)_t + u \cdot \nabla b\varphi - \Delta b\varphi = b \cdot \nabla u\varphi. \tag{3.2}$$

Multiplying equations (3.1) and (3.2) by $u\varphi$ and $b\varphi$ respectively, and integrating the result equations on $B(r)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\varphi\|_{L^2}^2 + \frac{1}{2}\int u \cdot \nabla |u|^2 \varphi^2 \mathrm{d}x = -\int \nabla p u \varphi^2 \mathrm{d}x + \int b \cdot \nabla b u \varphi^2 \mathrm{d}x$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b\varphi\|_{L^2}^2 + \frac{1}{2}\int u \cdot \nabla |b|^2 \varphi^2 \mathrm{d}x - \int \Delta b b \varphi^2 \mathrm{d}x = \int b \cdot \nabla u b \varphi^2 \mathrm{d}x.$$

Adding the two equations above, and applying integration by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\varphi\|_{L^2}^2 + \|b\varphi\|_{L^2}^2) + \int |\nabla b|^2 \varphi^2 \mathrm{d}x \\ =& \frac{1}{2}\int u \cdot \nabla \varphi^2 |u|^2 \mathrm{d}x + \frac{1}{2}\int u \cdot \nabla \varphi^2 |b|^2 \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int pu\cdot\nabla\varphi^2\mathrm{d}x+\int \nabla bb\nabla\varphi^2\mathrm{d}x-\int b\cdot\nabla\varphi^2(bu)\mathrm{d}x\\
&\leqslant c\|u\|^3_{L^4(B(\rho))}+c\|u\|_{L^4(B(\rho))}\|b\|^2_{L^4(B(\rho))}+c\|u\|_{L^4(B(\rho))}\|p\|_{L^2(B(\rho))}\\
&+\frac{1}{2}\int|\nabla b|^2\varphi^2\mathrm{d}x+c\|b\|^2_{L^4(B(\rho))}
\end{aligned}
$$

by using $\nabla\cdot u=\nabla\cdot b=0$. Absorbing $\frac{1}{2}\int|\nabla b|^2\varphi^2\mathrm{d}x$ on the right side of the inequality above and integrating the resulting differential inequality in time on $(-\rho,0)$, we prove this lemma. □

Let $\omega=\partial_1u_2-\partial_2u_1$ and $j=\partial_1b_2-\partial_2b_1$. Applying $\nabla^\perp\cdot$ to equations (1.1), we have

$$
\begin{cases}
\partial_t\omega+u\cdot\nabla\omega=b\cdot\nabla j,\\
\partial_t j+u\cdot\nabla j-\Delta j=b\cdot\nabla\omega+Q(\nabla u,\nabla b),
\end{cases}
\tag{3.3}
$$

where $Q(\nabla u,\nabla b)=2\partial_1b_1(\partial_2u_1+\partial_1u_2)-2\partial_1u_1(\partial_2b_1+\partial_1b_2)$.

Lemma 3.2. *Under the assumption of Theorem 1.1, for any $0<r<\rho$, we have $(\omega,j)\in L^\infty(-\rho,0;L^2(B(r)))$ and $\nabla j\in L^2(-\rho,0;L^2(B(r)))$.*

Remark 3.3. From Lemma 3.1 and Lemma 3.2, we obtain $(\nabla u,\nabla b)\in L^\infty(-\rho,0;L^2(B(r)))$, $\nabla^2b\in L^2(-\rho,0;L^2(B(r)))$ and $(u,b)\in L^\infty(-\rho,0;L^p(B(r)))$ for $r<\rho$ and $2\leqslant p<\infty$.

Remark 3.4. Let $\gamma=\max(\delta+\frac{q}{2q-2},\frac{q_1}{2(q_1-2)})>\frac{1}{2}$. From the estimates of I_1 and I_4, we can choose δ small enough and q,q_1 large enough so that γ close to $\frac{1}{2}$. Here Lemma 3.2 holds just asking for $\int_{-\rho}^0|\omega(\tau)|^\gamma_{BMO(B(\rho))}\mathrm{d}\tau<+\infty$.

Proof of Lemma 3.2. Let $0<r<\sigma<\rho$ be arbitrary chosen. Let $0<r<r_1<r_2<\sigma<\rho$. Let $\varphi\in C_c^\infty(B(r_2))$ be a cut off function such that $0\leqslant\varphi\leqslant1$ and $\varphi\equiv1$ on $B(\frac{r_1+r_2}{2})$ and $|\nabla\varphi|\leqslant c(r_2-r_1)^{-1}$. Multiplying equations (3.3) by φ, we have

$$
(\omega\varphi)_t+u\cdot\nabla\omega\varphi=b\cdot\nabla j\varphi \tag{3.4}
$$

and

$$
(j\varphi)_t+u\cdot\nabla j\varphi-\Delta j\varphi=b\cdot\nabla\omega\varphi+Q(\nabla u,\nabla b)\varphi. \tag{3.5}
$$

We multiply equations (3.4) and (3.5) by $\omega\varphi$ and $j\varphi$ respectively, integrate the equations on $B(r_2)\times(-\rho,0)$ and add the results together. This yields

$$
\begin{aligned}
&(\|\omega(t)\varphi\|^2_{L^2}+\|j(t)\varphi\|^2_{L^2})+2\int_{-\rho}^t\int|\nabla j|^2\varphi^2\mathrm{d}x\mathrm{d}\tau\\
&=(\|\omega(-\rho)\varphi\|^2_{L^2}+\|j(-\rho)\varphi\|^2_{L^2})-\int_{-\rho}^t\int u\cdot\nabla|\omega|^2\varphi^2\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$+\int_{-\rho}^{t}\int u\cdot\nabla|j|^2\varphi^2\mathrm{d}x\mathrm{d}\tau+2\int_{-\rho}^{t}\int\nabla jj\nabla\varphi^2\mathrm{d}x\mathrm{d}\tau$$

$$+2\int_{-\rho}^{t}\int b\cdot\nabla(j\omega)\varphi^2\mathrm{d}x\mathrm{d}\tau+2\int_{-\rho}^{t}\int Q(\nabla u,\nabla b)j\varphi^2\mathrm{d}x\mathrm{d}\tau$$

$$=(\|\omega(-\rho)\varphi\|_{L^2}^2+\|j(-\rho)\varphi\|_{L^2}^2)+\sum_{i=1}^{5}I_i. \tag{3.6}$$

Let δ be a small enough positive constant. We estimate

$$|I_1|=|\int_{-\rho}^{t}\int u\cdot\nabla\varphi^2\omega^2\mathrm{d}x\mathrm{d}\tau|\leqslant c\int_{-\rho}^{t}\|\omega\|_{L^2(B(r_2))}^2\|u\varphi\|_{L^\infty}\|\nabla\varphi\|_{L^\infty}\mathrm{d}\tau$$

$$\leqslant c(r_2-r_1)^{-1}\sup_{s\in(-\rho,0)}\|\omega(s)\|_{L^2(B(r_2))}^{2-\delta}\int_{-\rho}^{t}\|\omega(s)\|_{L^2(B(r_2))}^{\delta}\|u\|_{L^\infty(B(r_2))}\mathrm{d}\tau. \tag{3.7}$$

Let $\psi\in C_c^\infty(B(\sigma))$ be a cut off function such that $0\leqslant\psi\leqslant 1$ and $\psi\equiv 1$ on $B(r_2)$. By virtue of Gagliardo-Nirenberg's inequality, for some large enough positive constant q, we have

$$\|u\|_{L^\infty(B(r_2))}\leqslant\|u\psi^{\frac{q}{2}}\|_{L^\infty}\leqslant c\|u\psi^{\frac{q}{2}}\|_{L^2}^{\frac{q-2}{2q-2}}\|\nabla(u\psi^{\frac{q}{2}})\|_{L^q}^{\frac{q}{2q-2}}$$

$$\leqslant c\|u\psi^{\frac{q}{2}}\|_{L^2}^{\frac{q-2}{2q-2}}(\|\nabla u\psi^{\frac{q}{2}})\|_{L^q}^{\frac{q}{2q-2}}+\|u\psi^{\frac{q}{2}-1}\|_{L^q}^{\frac{q}{2q-2}}\|\nabla\psi\|_{L^\infty}^{\frac{q}{2q-2}}). \tag{3.8}$$

Recalling that $\nabla\cdot u=0$, we obtain $-\Delta(u\psi^m)=\nabla\times\nabla\times(u\psi^m)-\nabla(u\cdot\nabla\psi^m)$ for $m>1$. Using Calderón-Zygmund's inequality [29], we have

$$\|\nabla u\psi^m\|_{L^q}\leqslant\|\nabla(u\psi^m)\|_{L^q}+c\|\nabla\psi\|_{L^\infty}\|u\psi^{m-1}\|_{L^q}$$

$$\leqslant c\|\nabla\times(u\psi^m)\|_{L^q}+c\|\nabla\psi\|_{L^\infty}\|u\psi^{m-1}\|_{L^q}$$

$$\leqslant c\|\omega\psi^m\|_{L^q}+c\|\nabla\psi\|_{L^\infty}\|u\psi^{m-1}\|_{L^q}. \tag{3.9}$$

Let $m=\frac{q}{2}$. We get

$$\|u\psi^{\frac{q}{2}-1}\|_{L^q}=(\int|u|^2|u\psi^{\frac{q}{2}}|^{q-2}\mathrm{d}x)^{\frac{1}{q}}\leqslant c\|u\|_{L^2(B(\rho))}^{\frac{2}{q}}\|u\psi^{\frac{q}{2}}\|_{L^\infty}^{1-\frac{2}{q}}. \tag{3.10}$$

Inserting (3.9) and (3.10) into (3.8) and using Young's inequality, we have

$$\|u\|_{L^\infty(B(r_2))}\leqslant\|u\psi^{\frac{q}{2}}\|_{L^\infty}\leqslant c\|u\|_{L^2(B(\rho))}^{\frac{q-2}{2q-2}}\|\omega\|_{L^q(B(\rho))}^{\frac{q}{2q-2}}+c\|u\|_{L^2(B(\rho))}$$

$$\leqslant c(\rho^{\frac{2}{q}}\|\omega\|_{BMO(B(\rho))})^{\frac{q}{2q-2}}+c \tag{3.11}$$

by taking advantage of Lemma 2.7. We also obtain

$$\|\omega\|_{L^2(B(r_2))} \leqslant c\|\omega\|_{BMO(B(\rho))} \tag{3.12}$$

by applying Lemma 2.7. Here we have used the assumption $u \in L^\infty(-\rho, 0; L^4(B(\rho)))$ in Theorem 1.1, which implies that $\|u\|_{L^2(B(\rho))}$ is bounded. Inserting (3.11) and (3.12) into (3.7) and applying for Young's inequality, we are led to

$$\begin{aligned}
|I_1| &\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)}\|\omega(s)\|^2_{L^2(B(r_2))} + c(r_2-r_1)^{-\frac{2}{\delta}}\Big(\int_{-\rho}^{t}(\|\omega\|^{\delta+\frac{q}{2q-2}}_{BMO(B(\rho))}+c)\mathrm{d}\tau\Big)^{\frac{2}{\delta}} \\
&\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)}\|\omega(s)\|^2_{L^2(B(r_2))} + c(r_2-r_1)^{-\frac{2}{\delta}}
\end{aligned}$$

by using Remark 3.4.

I_2 is estimated as follows:

$$\begin{aligned}
|I_2| &= |\int_{-\rho}^{t}\int u\cdot\nabla|j|^2\varphi^2\mathrm{d}x\mathrm{d}\tau| \leqslant c\int_{-\rho}^{t}\|u\|_{L^4}\|\nabla j\varphi\|_{L^2}\|j\varphi\|_{L^4}\mathrm{d}\tau \\
&\leqslant c\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}\|j\varphi\|^{\frac{1}{2}}_{L^2}(\|\nabla j\varphi\|^{\frac{1}{2}}_{L^2}+\|j\nabla\varphi\|^{\frac{1}{2}}_{L^2})\mathrm{d}\tau \\
&\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|^2_{L^2}\mathrm{d}\tau + c\int_{-\rho}^{t}(\|j\varphi\|^2_{L^2}+\|j\|^2_{L^2(B(r_2))}\|\nabla\varphi\|^2_{L^\infty})\mathrm{d}\tau \\
&\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|^2_{L^2}\mathrm{d}\tau + c(r_2-r_1)^{-2}+c,
\end{aligned}$$

where we use Lemma 3.1.

It is easy to estimate I_3 by using Young's inequality and Lemma 3.1:

$$\begin{aligned}
|I_3| &= 2|\int_{-\rho}^{t}\int\nabla jj\nabla\varphi^2\mathrm{d}x\mathrm{d}\tau| \leqslant c\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}\|j\nabla\varphi\|_{L^2}\mathrm{d}\tau \\
&\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|^2_{L^2}\mathrm{d}\tau + c(r_2-r_1)^{-2}.
\end{aligned}$$

For some large enough positive constant q_1, we see that

$$|I_4| = 4|\int_{-\rho}^{t}\int b\cdot\nabla\varphi j\omega\varphi\mathrm{d}x\mathrm{d}\tau|$$

$$
\begin{aligned}
&\leqslant c\int_{-\rho}^{t}\|j\varphi\|_{L^2}\|\omega\|_{L^4(B(r_2))}\|b\|_{L^4(B(r_2))}\|\nabla\varphi\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant c(r_2-r_1)^{-1}\int_{-\rho}^{t}\|j\varphi\|_{L^2}\|\omega\|_{L^2(B(r_2))}^{\frac{q_1-4}{2(q_1-2)}}\|\omega\|_{L^{q_1}(B(r_2))}^{\frac{q_1}{2(q_1-2)}}\mathrm{d}\tau\\
&\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r_2))}^2+\|j(s)\|_{L^2(B(r_2))}^2)\\
&+c(r_2-r_1)^{-\frac{4(q_1-2)}{q_1}}(\int_{-\rho}^{t}\|\omega\|_{BMO(B(\rho))}^{\frac{q_1}{2(q_1-2)}}\mathrm{d}\tau)^{\frac{4(q_1-2)}{q_1}}\\
&\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r_2))}^2+\|j(s)\|_{L^2(B(r_2))}^2)+c(r_2-r_1)^{-\frac{4(q_1-2)}{q_1}},
\end{aligned}
$$

where we use Young's inequality, Lemma 2.7 and Remark 3.4.

$$
\begin{aligned}
|I_5|&\leqslant c|\int_{-\rho}^{t}\int Q(u,\nabla\nabla b)j\varphi^2\mathrm{d}x\mathrm{d}\tau|+c|\int_{-\rho}^{t}\int Q(u,\nabla b)\nabla j\varphi^2\mathrm{d}x\mathrm{d}\tau|\\
&+c|\int_{-\rho}^{t}\int Q(u,\nabla b)j\nabla\varphi^2\mathrm{d}x\mathrm{d}\tau|\\
&\leqslant c\int_{-\rho}^{t}\|u\|_{L^4(B(\rho))}\|\nabla^2 b\varphi\|_{L^2}\|j\varphi\|_{L^4}\mathrm{d}\tau+c\int_{-\rho}^{t}\|u\|_{L^4(B(\rho))}\|\nabla b\varphi\|_{L^4}\|\nabla j\varphi\|_{L^2}\mathrm{d}\tau\\
&+c\int_{-\rho}^{t}\|u\|_{L^4(B(\rho))}\|\nabla b\nabla\varphi\|_{L^2}\|j\varphi\|_{L^4}\mathrm{d}\tau.
\end{aligned}
$$

Recalling that $\nabla\cdot b=0$, we obtain $-\Delta(\partial_i b\varphi)=\nabla\times\nabla\times(\partial_i b\varphi)-\nabla(\partial_i b\cdot\nabla\varphi)$. Using Calderón-Zygmund's inequality [29], we have

$$
\begin{aligned}
\|\nabla^2 b\varphi\|_{L^2}&\leqslant c\|\nabla(\nabla b\varphi)\|_{L^2}+c\|\nabla b\|_{L^2(B(r_2))}\|\nabla\varphi\|_{L^\infty}\\
&\leqslant c\sum_{i=1}^{2}\|\nabla\times(\partial_i b\varphi)\|_{L^2}+c\|\nabla b\|_{L^2(B(r_2))}\|\nabla\varphi\|_{L^\infty}\\
&\leqslant c\|\nabla j\varphi\|_{L^2}+c\|\nabla b\|_{L^2(B(r_2))}\|\nabla\varphi\|_{L^\infty}.
\end{aligned}
$$

Thanks to Gagliardo-Nirenberg's inequality, we obtain

$$
\|j\varphi\|_{L^4}\leqslant c\|j\varphi\|_{L^2}^{\frac{1}{2}}(\|\nabla j\varphi\|_{L^2}^{\frac{1}{2}}+\|j\|_{L^2}^{\frac{1}{2}}\|\nabla\varphi\|_{L^\infty}^{\frac{1}{2}})
$$

and

$$\begin{aligned}\|\nabla b\varphi\|_{L^4} &\leqslant c\|\nabla(b\varphi)\|_{L^4} + c\|b\|_{L^4(B(\rho))}\|\nabla\varphi\|_{L^\infty}\\ &\leqslant c\|j\varphi\|_{L^4} + c\|b\|_{L^4(B(\rho))}\|\nabla\varphi\|_{L^\infty}.\end{aligned}$$

Combining the above estimates and using Young's inequality, we see that

$$\begin{aligned}|I_5| &\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}^2 \mathrm{d}\tau + c(1+(r_2-r_1)^{-1}+(r_2-r_1)^{-\frac{3}{2}})\int_{-\rho}^{t}\|j\|_{L^2(B(r_2))}^2 \mathrm{d}\tau\\ &\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}^2 \mathrm{d}\tau + c(1+(r_2-r_1)^{-1}+(r_2-r_1)^{-\frac{3}{2}})\end{aligned}$$

by the aid of Lemma 3.1.

Inserting the estimates of $I_1 - I_5$ into the right side of (3.6), we get

$$\begin{aligned}&\operatorname{ess\,sup}_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r_1))}^2 + \|j(s)\|_{L^2(B(r_1))}^2)\\ &\leqslant \frac{1}{2}\operatorname{ess\,sup}_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r_2))}^2 + \|j(s)\|_{L^2(B(r_2))}^2) + (\|\omega(-\rho)\|_{L^2(B(\rho))}^2 + \|j(-\rho)\|_{L^2(B(\rho))}^2)\\ &+c + c(r_2-r_1)^{-\frac{2}{\delta}}.\end{aligned}$$

Thanks to the assumption in Theorem 1.1, we have $(u,b)\in C([-\rho,0);W^{2,\kappa}(B(\rho)))$ and thus $(\omega,j)\in C([-\rho,0);W^{1,\kappa}(B(\rho)))$. By the continuity of (ω,j) in time and an iteration argument, we deduce that

$$\begin{aligned}&\operatorname{ess\,sup}_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r))}^2 + \|j(s)\|_{L^2(B(r))}^2)\\ &\leqslant c(\sigma-r)^{-\frac{2}{\delta}}(\|\omega(-\rho)\|_{L^2(B(\rho))}^2 + \|j(-\rho)\|_{L^2(B(\rho))}^2 + c).\end{aligned}$$

Therefore, for any $0<r<\rho$, we have $(\omega,j)\in L^\infty(-\rho,0;L^2(B(r)))$. Furthermore, in virtue of (3.6), we obtain $\nabla j\in L^2(-\rho,0;L^2(B(r)))$ for any $0<r<\rho$. □

Lemma 3.5. *Under the assumption of Theorem 1.1, for any $0<r<\rho$, we have $b\in L^\infty(-\rho,0;L^\infty(B(r)))$ and $\nabla b\in L^\infty(-\rho,0;L^q(B(r)))$ for $2<q<\infty$.*

Proof. Let $0<r<r_1<\rho$ be arbitrary chosen. Let $\varphi\in C_c^\infty(B(r_1))$ be a cut off function such that $0\leqslant\varphi\leqslant 1$ and $\varphi\equiv 1$ on $B(r)$. We rewrite (3.2) as

$$(b\varphi)_t - \Delta(b\varphi) = b\cdot\nabla u\varphi - u\cdot\nabla b\varphi - 2\nabla b\nabla\varphi - b\Delta\varphi. \tag{3.13}$$

Writing this equation under its integral form, for $t\in[-\rho,0)$, we have

$$b(t)\varphi = e^{(\rho+t)\Delta}(b(-\rho)\varphi) + \int_{-\rho}^{t} e^{(t-\tau)\Delta}\Big((b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi$$
$$-2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\Big)\mathrm{d}\tau. \tag{3.14}$$

Then we estimate

$$\begin{aligned}
&\|b(t)\varphi\|_{L^\infty}\\
&\leqslant \|b(-\rho)\varphi\|_{L^\infty} + \int_{-\rho}^{t} \|e^{(t-\tau)\Delta}\Big((b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi\\
&\quad -2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\Big)\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant \|b(-\rho)\|_{L^\infty} + c\int_{-\rho}^{t} (t-\tau)^{-\frac{2}{3}}\|(b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi\\
&\quad -2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\|_{L^{\frac{3}{2}}}\mathrm{d}\tau\\
&\leqslant c + c\int_{-\rho}^{t} (t-\tau)^{-\frac{2}{3}}(\|b(\tau)\|_{L^6(B(r_1))}\|\nabla u(\tau)\varphi\|_{L^2}\\
&\quad +\|u(\tau)\|_{L^6(B(r_1))}\|\nabla b(\tau)\varphi\|_{L^2} + \|\nabla b(\tau)\nabla\varphi\|_{L^{\frac{3}{2}}} + \|b(\tau)\Delta\varphi\|_{L^{\frac{3}{2}}})\mathrm{d}\tau.
\end{aligned}$$

Similar to inequality (3.9), we have

$$\|\nabla u\varphi\|_{L^2} \leqslant c\|\omega\varphi\|_{L^2} + c\|u\|_{L^2(B(\rho))}\|\nabla\varphi\|_{L^\infty},$$
$$\|\nabla b\varphi\|_{L^2} \leqslant c\|j\varphi\|_{L^2} + c\|b\|_{L^2(B(\rho))}\|\nabla\varphi\|_{L^\infty},$$

and

$$\|\nabla b\nabla\varphi\|_{L^{\frac{3}{2}}} \leqslant c\|j\varphi\|_{L^{\frac{3}{2}}} + c\|b\|_{L^{\frac{3}{2}}(B(\rho))}\|\nabla^2\varphi\|_{L^\infty}.$$

As a consequence, for any $0 < \sigma < \rho$, we obtain

$$\|b\|_{L^\infty(-\rho,0;L^\infty(B(\sigma)))} \leqslant c$$

by making use of Remark 3.3.

By virtue of (3.14), for $t \in [-\rho, 0)$, we have

$$\nabla(b(t)\varphi) = e^{(\rho+t)\Delta}\nabla(b(-\rho)\varphi) + \int_{-\rho}^{t} \nabla e^{(t-\tau)\Delta}\Big((b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi$$
$$-2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\Big)\mathrm{d}\tau.$$

Let $\frac{2q}{q+2} < s < 2$. Then we have

$$\begin{aligned}
&\|\nabla(b(t)\varphi)\|_{L^q}\\
\leqslant\ &\|\nabla(b(-\rho)\varphi)\|_{L^q} + \int_{-\rho}^{t} \|\nabla e^{(t-\tau)\Delta}\Big((b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi\\
&-2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\Big)\|_{L^q}\mathrm{d}\tau\\
\leqslant\ &c + c\int_{-\rho}^{t} (t-\tau)^{-\frac{1}{2}+\frac{1}{q}-\frac{1}{s}} \|(b\cdot\nabla u)(\tau)\varphi - (u\cdot\nabla b)(\tau)\varphi\\
&-2\nabla b(\tau)\nabla\varphi - b(\tau)\Delta\varphi\|_{L^s}\mathrm{d}\tau\\
\leqslant\ &c + c\int_{-\rho}^{t} (t-\tau)^{-\frac{1}{2}+\frac{1}{q}-\frac{1}{s}} (\|b(\tau)\|_{L^{\frac{2s}{2-s}}(B(r_1))}\|\nabla u(\tau)\|_{L^2(B(r_1))}\|\varphi\|_{L^\infty}\\
&+\|u(\tau)\|_{L^{\frac{2s}{2-s}}(B(r_1))}\|\nabla b(\tau)\|_{L^2(B(r_1))}\|\varphi\|_{L^\infty} + \|\nabla b(\tau)\|_{L^2(B(r_1))}\|\nabla\varphi\|_{L^{\frac{2s}{2-s}}}\\
&+\|b(\tau)\|_{L^2(B(r_1))}\|\Delta\varphi\|_{L^{\frac{2s}{2-s}}})\mathrm{d}\tau\\
\leqslant\ &c.
\end{aligned}$$

Therefore, we prove $\nabla b \in L^\infty(-\rho, 0; L^q(B(r)))$ for any $0 < r < \rho$ and $2 < q < \infty$. □

In order to obtain better estimates for ω and $\nabla^2 b$, we borrow some arguments used in [19] and then use iteration inequalities to prove the following lemma.

Lemma 3.6. *Under the assumption of Theorem 1.1, then for any $0 < r < \rho$, we have $\omega \in L^\infty(-\rho, 0; L^q(B(r)))$, $\nabla u \in L^\infty(-\rho, 0; L^q(B(r)))$, $u \in L^\infty(-\rho, 0; L^\infty(B(r)))$, $\nabla^2 b \in L^{p_1}(-\rho, 0; L^q(B(r))) \cap L^{p_2}(-\rho, 0; L^\kappa(B(r)))$ and $(p, \nabla p) \in L^\infty(-\rho, 0; L^q(B(r)))$ for any $1 < p_1 < 2$ and $2 \leqslant p_2, q < \infty$.*

Remark 3.7. Let $\gamma = \delta + \frac{2p_1q_2}{q_1q_2+2(q_2-q_1)} > 0$. Here γ can be chosen close to 0. From the (3.17), we can choose δ small enough and q_1, q_2 large enough so that γ close to 0. Here Lemma 3.6 holds just asking for $\int_{-\rho}^{0} |\omega(\tau)|^\gamma_{BMO(B(\rho))}\mathrm{d}\tau < +\infty$.

Proof of Lemma 3.6. Let $0 < r < r_1 < r_2 < \sigma < \rho$. Let $\varphi \in C_c^\infty(B(r_2))$ be a cut off function such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B(\frac{r_1+r_2}{2})$ and $|\nabla\varphi| \leqslant c(r_2-r_1)^{-1}$ and $|\nabla^2\varphi| \leqslant c(r_2-r_1)^{-2}$. By virtue of Lemma 2.5 and (3.13), we obtain

$$\begin{aligned}
\int_{-\rho}^{t} \|\Delta(b(\tau)\varphi)\|_{L^q}^{p_1}\mathrm{d}\tau &\leqslant c\|\nabla(b(-\rho)\varphi)\|_{L^q}^{p_1} + c\int_{-\rho}^{t} (\|(b\cdot\nabla u)(\tau)\varphi\|_{L^q}^{p_1} + \|(u\cdot\nabla b)(\tau)\varphi\|_{L^q}^{p_1}\\
&+\|\nabla b(\tau)\nabla\varphi\|_{L^q}^{p_1} + \|b(\tau)\Delta\varphi\|_{L^q}^{p_1})\mathrm{d}\tau. \qquad (3.15)
\end{aligned}$$

We rewrite (3.4) as follows:

$$(\omega\varphi)_t + u\cdot\nabla(\omega\varphi) = b\cdot\nabla j\varphi + u\cdot\nabla\varphi\omega. \tag{3.16}$$

Multiplying equation (3.16) by $\omega\varphi|\omega\varphi|^{q-2}$, and integrating on $B(r_1)$, we have

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\varphi\|_{L^q}^q \leqslant \|b\cdot\nabla j\varphi\|_{L^q}\|\omega\varphi\|_{L^q}^{q-1} + \|u\cdot\nabla\varphi\omega\|_{L^q}\|\omega\varphi\|_{L^q}^{q-1}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\varphi\|_{L^q}^{p_1} \leqslant c\|b\cdot\nabla j\varphi\|_{L^q}\|\omega\varphi\|_{L^q}^{p_1-1} + c\|u\cdot\nabla\varphi\omega\|_{L^q}\|\omega\varphi\|_{L^q}^{p_1-1}.$$

Integrating in time on $[-\rho, 0)$ and applying Young's inequality, we have

$$\begin{aligned}\|\omega(t)\varphi\|_{L^q}^{p_1} \leqslant{}& \|\omega(-\rho)\varphi\|_{L^q}^{p_1} + c\int_{-\rho}^{t}(\|b\|_{L^\infty(B(\sigma))}^{p_1}\|\nabla j\varphi\|_{L^q}^{p_1} + \|\omega\varphi\|_{L^q}^{p_1}\\ &+\|u\|_{L^\infty(B(\sigma))}^{p_1}\|\nabla\varphi\|_{L^\infty}^{p_1}\|\omega\|_{L^q(B(r_2))}^{p_1})\mathrm{d}\tau.\end{aligned}$$

Thanks to

$$\begin{aligned}\|\nabla u\varphi\|_{L^q} &\leqslant c\|\omega\varphi\|_{L^q} + c\|u\|_{L^q(B(\rho))}\|\nabla\varphi\|_{L^\infty},\\ \|\nabla j\varphi\|_{L^q} &\leqslant c\|\nabla^2(b\varphi)\|_{L^q} + c\|\nabla b\nabla\varphi\|_{L^q} + c\|b\nabla^2\varphi\|_{L^q}\\ &\leqslant c\|\Delta(b\varphi)\|_{L^q} + c\|\nabla b\|_{L^q}\|\nabla\varphi\|_{L^\infty} + c\|b\|_{L^q}\|\nabla^2\varphi\|_{L^\infty},\end{aligned}$$

and $\|b\|_{L^\infty(-\rho,0;L^\infty(B(\sigma)))} \leqslant c$, for some small enough positive constant δ, we have

$$\begin{aligned}&\|\omega(t)\varphi\|_{L^q}^{p_1}\\ \leqslant{}& \|\omega(-\rho)\varphi\|_{L^q}^{p_1} + c + c\int_{-\rho}^{t}(\|(b\cdot\nabla u)(\tau)\varphi\|_{L^q}^{p_1} + \|(u\cdot\nabla b)(\tau)\varphi\|_{L^q}^{p_1}\\ &+\|\nabla b(\tau)\nabla\varphi\|_{L^q}^{p_1} + \|b(\tau)\Delta\varphi\|_{L^q}^{p_1})\mathrm{d}\tau\\ &+c\int_{-\rho}^{t}(\|\omega\varphi\|_{L^q}^{p_1} + \|u\|_{L^\infty(B(\sigma))}^{p_1}\|\nabla\varphi\|_{L^\infty}^{p_1}\|\omega\|_{L^q(B(r_2))}^{p_1})\mathrm{d}\tau\\ \leqslant{}& c + c\int_{-\rho}^{t}\|\omega\|_{L^q(B(r_2))}^{p_1}(1 + \|u\|_{L^\infty(B(\sigma))}^{p_1}\|\nabla\varphi\|_{L^\infty}^{p_1})\mathrm{d}\tau\\ &+\int_{-\rho}^{t}(\|\nabla\varphi\|_{L^\infty}^{p_1} + \|\nabla^2\varphi\|_{L^\infty}^{p_1})\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\leqslant c + c(r_2 - r_1)^{-p_1} + c(r_2 - r_1)^{-2p_1} \\
&\quad + \sup_{s\in(-\rho,0)} \|\omega(s)\|^{p_1-\delta}_{L^q(B(r_2))} \int_{-\rho}^{t} \|\omega(s)\|^{\delta}_{L^q(B(r_2))} (1 + (r_2 - r_1)^{-p_1} \|u\|^{p_1}_{L^\infty(B(\sigma))}) \mathrm{d}\tau \\
&\leqslant \frac{1}{2} \sup_{s\in(-\rho,0)} \|\omega(s)\|^{p_1}_{L^q(B(r_2))} + c + c(r_2 - r_1)^{-p_1} + c(r_2 - r_1)^{-2p_1} \\
&\quad + \Big(\int_{-\rho}^{t} \|\omega(s)\|^{\delta}_{BMO(B(\rho))} (1 + (r_2 - r_1)^{-p_1} \|u\|^{p_1}_{L^\infty(B(\sigma))}) \mathrm{d}\tau \Big)^{\frac{p_1}{\delta}}
\end{aligned}
$$

by using Lemma 3.5, Lemma 2.7 and Young's inequality. Let $\sigma < \varrho < \rho$. Let $\psi \in C_c^\infty(B(\varrho))$ be a cut off function such that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $B(\sigma)$. For some large enough positive constant q_1, q_2, we have

$$
\begin{aligned}
&\|u\|_{L^\infty(B(r_2))} \leqslant \|u\psi\|_{L^\infty} \leqslant c \|u\psi\|^{\frac{q_1q_2-2q_1}{q_1q_2+2(q_2-q_1)}}_{L^{q_1}} \|\nabla(u\psi)\|^{\frac{2q_2}{q_1q_2+2(q_2-q_1)}}_{L^{q_2}} \\
&\leqslant c \|\omega\psi\|^{\frac{2q_2}{q_1q_2+2(q_2-q_1)}}_{L^{q_2}} + \|u\|^{\frac{2q_2}{q_1q_2+2(q_2-q_1)}}_{L^{q_2}(B(\varrho))} \|\nabla\psi\|^{\frac{2q_2}{q_1q_2+2(q_2-q_1)}}_{L^\infty} \\
&\leqslant c \Big(\rho^{\frac{q_1q_2+2(q_2-q_1)}{q_2}} \|\omega\|_{BMO(B(\rho))} \Big)^{\frac{2q_2}{q_1q_2+2(q_2-q_1)}} + c
\end{aligned}
$$

by virtue of Lemma 2.7. Thus,

$$
\begin{aligned}
&\|\omega(t)\varphi\|^{p_1}_{L^q} \\
&\leqslant \frac{1}{2} \sup_{s\in(-\rho,0)} \|\omega(s)\|^{p_1}_{L^q(B(r_2))} + c + c(r_2 - r_1)^{-p_1} + c(r_2 - r_1)^{-2p_1} \\
&\quad + c(1 + (r_2 - r_1)^{-\frac{p_1^2}{\delta}}) \Big(\int_{-\rho}^{t} \|\omega(s)\|^{\delta}_{BMO(B(\rho))} \mathrm{d}\tau \Big)^{\frac{p_1}{\delta}} \\
&\quad + c(r_2 - r_1)^{-\frac{p_1^2}{\delta}} \Big(\int_{-\rho}^{t} \|\omega(s)\|^{\delta + \frac{2p_1q_2}{q_1q_2+2(q_2-q_1)}}_{BMO(B(\rho))} \mathrm{d}\tau \Big)^{\frac{p_1}{\delta}}.
\end{aligned} \tag{3.17}
$$

Using Remark 3.7, we see that

$$
\begin{aligned}
&\operatorname{ess\,sup}_{s\in(-\rho,0)} \|\omega(s)\|^{p_1}_{L^q(B(r_1))} \\
&\leqslant \frac{1}{2} \operatorname{ess\,sup}_{s\in(-\rho,0)} \|\omega(s)\|^{p_1}_{L^q(B(r_2))} + c + c(r_2 - r_1)^{-\frac{p_1^2}{\delta}}.
\end{aligned}
$$

By using a standard iteration argument, we deduce that

$$\begin{aligned}&\operatorname{ess\,sup}_{s\in(-\rho,0)}\|\omega(s)\|_{L^q(B(r))}^{p_1}\\ \leqslant\ & c(\sigma-r)^{-\frac{p_1^2}{\delta}}.\end{aligned}$$

Therefore, we obtain $\|\omega(s)\|_{L^\infty(-\rho,0;L^q(B(r)))}$ for any $0<r<\rho$. Furthermore, by virtue of (3.15) and Lemma 3.5, if $1<p_1<2, 2\leqslant q<\infty$, we can prove $\|\nabla^2 b(s)\|_{L^{p_1}(-\rho,0;L^q(B(r)))}$ for any $0<r<\rho$.

By virtue of Lemma 2.5 and (3.13), we also obtain

$$\begin{aligned}&\int_{-\rho}^{t}\|\Delta(b(\tau)\varphi)\|_{L^\kappa}^{p_2}\mathrm{d}\tau\\ \leqslant\ & c\|\Delta(b(-\rho)\varphi)\|_{L^\kappa}^{p_2}+c\int_{-\rho}^{t}(\|(b\cdot\nabla u)(\tau)\varphi\|_{L^\kappa}^{p_2}+\|(u\cdot\nabla b)(\tau)\varphi\|_{L^\kappa}^{p_2}\\ &+\|\nabla b(\tau)\nabla\varphi\|_{L^\kappa}^{p_2}+\|b(\tau)\Delta\varphi\|_{L^\kappa}^{p_2})\mathrm{d}\tau\\ \leqslant\ & c.\end{aligned}$$

As a consequence, we have $\|\nabla^2 b(s)\|_{L^{p_2}(-\rho,0;L^\kappa(B(r)))}$ for any $0<r<\rho$ and for any $2<p_2<\infty$. Now we estimate the pressure p. Applying $\nabla\cdot$ to $(1.1)_1$, we have

$$\Delta p=\nabla\cdot(b\cdot\nabla b-u\cdot\nabla u). \tag{3.18}$$

Multiplying (3.18) by $p\psi^2$, integrating in x and applying integration by parts, we obtain

$$\int|\nabla p|^2\psi^2\mathrm{d}x+2\int\nabla p\cdot\nabla\psi p\psi\mathrm{d}x=\int(b\cdot\nabla b-u\cdot\nabla u)\cdot(\nabla p\psi+p\nabla\psi)\mathrm{d}x.$$

Then,

$$\|\nabla p\psi\|_{L^2}^2\leqslant\frac{1}{2}\|\nabla p\psi\|_{L^2}^2+c\|p\nabla\psi\|_{L^2}^2+c\|b\cdot\nabla b-u\cdot\nabla u\|_{L^2(B(r_3))}^2.$$

Therefore, for any $0<r<\rho$, $\|\nabla p\|_{L^\infty(-\rho,0;(L^2(B(r))))}\leqslant c$. By interpolation inequality, we have $\|p\|_{L^\infty(-\rho,0;(L^q(B(r))))}\leqslant c$.

Multiplying (3.18) by ψ, it yields

$$\Delta(p\psi)=2\nabla\cdot(p\nabla\psi)-p\Delta\psi+\nabla\cdot\big((b\cdot\nabla b-u\cdot\nabla u)\psi\big)-(b\cdot\nabla b-u\cdot\nabla u)\cdot\nabla\psi.$$

Using Calderón-Zygmund's inequality [29] and the following Hardy-Littlewood-Sobolev inequality

$$\|(-\Delta)^{-s/2}f\|_{L^q(\mathbb{R}^n)}\lesssim\|f\|_{L^p(\mathbb{R}^n)},\quad\text{where }\frac{1}{p}-\frac{1}{q}=\frac{s}{n},\quad 0<s<n$$

for any $2<q_3<\infty$, we have

$$
\begin{aligned}
&\|\nabla p\psi\|_{L^{q_3}}\\
&\leqslant c\|p\nabla\psi\|_{L^{q_3}} + c\|p\Delta\psi\|_{L^{\frac{2q_3}{q_3+2}}} + c\|(b\cdot\nabla b - u\cdot\nabla u)\psi\|_{L^{q_3}}\\
&\quad + c\|(b\cdot\nabla b - u\cdot\nabla u)\nabla\psi\|_{L^{\frac{2q_3}{q_3+2}}}\\
&\leqslant c.
\end{aligned}
$$

This completes the proof of the lemma. □

Using the similar argument in [36], we have

Lemma 3.8. *Under the assumption of Theorem 1.1, for any $0 < r < \rho$, we have $\Delta b \in L^\infty(-\rho, 0; L^\kappa(B(r)))$ and $\nabla(\Delta b + b\cdot\nabla u) \in L^p(-\rho, 0; L^\kappa(B(r)))$ for $1 < p < 2$.*

Proof. Applying $b\cdot\nabla$ and Δ to $(1.1)_1$ and $(1.1)_2$ respectively, and multiplying $(1.1)_2$ by ∇u, then adding the resulting equations together, it yields

$$
\begin{aligned}
&(\Delta b + b\cdot\nabla u)_t - \Delta(\Delta b + b\cdot\nabla u)\\
&= -b\cdot\nabla(u\cdot\nabla u) + b\cdot\nabla(b\cdot\nabla b) - (u\cdot\nabla b)\cdot\nabla u + (b\cdot\nabla u)\cdot\nabla u\\
&\quad + \Delta b\cdot\nabla u - b\cdot\nabla(\nabla p) - \Delta(u\cdot\nabla b).
\end{aligned}
\tag{3.19}
$$

Let $0 < r < r_1 < \rho$. Let $\varphi \in C_c^\infty(B(r_1))$ be a cut off function such that $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on $B(r)$. Multiplying (3.19) by φ, we obtain

$$
\begin{aligned}
&\big((\Delta b + b\cdot\nabla u)\varphi\big)_t - \Delta\big((\Delta b + b\cdot\nabla u)\varphi\big)\\
&= -2\nabla\cdot\big((\Delta b + b\cdot\nabla u)\nabla\varphi\big) + (\Delta b + b\cdot\nabla u)\Delta\varphi - b\cdot\nabla\big((u\cdot\nabla u)\varphi\big) + (b\cdot\nabla\varphi)(u\cdot\nabla u)\\
&\quad + b\cdot\nabla(b\cdot\nabla b)\varphi - (u\cdot\nabla b)\cdot\nabla u\varphi + (b\cdot\nabla u)\cdot\nabla u\varphi\\
&\quad + \Delta b\cdot\nabla u\varphi - b\cdot\nabla(\nabla p\varphi) + b\cdot\nabla\varphi(\nabla p) - \nabla\cdot\big(\nabla(u\cdot\nabla b)\varphi\big) + \nabla(u\cdot\nabla b)\nabla\varphi.
\end{aligned}
$$

We write this equation in its integral form, for $t \in [-\rho, 0)$ we have

$$
\begin{aligned}
&(\Delta b + b\cdot\nabla u)(t)\varphi\\
&= e^{(\rho+t)\Delta}((\Delta b + b\cdot\nabla u)(-\rho)\varphi) + \int_{-\rho}^{t} e^{(t-\tau)\Delta}\Big(-2\nabla\cdot\big((\Delta b + b\cdot\nabla u)(\tau)\nabla\varphi\big)\\
&\quad + (\Delta b + b\cdot\nabla u)(\tau)\Delta\varphi - b(\tau)\cdot\nabla\big((u\cdot\nabla u)(\tau)\varphi\big) + (b\cdot\nabla\varphi)(\tau)(u\cdot\nabla u)(\tau)\\
&\quad + b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau)\varphi - (u\cdot\nabla b)(\tau)\cdot\nabla u(\tau)\varphi + (b\cdot\nabla u)(\tau)\cdot\nabla u(\tau)\varphi\\
&\quad + \Delta b(\tau)\cdot\nabla u(\tau)\varphi - b(\tau)\cdot\nabla(\nabla p(\tau)\varphi) + b(\tau)\cdot\nabla\varphi(\nabla p(\tau))\\
&\quad - \nabla\cdot\big(\nabla(u\cdot\nabla b)(\tau)\varphi\big) + \nabla(u\cdot\nabla b)(\tau)\nabla\varphi\Big)\mathrm{d}\tau.
\end{aligned}
$$

We estimate

$$
\begin{aligned}
&\|(\Delta b + b\cdot\nabla u)(t)\varphi\|_{L^\kappa}\\
\leqslant\ &\|(\Delta b + b\cdot\nabla u)(-\rho)\varphi\|_{L^\kappa} + \int_{-\rho}^{t}(t-\tau)^{-\frac{1}{2}}\Big(\|(\Delta b + b\cdot\nabla u)(\tau)\nabla\varphi\|_{L^\kappa}\\
&+\|b(\tau)(u\cdot\nabla u)(\tau)\varphi\|_{L^\kappa} + \|\nabla(u\cdot\nabla b)(\tau)\varphi\|_{L^\kappa} + \|b(\tau)\nabla p(\tau)\varphi\|_{L^\kappa}\Big)\mathrm{d}\tau\\
&+\int_{-\rho}^{t}\|(\Delta b + b\cdot\nabla u)(\tau)\Delta\varphi + (b\cdot\nabla\varphi)(\tau)(u\cdot\nabla u)(\tau) + b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau)\varphi\\
&-(u\cdot\nabla b)(\tau)\cdot\nabla u(\tau)\varphi + (b\cdot\nabla u)(\tau)\cdot\nabla u(\tau)\varphi + \Delta b(\tau)\cdot\nabla u(\tau)\varphi\\
&+b(\tau)\cdot\nabla\varphi(\nabla p(\tau)) + \nabla(u\cdot\nabla b)(\tau)\nabla\varphi\|_{L^\kappa}\mathrm{d}\tau\\
\leqslant\ &c.
\end{aligned}
$$

As a consequence, $\Delta b + b\cdot\nabla u \in L^\infty(-\rho,0;L^\kappa(B(r)))$. By Lemma 3.6, we have $\Delta b \in L^\infty(-\rho,0;L^\kappa(B(r)))$.

Furthermore, we estimate

$$
\begin{aligned}
&\|\nabla((\Delta b + b\cdot\nabla u)\varphi)\|_{L^p(-\rho,0;L^\kappa)}\\
\leqslant\ &c\|(\Delta b + b\cdot\nabla u)(-\rho)\varphi\|_{L^\kappa} + \|(\Delta b + b\cdot\nabla u)\nabla\varphi\|_{L^p(-\rho,0;L^\kappa)}\\
&+\|b(u\cdot\nabla u)\varphi\|_{L^p(-\rho,0;L^\kappa)} + \|\nabla(u\cdot\nabla b)\varphi\|_{L^p(-\rho,0;L^\kappa)} + \|b\nabla p\varphi\|_{L^p(-\rho,0;L^\kappa)}\\
&+\|(\Delta b + b\cdot\nabla u)\Delta\varphi + (b\cdot\nabla\varphi)(u\cdot\nabla u) + b\cdot\nabla(b\cdot\nabla b)\varphi - (u\cdot\nabla b)\cdot\nabla u\varphi\\
&+(b\cdot\nabla u)\cdot\nabla u\varphi + \Delta b\cdot\nabla u\varphi + b\cdot\nabla\varphi(\nabla p) + \nabla(u\cdot\nabla b)\nabla\varphi\|_{L^p(-\rho,0;L^\kappa)}\\
\leqslant\ &c,
\end{aligned}
$$

where we use Lemma 2.5.

This completes the proof of this lemma. □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. In order to prove (1.5), we need to estimate high order local estimates of (u,b) or (ω,j). Here we can use the special structure of 2D resistive MHD equations to deal only with the vorticity equation. Due to the localization of (1.1), it seems difficult to estimate directly the convection term $u\cdot\nabla\omega$. We borrow the arguments [12] to transform equations (1.1) into new equations to avoid this difficulty and use a iteration scheme of the Gronwall type to prove the conclusion.

Proof of Theorem 1.1. Let $0<r<\rho$ be arbitrarily chosen. Let us choose $-\rho<t'<0$ such that

$$
\frac{1}{2}\rho^2(-t')^{-\frac{1}{2}} - 4c_0 > 0. \tag{4.1}
$$

Let $t' < t_0 < 0$. Consider the transformation of $(u, p) \mapsto (v, P)$ defined by $v(y,t) = u((1+(-t)^{\frac{1}{2}})y, t) = u(x,t)$ or $v(y,t) = v(\frac{x}{1+(-t)^{\frac{1}{2}}}, t) = u(x,t)$ and $P(y,t) = \frac{1}{1+(-t)^{\frac{1}{2}}} p((1+(-t)^{\frac{1}{2}})y, t)$. Let $\varrho := \frac{\rho}{(1+(-t_0)^{\frac{1}{2}})} > \frac{\rho}{2}$. Using Lemma 3.6, for $t \in [t_0, 0)$, we have

$$\|v(t)\cdot y\|_{L^\infty(B(\varrho))} \leqslant c_0. \tag{4.2}$$

Here we can choose $|t_0|$ small such that $\varrho > r > \frac{\rho}{2}$ is arbitrarily chosen.

If $(y,t) \in \Big(\overline{B(\varrho)} \setminus B(\rho/2)\Big) \times (t_0, 0)$, we have

$$\begin{aligned}
&\frac{1}{2}(-t)^{-\frac{1}{2}}|y|^2 + v(y,t)\cdot y \geqslant \frac{1}{2}(-t)^{-\frac{1}{2}}\frac{\rho^2}{4} + v(y,t)\cdot y\\
&\geqslant \frac{1}{4}\Big(\frac{1}{2}\rho^2(-t)^{-\frac{1}{2}} - 4c_0\Big) \geqslant \frac{1}{4}\Big(\frac{1}{2}\rho^2(-t_*)^{-\frac{1}{2}} - 4c_0\Big) > 0,
\end{aligned}$$

by using (4.1) and (4.2). As a consequence, for $(y,t) \in \Big(\overline{B(\varrho)} \setminus B(\rho/2)\Big) \times (t_0, 0)$, we see that

$$\frac{\frac{1}{2}(-t)^{-1/2}y + v(y,t)}{1+(-t)^{1/2}} \cdot y > 0. \tag{4.3}$$

Using the chain rule and noting that $x = (1+(-t)^{\frac{1}{2}})y$, equation $(1.1)_1$ turns into the following equation

$$\begin{aligned}
&\partial_t v(y,t) + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}} y\cdot\nabla_y v(y,t) + \frac{1}{1+(-t)^{\frac{1}{2}}} v(y,t)\cdot\nabla_y v(y,t) + \nabla_y P(y,t)\\
&= b(x,t)\cdot\nabla_x b(x,t),
\end{aligned}$$

which hold for $(y,t) \in B(\varrho)\times(t_0,0)$. Denote $\Omega(y,t) = \nabla_y^{\perp} v(y,t)$. Thanks to $x = (1+(-t)^{\frac{1}{2}})y$, we have

$$\begin{aligned}
&\partial_t \Omega(y,t) + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\Omega(y,t) + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}} y\cdot\nabla_y\Omega(y,t)\\
&+\frac{1}{1+(-t)^{\frac{1}{2}}} v(y,t)\cdot\nabla_y\Omega(y,t)\\
&= (1+(-t)^{\frac{1}{2}})b(x,t)\cdot\nabla_x j(x,t))\\
&= (1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)\Delta_x b_2(x,t) - b_2(x,t)\Delta_x b_1(x,t)\Big)\\
&= (1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)\Big(\Delta_x b_2(x,t) + b(x,t)\cdot\nabla_x u_2(x,t)\Big)\Big)\\
&\quad -(1+(-t)^{\frac{1}{2}})\Big(b_2(x,t)\Big(\Delta_x b_1(x,t) + b(x,t)\cdot\nabla_x u_1(x,t)\Big)\Big)\\
&\quad -(1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)b(x,t)\cdot\nabla_x u_2(x,t) - b_2(x,t)b(x,t)\cdot\nabla_x u_1(x,t)\Big).
\end{aligned} \tag{4.4}$$

Applying the operator ∂_{y_i} to (4.4), it yields

$$\begin{aligned}
&\partial_t\partial_{y_i}\Omega(y,t)+\frac{(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\partial_{y_i}\Omega(y,t)+\frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}y\cdot\nabla_y\partial_{y_i}\Omega(y,t)\\
&+\frac{1}{1+(-t)^{\frac{1}{2}}}v(y,t)\cdot\nabla_y\partial_{y_i}\Omega(y,t)\\
=&-\frac{1}{1+(-t)^{\frac{1}{2}}}\partial_{y_i}v(y,t)\cdot\nabla_y\Omega(y,t)\\
&+(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_1(x,t)\Big(\Delta_x b_2(x,t)+b(x,t)\cdot\nabla_x u_2(x,t)\Big)\Big)\\
&-(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_2(x,t)\Big(\Delta_x b_1(x,t)+b(x,t)\cdot\nabla_x u_1(x,t)\Big)\Big)\\
&-(1+(-t)^{\frac{1}{2}})^2\Big(\partial_{x_i}\big(b_1(x,t)b(x,t)\big)\cdot\nabla_x u_2(x,t)+b_1(x,t)b(x,t)\cdot\nabla_x\partial_{x_i}u_2(x,t)\Big)\\
&+(1+(-t)^{\frac{1}{2}})^2\Big(\partial_{x_i}\big(b_2(x,t)b(x,t)\big)\cdot\nabla_x u_1(x,t)+b_2(x,t)b(x,t)\cdot\nabla_x\partial_{x_i}u_1(x,t)\Big).
\end{aligned}\tag{4.5}$$

Let

$$\begin{aligned}
f_{1i}(x,t)&=(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_1(x,t)\Big(\Delta_x b_2(x,t)+b(x,t)\cdot\nabla_x u_2(x,t)\Big)\Big),\\
f_{2i}(x,t)&=-(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_2(x,t)\Big(\Delta_x b_1(x,t)+b(x,t)\cdot\nabla_x u_1(x,t)\Big)\Big),\\
f_{3i}(x,t)&=-(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_1(x,t)b(x,t)\Big)\cdot\nabla_x u_2(x,t),\\
f_{4i}(x,t)&=(1+(-t)^{\frac{1}{2}})^2\partial_{x_i}\Big(b_2(x,t)b(x,t)\Big)\cdot\nabla_x u_1(x,t),\\
g_1(x,t)&=-b_1(x,t)b(x,t),\ g_2(x,t)=b_2(x,t)b(x,t),
\end{aligned}$$

and $F_{ji}(y,t)=f_{ji}((1+(-t)^{\frac{1}{2}})y,t)$, $G_l(y,t)=g_l((1+(-t)^{\frac{1}{2}})y,t)$. Then (4.5) turns into the following equation

$$\begin{aligned}
&\partial_t\partial_{y_i}\Omega(y,t)+\frac{(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\partial_{y_i}\Omega(y,t)+\frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}y\cdot\nabla_y\partial_{y_i}\Omega(y,t)\\
&+\frac{1}{1+(-t)^{\frac{1}{2}}}v(y,t)\cdot\nabla_y\partial_{y_i}\Omega(y,t)\\
=&-\frac{1}{1+(-t)^{\frac{1}{2}}}\partial_{y_i}v(y,t)\cdot\nabla_y\Omega(y,t)+\sum_{j=1}^{4}F_{ji}(y,t)+G_1(y,t)\cdot\nabla_y\partial_{y_i}v_2(y,t)\\
&+G_2(y,t)\cdot\nabla_y\partial_{y_i}v_1(y,t).
\end{aligned}\tag{4.6}$$

Set $\sigma = \frac{r+\varrho}{2}$, and define

$$r_m := r + (\sigma - r)(1 - \theta^m), \quad m \in \mathbb{N} \cup \{0\},$$

for some $0 < \theta < 1$. It is easy to see that $r_0 = r$,

$$r_{m+1} - r_m = (\sigma - r)\theta^m(1 - \theta),$$

and $r_m \nearrow \sigma$.

Let $\zeta_m \in C^\infty([0, +\infty))$ denote a cut off function such that $0 \leqslant \zeta_m \leqslant 1$ on $[0, +\infty)$, $\zeta_m \equiv 1$ on $[0, r_m]$, $\zeta_m \equiv 0$ in $(r_{m+1}, +\infty)$, and $0 \leqslant -\zeta_m' \leqslant \frac{2}{r_{m+1}-r_m} = \frac{2}{(\sigma-r)(1-\theta)}(\frac{1}{\theta})^m$. For convenience, we omit subscript y. Let $\varphi_m(y) = \zeta_m(|y|)$. Multiplying both side of (4.6) by $\partial_i\Omega|\nabla\Omega|^{\kappa-2}\varphi_m^\kappa$ and integrating on $B(r_{m+1})$, then summing over $i = 1, 2$, it yields

$$\begin{aligned}
&\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Omega(t)\varphi_m\|_{L^\kappa}^\kappa + (1 - \frac{1}{\kappa})\frac{(-t)^{-\frac{1}{2}}}{1 + (-t)^{\frac{1}{2}}}\int\limits_{B(r_{m+1})} |\nabla\Omega(t)|^\kappa\varphi_m^\kappa \mathrm{d}y \\
&- \int\limits_{B(r_{m+1})\setminus B(r_m)} \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}y + v(t)}{1 + (-t)^{\frac{1}{2}}} \cdot y\frac{1}{|y|}|\nabla\Omega(t)|^\kappa\varphi_m^{\kappa-1}\zeta_m'(|y|)\mathrm{d}y \\
&= \int\limits_{B(r_{m+1})} \sum_{i=1}^{2}\Big(- \frac{1}{1 + (-t)^{\frac{1}{2}}}\partial_i v(t) \cdot \nabla\Omega(t) + \sum_{j=1}^{4} F_{ji}(t) + G_1(t) \cdot \nabla\partial_i v_2(t) \\
&+ G_2(t) \cdot \nabla\partial_i v_1(t)\Big)\partial_i\Omega(t)|\nabla\Omega(t)|^{\kappa-2}\varphi_m^\kappa \mathrm{d}y \\
&\leqslant c\|\nabla v\|_{L^\infty(B(r_{m+1}))}\|\nabla\Omega\varphi_m\|_{L^\kappa}^\kappa + c\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}\varphi_m\|_{L^\kappa}\|\nabla\Omega\varphi_m\|_{L^\kappa}^{\kappa-1} \\
&+ c\|\nabla^2 v\varphi_m\|_{L^\kappa}\|\nabla\Omega\varphi_m\|_{L^\kappa}^{\kappa-1}
\end{aligned} \tag{4.7}$$

by applying the integration by parts and using Lemma 3.5. Noting that $B(r_{m+1})\setminus B(r_m) \subset B(\varrho)\setminus B(\frac{\varrho}{2})$ and $\zeta_m' \leqslant 0$, the third term on the left side of (4.7) is nonnegative by using (4.3). As a consequence, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Omega(t)\varphi_m\|_{L^\kappa} &\leqslant c\|\nabla v\|_{L^\infty(B(r_{m+1}))}\|\nabla\Omega\varphi_m\|_{L^\kappa} + c\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}\varphi_m\|_{L^\kappa} \\
&+ c\|\nabla^2 v\varphi_m\|_{L^\kappa}.
\end{aligned} \tag{4.8}$$

Integrating on (t_0, t) for $t_0 < t < 0$, we find that

$$\|\nabla\Omega(t)\varphi_m\|_{L^\kappa} \leqslant \|\nabla\Omega(t_0)\varphi_m\|_{L^\kappa} + c\int\limits_{t_0}^{t}\|\nabla v(\tau)\|_{L^\infty(B(r_{m+1}))}\|\nabla\Omega(\tau)\varphi_m\|_{L^\kappa}\mathrm{d}\tau$$

$$+c\int_{t_0}^{t}\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}(\tau)\varphi_m\|_{L^\kappa}\mathrm{d}\tau+c\int_{t_0}^{t}\|\nabla^2 v(\tau)\varphi_m\|_{L^\kappa}\mathrm{d}\tau.$$

Similar to inequality (3.9), we have

$$\|\nabla^2 v(t)\varphi_m\|_{L^\kappa}\leqslant c\|\nabla\Omega(t)\varphi_m\|_{L^\kappa}+\|\nabla v(t)\|_{L^\kappa(B(r_{m+1}))}\,\|\nabla\varphi_m\|_{L^\infty}.$$

Combining the two inequalities above, we see that

$$\begin{aligned}
&\|\nabla^2 v(t)\varphi_m\|_{L^\kappa}\\
&\leqslant c\|\nabla\Omega(t_0)\varphi_m\|_{L^\kappa}+c+\frac{c}{(\sigma-r)(1-\theta)}(\frac{1}{\theta})^m\\
&\quad+c\int_{t_0}^{t}(\|\nabla v(\tau)\|_{L^\infty(B(r_{m+1}))}+1)\|\nabla^2 v(\tau)\varphi_m\|_{L^\kappa}\mathrm{d}\tau\\
&\leqslant(\frac{c}{\theta})^m+c\int_{t_0}^{t}(\|\nabla v(\tau)\|_{L^\infty(B(r_{m+1}))}+1)\|\nabla^2 v(\tau)\varphi_m\|_{L^\kappa}\mathrm{d}\tau
\end{aligned}\tag{4.9}$$

by using Lemma 3.6 and Lemma 3.8. Here we have used the following bound

$$\int_{t_0}^{t}\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}(\tau)\varphi_m\|_{L^\kappa}\,\mathrm{d}\tau\leq c,$$

where $1\lesssim 1+(-t)^{1/2}\lesssim c(\rho)$ and $c(\rho)$ depends only on ρ. This bound can be obtained by a change of variables and applying Lemmas 3.6 and 3.8. For instance,

$$\int_{t_0}^{t}\|F_{1i}(\tau)\varphi_m\|_{L^\kappa}\,\mathrm{d}\tau\lesssim\int_{-\rho}^{0}\|\partial_{x_i}\left(b_1(x,t)(\Delta_x b_2(x,t)+b(x,t)\cdot\nabla_x u_2(x,t))\right)\|_{L^\kappa_x(B(r))}\,\mathrm{d}t$$

for some $0<r<\rho$. Lemma 3.8 ensures that $\nabla(\Delta b+b\cdot\nabla u)\in L^p(-\rho,0;L^\kappa(B(r)))$ for $1<p<2$, from which the desired estimate follows. By virtue of Lemma 2.6, Lemma 2.8 and Lemma 3.6, we find that

$$\begin{aligned}
&\|\nabla v(\tau)\|_{L^\infty(B(r_{m+1}))}\\
&\leqslant c(1+\|\nabla v(\tau)\|_{BMO(B(\sigma))})\log(e+\|\nabla^2 v(\tau)\|_{L^\kappa(B(r_{m+1}))}+\|\nabla v(\tau)\|_{L^2(B(r_{m+1}))})\\
&\leqslant c(1+\|\nabla v(\tau)\|_{L^1(B(\sigma))}+|\nabla v(\tau)|_{BMO(B(\sigma))})\log(e+\|\nabla^2 v(\tau)\varphi_{m+1}\|_{L^\kappa}).
\end{aligned}\tag{4.10}$$

Now we estimate $|\nabla v(\tau)|_{BMO(B(\sigma))}$. Let $\eta\in C^\infty([0,+\infty))$ be a cut off function such that $0\leqslant\eta\leqslant 1$ in $[0,+\infty)$, $\eta\equiv 1$ on $[0,\frac{(2k+1)\sigma}{2k}]$, $\eta\equiv 0$ in $(\frac{(k+1)\sigma}{k},+\infty)$, and $|\eta'|\leqslant\frac{c(k)}{\sigma}$. Let $\psi(y)=\eta(|y|)$. By means of Jensen's inequality we get

$$
\begin{aligned}
&\sup_{\substack{z\in B(\sigma)\\ 0<s\leqslant 2\sigma B(\sigma)\cap B(z,s)}} \fint |\nabla v(\tau)-(\nabla v(\tau))_{B(\sigma)\cap B(z,\sigma)}|\mathrm{d}y\\
&\leqslant \sup_{\substack{z\in B(\sigma)\\ 0<s\leqslant 2\sigma B(\sigma)\cap B(z,s)}} \fint_{B(\sigma)\cap B(z,s)} \fint_{B(\sigma)\cap B(z,s)} |\nabla v(y,\tau)-\nabla v(y',\tau)|\mathrm{d}y\mathrm{d}y'\\
&\leqslant \sup_{\substack{z\in B(\sigma)\\ s\geqslant \frac{1}{2k}\sigma B(\sigma)\cap B(z,s)}} \fint_{B(\sigma)\cap B(z,s)} \fint_{B(\sigma)\cap B(z,s)} |\nabla v(y,\tau)-\nabla v(y',\tau)|\mathrm{d}y\mathrm{d}y'\\
&+ \sup_{\substack{z\in B(\sigma)\\ 0<s\leqslant \frac{1}{2k}\sigma B(\sigma)\cap B(z,s)}} \fint_{B(\sigma)\cap B(z,s)} \fint_{B(\sigma)\cap B(z,s)} |\nabla v(y,\tau)-\nabla v(y',\tau)|\mathrm{d}y\mathrm{d}y'\\
&\leqslant c(k)\|\nabla v(\tau)\|_{L^1(B(\sigma))}+c\sup_{\substack{z\in B(\sigma)\\ 0<s\leqslant \frac{1}{2k}\sigma B(z,s)}} \fint |(\nabla v(\tau))\psi-(\nabla v(\tau)\psi)_{B(z,s)}|\mathrm{d}y.
\end{aligned}
$$

Choosing k large enough so that $(\frac{k+1}{k})^2\sigma<\varrho$, it yields

$$
\begin{aligned}
&|\nabla v(\tau)|_{BMO(B(\sigma))}\\
&\leqslant c(k)\|\nabla v(\tau)\|_{L^1(B(\sigma))}+c\|\nabla v(\tau)\psi\|_{BMO(\mathbb{R}^2)}\\
&\leqslant c(k)\|\nabla v(\tau)\|_{L^1(B(\sigma))}+c(k)\Big(1+\sigma\|\nabla\psi\|_{L^\infty}\Big)\Big((|\Omega(\tau)|_{BMO(B((\frac{k+1}{k})^2\sigma))}\\
&\quad+\sigma^{-1}\|\nabla\times v(\tau)\|_{L^2(B((\frac{k+1}{k})^2\sigma))}+\sigma^{-2}\|v(\tau)\|_{L^2(B((\frac{k+1}{k})^2\sigma))}\big)\Big)\\
&\quad+c\|\nabla\psi\|_{L^\infty}\|v(\tau)\|_{L^\infty(B((\frac{k+1}{k})^2\sigma)))}\\
&\leqslant c(k)\|\nabla v(\tau)\|_{L^1(B(\varrho))}+c(k)\Big(1+\sigma\|\nabla\psi\|_{L^\infty}\Big)\Big((|\Omega(\tau)|_{BMO(B((\varrho))}\\
&\quad+\sigma^{-1}\|\nabla\times v(\tau)\|_{L^2(B((\frac{k+1}{k})^2\sigma))}+\sigma^{-2}\|v(\tau)\|_{L^2(B((\varrho)))}\big)\Big)\\
&\quad+c\|\nabla\psi\|_{L^\infty}\|v(\tau)\|_{L^\infty(B((\frac{k+1}{k})^2\sigma)))}\\
&\leqslant c(k)\|\nabla u(\tau)\|_{L^1(B(\rho))}+c(k)\Big((|\omega(\tau)|_{BMO(B((\rho))}\\
&\quad+\sigma^{-1}\|\nabla\times u(\tau)\|_{L^2(B((1+(-t)^{\frac12})(\frac{k+1}{k})^2\sigma))}+\sigma^{-2}\|u(\tau)\|_{L^2(B((\rho)))}\big)\Big)\\
&\quad+c(k)\sigma^{-1}\|u(\tau)\|_{L^\infty(B(((1+(-t)^{\frac12})(\frac{k+1}{k})^2\sigma)))}\\
&\leqslant c(|\omega(\tau)|_{BMO(B(\rho))}+1),
\end{aligned}
\tag{4.11}
$$

where we use Corollary 2.12, Lemma 2.8, Lemma 2.9, Remark 3.3 and Lemma 3.6. Combining (4.10) and (4.11), we have

$$
\|\nabla v(\tau)\|_{L^\infty(B(r_{m+1}))}\leq c\Big\{1+|\omega(\tau)|_{BMO(B(\rho))}\Big\}\log(e+\|\nabla^2 v(\tau)\varphi_{m+1}\|_{L^\kappa}).
$$

This together with (4.9) yields

$$\begin{aligned}
&e+\|\nabla^2 v(t)\varphi_m\|_{L^\kappa}\\
&\leqslant (\frac{c}{\theta})^m + c\int_{t_0}^{t}(1+|\omega(\tau)|_{BMO(B(\rho))})(e+\|\nabla^2 v(t)\varphi_m\|_{L^\kappa})\log(e+\|\nabla^2 v(\tau)\varphi_{m+1}\|_{L^\kappa})\mathrm{d}\tau\\
&= X_m(t). \qquad (4.12)
\end{aligned}$$

As a consequence, we have the following differential inequality

$$X_m'(t)\leqslant c(1+|\omega(t)|_{BMO(B(\rho))})X_m(t)\log(X_{m+1}(t)).$$

Let $Y_m(t)=\log(X_m(t))$. Then

$$Y_m'(t)\leqslant c(1+|\omega(t)|_{BMO(B(\rho))})Y_{m+1}(t).$$

Integrating the both sides over (t_0,t) with $t_0\leq t<0$, we arrive at

$$Y_m(t)\leq m\log\frac{c}{\theta}+\int_{t_0}^{t}c(1+|\omega(t)|_{BMO(B(\rho))})Y_{m+1}(\tau)\mathrm{d}\tau.$$

We now verify that the sequence $Y_m(t)$ satisfy the condition (2.2) in Lemma 2.13. In fact, from (4.12), we have

$$\begin{aligned}
&Y_m(t)=\log(X_m(t))\\
&\quad\leq m\log\frac{c}{\theta}+\log\Big(\int_{t_0}^{t_1}c(1+|\omega(\tau)|_{BMO(B(\rho))})\mathrm{d}\tau\Big)\\
&\quad\times\sup_{t_0<s<t}\log\Big\{(e+\|\nabla^2 v(t)\varphi_m\|_{L^\kappa})\log(e+\|\nabla^2 v(\tau)\varphi_{m+1}\|_{L^\kappa})\Big\}\Big)+1\\
&\quad\leq m\log\frac{c}{\theta}\\
&\qquad+c\log\Big(\int_{t_0}^{t_1}(1+|\omega(\tau)|_{BMO(B(\rho))})\mathrm{d}\tau\Big)\log\Big\{\sup_{t_0<s<t}\|\nabla^2 v(s)\|_{L^\kappa(B(\varrho))}+e\Big\}\Big)+1\\
&\quad\leq M(t)^m<+\infty,
\end{aligned}$$

for all $m\in\mathbb{N}$ and $t\in[t_0,t)$.

Using Lemma 2.13, one has

$$\begin{aligned}
&\log(e+\sup_{\tau\in(-\rho,t)}\|\nabla^2 v(\tau)\|_{L^\kappa(B(\varrho))})\\
&\leqslant Y_0(t)\\
&\leqslant \log\frac{c}{\theta}\int_{t_0}^{t} c(1+|\omega(\tau)|_{BMO(B(\rho))})\mathrm{d}\tau e^{\int_{t_0}^{t}(1+|\omega(\tau)|_{BMO(B(\rho))})\mathrm{d}\tau}.
\end{aligned}$$

This yields

$$\sup_{t\in(-\rho,0)}\|\nabla^2 u(t)\|_{L^\kappa(B(r))}\leq\sup_{t\in(-\rho,0)}\|\nabla^2 v(t)\|_{L^k(B(\varrho))}<+\infty.$$

This together with Lemma 3.8 completes the proof of Theorem 1.1. □

5. Proof of Theorem 1.2

In this section, we prove localized version of non blow up criterion in [2], namely Theorem 1.2. Similar to the proof of Theorem 1.1, in order to prove (1.8), we need to estimate high order local estimates of u or ω, namely $\|\omega(t)\psi\|_{L^\infty}$ and $\|\nabla\omega(t)\psi\|_{L^\kappa}$. Consequently, we obtain new iteration inequalities to get the conclusion where the iterated integration balls are depend on time.

Proposition 5.1. *Under the assumption of Theorem 1.2, it still holds for Lemma 3.1, Lemma 3.2, Remark 3.3, Lemma 3.5, Lemma 3.6 and Lemma 3.8.*

Proof. Noting that $|\omega(\tau)|_{BMO(B(\rho))}\leqslant c\|\nabla u(\tau)\|_{L^\infty(B(\rho))}$. According to Remark 3.4 and Remark 3.7, we need only to re-estimate I_1 and I_4 in Lemma 3.2.

Similar to (3.8), for some large positive constant q, we have

$$\begin{aligned}
&\|u\|_{L^\infty(B(r_2))}\leqslant\|u\psi^{\frac{q}{2}}\|_{L^\infty}\leqslant c\|u\psi^{\frac{q}{2}}\|_{L^4}^{\frac{2q-4}{3q-4}}\|\nabla(u\psi^{\frac{q}{2}})\|_{L^q}^{\frac{q}{3q-4}}\\
&\leqslant c(\|\nabla u\psi^{\frac{q}{2}})\|_{L^q}^{\frac{q}{3q-4}}+\|u\psi^{\frac{q}{2}-1}\|_{L^q}^{\frac{q}{3q-4}}\|\nabla\psi\|_{L^\infty}^{\frac{q}{3q-4}})\\
&\leqslant c(\|\nabla u\|_{L^\infty(B(\rho))}^{\frac{q}{3q-4}}+\|u\psi^{\frac{q}{2}}\|_{L^\infty}^{\frac{q-2}{3q-4}}),
\end{aligned}$$

where we use (3.10). Applying Young's inequality, we obtain

$$\|u\|_{L^\infty(B(r_2))}\leqslant\|u\psi^{\frac{q}{2}}\|_{L^\infty}\leqslant c\|\nabla u\|_{L^\infty(B(\rho))}^{\frac{q}{3q-4}}+c.$$

Therefore, we have

$$|I_1| \leqslant \frac{1}{4} \sup_{s\in(-\rho,0)} \|\omega(s)\|^2_{L^2(B(r_2))} + c(r_2-r_1)^{-\frac{2}{\delta}} \Big(\int_{-\rho}^{t} (\|\nabla u\|^{\delta+\frac{q}{3q-4}}_{L^\infty(B(\rho))} + c)\mathrm{d}\tau\Big)^{\frac{2}{\delta}}$$

$$\leqslant \frac{1}{4} \sup_{s\in(-\rho,0)} \|\omega(s)\|^2_{L^2(B(r_2))} + c(r_2-r_1)^{-\frac{2}{\delta}} \tag{5.1}$$

by using (3.7), (5.1) and the assumption (1.7). Here we can choose q large enough such that $\delta + \frac{q}{3q-4} < \frac{1}{2}$.

Thanks to (1.7), we estimate I_4 as

$$|I_4| = 4\Big|\int_{-\rho}^{t}\int b\cdot\nabla\varphi j\omega\varphi \mathrm{d}x\mathrm{d}\tau\Big|$$

$$\leqslant c\int_{-\rho}^{t} \|j\varphi\|_{L^2}\|\omega\|_{L^4(B(r_2))}\|b\|_{L^4(B(r_2))}\|\nabla\varphi\|_{L^\infty}\mathrm{d}\tau$$

$$\leqslant c(r_2-r_1)^{-1}\int_{-\rho}^{t} \|j\varphi\|_{L^2}\|\omega\|^{\frac{1}{2}}_{L^2(B(r_2))}\|\nabla u\|^{\frac{1}{2}}_{L^\infty(B(r_2))}\mathrm{d}\tau$$

$$\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)} (\|\omega(s)\|^2_{L^2(B(r_2))} + \|j(s)\|^2_{L^2(B(r_2))})$$

$$+c(r_2-r_1)^{-4}\Big(\int_{-\rho}^{t} \|\nabla u\|^{\frac{1}{2}}_{L^\infty(B(\rho))}\mathrm{d}\tau\Big)^4$$

$$\leqslant \frac{1}{4}\sup_{s\in(-\rho,0)} (\|\omega(s)\|^2_{L^2(B(r_2))} + \|j(s)\|^2_{L^2(B(r_2))}) + c(r_2-r_1)^{-4}.$$

Once these two bounds for I_1 and I_4 are established, a complete proof then follows from Section 3. □

By means of Sobolev's embedding theorem, we deduce from Lemma 3.8 and Proposition 5.1 the following corollary.

Corollary 5.2. *Under the assumption of Theorem 1.2, for any $0 < r < \rho$, we have $\Delta b + b\cdot\nabla u \in L^p(-\rho, 0; L^\infty(B(r)))$ for $1 < p < 2$.*

We now prove Theorem 1.2.

Proof of Theorem 1.2. For simplicity, we use all notations from (4.1) to (4.6). Let $0 < r < \varrho$ be arbitrarily chosen. Let $\frac{\rho}{2} < r \leqslant r_1 < r_2 \leqslant \rho' < \varrho$. Let $\varphi \in C_c^\infty(r_2)$ and $0 \leqslant \varphi \leqslant 1$, $\varphi(z) = 1$ for $z \in B(r_1)$. Set $\psi(x,t) = \varphi(\frac{x}{1+(-t)^{\frac{1}{2}}})$. Thanks to $\nabla\cdot u = 0$, we see that $-\Delta(u\psi) = \nabla\times\nabla\times(u\psi) - \nabla(u\cdot\nabla\psi)$. This combined with (2.1) and Lemma 2.2 yields

$$
\begin{aligned}
&\|\nabla(u\psi)\|_{L^\infty}\\
\leqslant\ &\|\Delta_{-1}\nabla(u\psi)\|_{L^\infty}+c\sum_{j=0}^{N-1}\|\Delta_j\nabla\times(u\psi)\|_{L^\infty}\\
&+c\sum_{j=0}^{N-1}\|\Delta_j(u\cdot\nabla\psi)\|_{L^\infty}+\sum_{j=N}^{\infty}\|\Delta_j\nabla(u\psi)\|_{L^\infty}\\
\leqslant\ &c\|\Delta_{-1}(u\psi)\|_{L^2}+c\sum_{j=0}^{N-1}\|\Delta_j(\omega\psi)\|_{L^\infty}\\
&+c\sum_{j=0}^{N-1}2^{(\frac{2}{\kappa}-1)j}\|\Delta_j\nabla(u\cdot\nabla\psi)\|_{L^\kappa}+\sum_{j=N}^{\infty}2^{(\frac{2}{\kappa}-1)j}\|\Delta_j\nabla^2(u\psi)\|_{L^\kappa}\\
\leqslant\ &c\|u\psi\|_{L^2}+cN\|\omega\psi\|_{L^\infty}+c(\|\nabla u\nabla\psi\|_{L^\kappa}+\|u\nabla^2\psi\|_{L^\kappa})\\
&+2^{(\frac{2}{\kappa}-1)N}\|\nabla^2u\psi\|_{L^\kappa}.
\end{aligned}
$$

Choosing $2^{(\frac{2}{\kappa}-1)N}(e+\|\nabla^2u\psi\|_{L^\kappa})\approx 1$, we get

$$
\begin{aligned}
&\|\nabla u\psi\|_{L^\infty}\\
\leqslant\ &c+c\|u\psi\|_{L^2}+c(\|u\nabla\psi\|_{L^\infty}+\|\nabla u\nabla\psi\|_{L^\kappa}+\|u\nabla^2\psi\|_{L^\kappa})\\
&+c\|\omega\psi\|_{L^\infty}\log(e+\|\nabla^2u\psi\|_{L^\kappa})\\
\leqslant\ &c+\frac{c}{r_2-r_1}+\frac{c}{(r_2-r_1)^2}+c\|\Omega\varphi\|_{L^\infty}\log(e+c\|\nabla^2v\varphi\|_{L^\kappa})
\end{aligned}
\tag{5.2}
$$

by taking advantage of Proposition 5.1 and Lemma 3.6. Thanks to (4.4), we have

$$
\begin{aligned}
&\partial_t\Omega(y,t)+\frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\Omega(y,t)+\frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}y\cdot\nabla_y\Omega(y,t)\\
&+\frac{1}{1+(-t)^{\frac{1}{2}}}v(y,t)\cdot\nabla_y\Omega(y,t)\\
=\ &(1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)\Big(\Delta_xb_2(x,t)+b(x,t)\cdot\nabla_xu_2(x,t)\Big)\Big)\\
&-(1+(-t)^{\frac{1}{2}})\Big(b_2(x,t)\Big(\Delta_xb_1(x,t)+b(x,t)\cdot\nabla_xu_1(x,t)\Big)\Big)\\
&-(1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)b(x,t)\cdot\nabla_xu_2(x,t)-b_2(x,t)b(x,t)\cdot\nabla_xu_1(x,t)\Big).
\end{aligned}
$$

Let

$$
\begin{aligned}
f(x,t)=\ &(1+(-t)^{\frac{1}{2}})\Big(b_1(x,t)\Big(\Delta_xb_2(x,t)+b(x,t)\cdot\nabla_xu_2(x,t)\Big)\Big)\\
&-(1+(-t)^{\frac{1}{2}})\Big(b_2(x,t)\Big(\Delta_xb_1(x,t)+b(x,t)\cdot\nabla_xu_1(x,t)\Big)\Big)
\end{aligned}
$$

and $F(y,t) = f((1+(-t)^{\frac{1}{2}})y,t)$. Then,

$$\partial_t \Omega(y,t) + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\Omega(y,t) + \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}} y\cdot\nabla_y\Omega(y,t) + \frac{1}{1+(-t)^{\frac{1}{2}}} v(y,t)\cdot\nabla_y\Omega(y,t)$$
$$= F(y,t) + G_1(y,t)\nabla_y v_2(y,t) + G_2(y,t)\nabla_y v_1(y,t). \tag{5.3}$$

Let $\eta \in C^\infty([0,+\infty))$ denote a cut off function such that $0 \leqslant \eta \leqslant 1$ on $[0,+\infty)$, $\eta \equiv 1$ on $[0,r_1]$, $\eta \equiv 0$ in $(r_2,+\infty)$, and $0 \leqslant -\eta' \leqslant \frac{c}{r_2-r_1}$. For convenience, we omit subscript y. Let $\varphi(y) = \eta(|y|)$. Multiplying both side of (5.3) by $\Omega|\Omega|^{q-2}\varphi^q$ for $q>2$ and integrating on $B(r_2)$, then summing over $i=1,2$ and applying the integration by parts, we find that

$$\begin{aligned}
&\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega(t)\varphi\|_{L^q}^q + (1-\frac{2}{q})\frac{\frac{1}{2}(-t)^{-\frac{1}{2}}}{1+(-t)^{\frac{1}{2}}}\int |\Omega(t)|^q\varphi^q \mathrm{d}y \\
&- \int_{B(r_2)\setminus B(r_1)} \frac{\frac{1}{2}(-t)^{-\frac{1}{2}}y + v(t)}{1+(-t)^{\frac{1}{2}}}\cdot y\frac{1}{|y|}|\Omega(t)|^q\varphi^{q-1}\eta'(|y|)\mathrm{d}y \\
&= \int \Big(F(t) + G_1(t)\cdot\nabla v_2(t) + G_2(t)\cdot\nabla v_1(t)\Big)\Omega(t)|\Omega(t)|^{q-2}\varphi^q \mathrm{d}y \\
&\leqslant c\|F\varphi\|_{L^q}\|\Omega\varphi\|_{L^q}^{q-1} + c\|\nabla v\varphi\|_{L^q}\|\Omega\varphi\|_{L^q}^{q-1},
\end{aligned}$$

where we use Proposition 5.1 and Lemma 3.5. Since $B(r_2)\setminus B(r_1) \subset B(\varrho)\setminus B(\frac{\varrho}{2})$ and $\eta' \leqslant 0$, the third term on the left side of the inequality above is nonnegative by using (4.3). Consequently, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Omega(t)\varphi\|_{L^q} \leqslant c\|F\varphi\|_{L^q} + c\|\nabla v\varphi\|_{L^q}. \tag{5.4}$$

Integrating on (t_0,t) for $t_0 < t < 0$ and letting $q\to\infty$, we have

$$\|\Omega(t)\varphi\|_{L^\infty} \leqslant \|\Omega(t_0)\varphi\|_{L^\infty} + c\int_{t_0}^{t}(\|F\varphi\|_{L^\infty} + \|\nabla v\varphi\|_{L^\infty})\mathrm{d}\tau. \tag{5.5}$$

Similar to (4.8), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Omega(t)\varphi\|_{L^\kappa} \leqslant c\|\nabla v\|_{L^\infty(B(r_2))}\|\nabla\Omega\varphi\|_{L^\kappa} + c\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}\varphi\|_{L^\kappa}$$
$$+c\|\nabla^2 v\varphi\|_{L^\kappa}.$$

Integrating over (t_0,t) for $t_0 < t < 0$, it yields

$$\|\nabla\Omega(t)\varphi\|_{L^\kappa} \leqslant \|\nabla\Omega(t_0)\varphi\|_{L^\kappa} + c\int_{t_0}^{t} \|\nabla v(\tau)\|_{L^\infty(B(r_2))}\|\nabla\Omega(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau$$

$$+c\int_{t_0}^{t}\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau + c\int_{t_0}^{t}\|\nabla^2 v(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau. \tag{5.6}$$

Similar to inequality (3.9), we have

$$\|\nabla^2 v(t)\varphi\|_{L^\kappa} \leqslant c\|\nabla\Omega(t)\varphi\|_{L^\kappa} + c\|\nabla v(t)\|_{L^\kappa(B(r_2))}\|\nabla\varphi\|_{L^\infty}. \tag{5.7}$$

Combining the two inequalities (5.6) and (5.7), we see that

$$\|\nabla^2 v(t)\varphi\|_{L^\kappa} \leqslant c + \frac{c}{r_2 - r_1} + c\int_{t_0}^{t}(\|\nabla v(\tau)\|_{L^\infty(B(r_2))} + 1)\|\nabla^2 v(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau,$$

by virtue of Proposition 5.1, Lemma 3.6 and Lemma 3.8. Using Gronwall's inequality, we have

$$\|\nabla^2 v(t)\varphi\|_{L^\kappa} \leqslant (c + \frac{c}{r_2 - r_1})e^{c\int_{t_0}^{t}(\|\nabla v(\tau)\|_{L^\infty(B(r_2))}+1)\mathrm{d}\tau}. \tag{5.8}$$

This combined with (5.2) and (5.5) yields

$$\begin{aligned}
&\|\nabla u\psi\|_{L^\infty}\\
&\leqslant c + \frac{c}{r_2 - r_1} + \frac{c}{(r_2 - r_1)^2} + c\Big(\|\Omega(t_0)\varphi\|_{L^\infty} + \int_{t_0}^{t}(\|F\varphi\|_{L^\infty}\\
&\quad +\|\nabla v\varphi\|_{L^\infty})\mathrm{d}\tau\Big)\Big(\log(c + \frac{c}{r_2 - r_1}) + c\int_{t_0}^{t}(\|\nabla v(\tau)\|_{L^\infty(B(r_2))} + 1)\mathrm{d}\tau\Big)\\
&\leqslant c + \frac{c}{(r_2 - r_1)^2} + (c + \frac{c}{r_2 - r_1})\int_{t_0}^{t}\|\nabla v(\tau)\|_{L^\infty(B(r_2))}\mathrm{d}\tau\\
&\quad +c(\int_{t_0}^{t}\|\nabla v(\tau)\|_{L^\infty(B(r_2))}\mathrm{d}\tau)^2\\
&\leqslant c + \frac{c}{(r_2 - r_1)^2} + (c + \frac{c}{r_2 - r_1})\int_{t_0}^{t}\|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))}\mathrm{d}\tau\\
&\quad +c(\int_{t_0}^{t}\|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))}\mathrm{d}\tau)^2.
\end{aligned}$$

Let $t_0 < T_0 < 0$, where T_0 is a constant determined later. Thanks to Sobolev's embedding theorem and the assumption $u \in C([-\rho, 0); W^{2,\kappa}(B(\rho)))$, then

$$
\begin{aligned}
&\|\nabla u\psi\|_{L^\infty} \\
&\leqslant c + \frac{c}{(r_2 - r_1)^2} + (c + \frac{c}{r_2 - r_1})\Big(\int_{t_0}^{T_0} \|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))} \mathrm{d}\tau \\
&\quad + \int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))} \mathrm{d}\tau \Big) + c(\int_{t_0}^{T_0} \|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))} \mathrm{d}\tau)^2 \\
&\quad + c(\int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty(B((1+(-\tau)^{\frac{1}{2}})r_2))} \mathrm{d}\tau)^2 \\
&\leqslant c(T_0) + \frac{c(T_0)}{(r_2 - r_1)^2} \\
&\quad + (c + \frac{c}{r_2 - r_1}) \mathrm{ess\ sup}_{s\in(T_0,0)} \|\nabla u(s)\|^{\frac{1}{2}}_{L^\infty(B((1+(-s)^{\frac{1}{2}})r_2))} \int_{T_0}^{t} (\|\nabla u(\tau)\|_{L^\infty(B(\rho))})^{\frac{1}{2}} \mathrm{d}\tau \\
&\quad + c\, \mathrm{ess\ sup}_{s\in(T_0,0)} \|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac{1}{2}})r_2))} \Big(\int_{T_0}^{t} (\|\nabla u(\tau)\|_{L^\infty(B(\rho))})^{\frac{1}{2}} \mathrm{d}\tau \Big)^2,
\end{aligned}
$$

where $c(T_0)$ is a constant depending on T_0. By virtue of (1.7), we can choose T_0 such that

$$
c(\int_{T_0}^{t} (\|\nabla u(\tau)\|_{L^\infty(B(\rho))})^{\frac{1}{2}} \mathrm{d}\tau)^2 < \frac{1}{4}.
$$

Consequently, using Young's inequality and (1.7), we have the following new iteration inequalities

$$
\begin{aligned}
&\mathrm{ess\ sup}_{s\in(T_0,0)} \|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac{1}{2}})r_1))} \\
&\leqslant \frac{1}{2} \mathrm{ess\ sup}_{s\in(T_0,0)} \|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac{1}{2}})r_2))} + c + \frac{c}{(r_2 - r_1)^2}.
\end{aligned} \tag{5.9}
$$

Consider the sequence ξ_i such that $\xi_0 = r$ and

$$
\xi_{i+1} - \xi_i = (1 - \mu)\mu^i(\rho' - r)
$$

with $0 < \mu < 1$. From (5.9) by induction we obtain

$$\begin{aligned}&\operatorname{ess\,sup}_{s\in(T_0,0)}\|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac12})r))}\\ \leqslant\ &\frac{1}{2^m}\operatorname{ess\,sup}_{s\in(T_0,0)}\|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac12})\xi_m))}+c+\sum_{i=0}^{m-1}(\frac12\mu^{-2})^i\frac{c}{(1-\mu)^2(\rho'-r)^2}.\end{aligned}$$

Choosing μ such that $\frac12\mu^{-2}<1$ and letting $m\to\infty$, we have

$$\begin{aligned}&\operatorname{ess\,sup}_{s\in(T_0,0)}\|\nabla u(s)\|_{L^\infty(B(r))}\\ \leqslant\ &\operatorname{ess\,sup}_{s\in(T_0,0)}\|\nabla u(s)\|_{L^\infty(B((1+(-s)^{\frac12})r))}\\ \leqslant\ &c+\frac{c}{(\rho'-r)^2}\end{aligned}$$

for any $0<r<\rho'$. This combined with the assumption $u\in C([-\rho,0);W^{2,\kappa}(B(\rho)))$ yields $\nabla u\in L^\infty(-\rho,0;L^\infty(B(\rho)))$ for any $0<r<\rho$. Furthermore, by (5.8), we have

$$\sup_{t\in(-\rho,0)}\|\nabla^2u(t)\|_{L^\kappa(B(r))}<+\infty.$$

This combined with Proposition 5.1 and Lemma 3.8 completes the proof of Theorem 1.2. □

6. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. First we give the following proposition.

Proposition 6.1. *Under the assumption of Theorem 1.2, it holds similarly for Lemma 3.1, Lemma 3.2, Remark 3.3, Lemma 3.5, Lemma 3.6, Lemma 3.8 and Corollary 5.2. In particular, we have*

$$\begin{aligned}&\|(u,b)\|_{H^1(-\rho,0;L^\infty(B(\frac89\rho)))}<c(\tilde c),\ \ \|b\|_{L^\infty(-\rho,0;L^\infty(B(\frac89\rho)))}<c(\tilde c),\\ &\|\nabla u\|_{L^\infty(-\rho,0;L^q(B(\frac89\rho)))}<c,\ \ \|\nabla b\|_{L^\infty(-\rho,0;L^q(B(\frac89\rho)))}<c,\\ &\|\Delta b+b\cdot\nabla u\|_{L^p(-\rho,0;L^\infty(B(\frac89\rho)))}<c,\ \ \|\nabla(\Delta b+b\cdot\nabla u)\|_{L^p(-\rho,0;L^\kappa(B(\frac89\rho)))}<c\end{aligned}\tag{6.1}$$

for $1<p<2$ and any $2\leqslant q<\infty$, where $c(\tilde c)$ is dependent on $\tilde c$, ρ, $\|(u(-\rho),b(-\rho)\|_{H^1(B(\rho))}$ and $\|b(-\rho)\|_{L^\infty(B(\rho))}$ and c is a constant depending on $\|(u,b)(-\rho)\|_{W^{2,\kappa}(B(\rho))}$ and $\tilde c$.

Proof. Similar to the proof of Proposition 5.1, we need only to re-estimate I_1 and I_4 in Lemma 3.2. Thanks to (1.9), it is easy to prove (5.1) namely the estimates for I_1.

Now we re-estimate I_4 in Lemma 3.2. Thanks to Lemma 3.1,

$$|I_4|=4|\int_{-\rho}^{t}\int b\cdot\nabla\varphi j\omega\varphi\mathrm{d}x\mathrm{d}\tau|$$

$$\leqslant c\int_{-\rho}^{t}\|j\varphi\|_{L^4}\|\omega\|_{L^2(B(r_2))}\|b\|_{L^4(B(r_2))}\|\nabla\varphi\|_{L^\infty}\mathrm{d}\tau$$

$$\leqslant c(r_2-r_1)^{-1}\int_{-\rho}^{t}\|j\varphi\|_{L^2}^{\frac{1}{2}}(\|\nabla j\varphi\|_{L^2}+\|j\nabla\varphi\|_{L^2})^{\frac{1}{2}}\|\omega\|_{L^2(B(r_2))}\mathrm{d}\tau$$

$$\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}^2\mathrm{d}\tau+\frac{1}{4}\sup_{s\in(-\rho,0)}\|\omega(s)\|_{L^2(B(r_2))}^2$$

$$+c(r_2-r_1)^{-4}\int_{-\rho}^{t}(\|j\varphi\|_{L^2(B(\rho))}^2+\|j\nabla\varphi\|_{L^2(B(\rho))}^2)\mathrm{d}\tau$$

$$\leqslant \frac{1}{4}\int_{-\rho}^{t}\|\nabla j\varphi\|_{L^2}^2\mathrm{d}\tau+\frac{1}{4}\sup_{s\in(-\rho,0)}(\|\omega(s)\|_{L^2(B(r_2))}^2+\|j(s)\|_{L^2(B(r_2))}^2)+c(r_2-r_1)^{-6}. \quad \square$$

We now prove Theorem 1.3.

Proof of Theorem 1.3. For simplicity, we use all notations in the proof of Theorem 1.2. Let $\tilde{C}$ be a positive independent constant. Let $r_1=\frac{3}{4}\rho$ and $r_2=\frac{5}{6}\rho$. Multiplying (5.4) by $-t$, integrating on (t_0,t) for $t_0<t<0$ and letting $q\to\infty$, we have

$$\begin{aligned}&-t\|\Omega(t)\varphi\|_{L^\infty}+\int_{t_0}^{t}\|\Omega(t)\varphi\|_{L^\infty}\mathrm{d}\tau\\ &\leqslant -t_0\|\Omega(t_0)\varphi\|_{L^\infty}+\tilde{C}\int_{t_0}^{t}(-\tau)\|F\varphi\|_{L^\infty}\mathrm{d}\tau+c(\tilde{c})\int_{t_0}^{t}(-\tau)\|\nabla v\varphi\|_{L^\infty}\mathrm{d}\tau\end{aligned}\tag{6.2}$$

by using (6.1). Similar to (5.2) and (5.6), we get by Proposition 6.1

$$\begin{aligned}&\|\nabla v\varphi\|_{L^\infty}\\ &\leqslant \tilde{C}+\tilde{C}\|v\varphi\|_{L^2}+\tilde{C}(\|v\nabla\varphi\|_{L^\infty}+\|\nabla v\nabla\varphi\|_{L^\kappa}+\|v\nabla^2\varphi\|_{L^\kappa})\\ &\quad+\tilde{C}\|\Omega\varphi\|_{L^\infty}\log(e+\|\nabla^2 v\varphi\|_{L^\kappa})\\ &\leqslant c+\tilde{C}\|\Omega\varphi\|_{L^\infty}\log(e+\|\nabla^2 v\varphi\|_{L^\kappa})\end{aligned}\tag{6.3}$$

and

$$\|\nabla\Omega(t)\varphi\|_{L^\kappa}\leqslant\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+\tilde{C}\int_{t_0}^{t}\|\nabla v(\tau)\|_{L^\infty(B(\frac{5}{6}\rho))}\|\nabla\Omega(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau$$

$$+\tilde{C}\int_{t_0}^{t}\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau+c(\tilde{c})\int_{t_0}^{t}\|\nabla^2 v(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau. \tag{6.4}$$

Using (5.7) and (6.4), we obtain by Proposition 6.1

$$\begin{aligned}
&\|\nabla^2 v(t)\varphi\|_{L^\kappa}\\
&\leqslant \|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+\tilde{C}\int_{t_0}^{t}\sum_{j=1}^{4}\sum_{i=1}^{2}\|F_{ji}(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau+\tilde{C}\|\nabla v(t)\|_{L^\kappa(B(\frac{5}{6}\rho))}\|\nabla\varphi\|_{L^\infty}\\
&\quad+\int_{t_0}^{t}(\tilde{C}\|\nabla v(\tau)\|_{L^\infty(B(\frac{5}{6}\rho))}+c(\tilde{c}))\|\nabla^2 v(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau\\
&\leqslant \|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c+\int_{t_0}^{t}(\tilde{C}\|\nabla v(\tau)\|_{L^\infty(B(\frac{5}{6}\rho))}+c(\tilde{c}))\|\nabla^2 v(\tau)\varphi\|_{L^\kappa}\mathrm{d}\tau.
\end{aligned}$$

Then we have

$$\|\nabla^2 v(t)\varphi\|_{L^\kappa}\leqslant(\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c)e^{\int_{t_0}^{t}(\tilde{C}\|\nabla v(\tau)\|_{L^\infty(B(\frac{5}{6}\rho))}+c(\tilde{c}))\mathrm{d}\tau} \tag{6.5}$$

by using Gronwall's inequality. (6.5) combined with (6.3) yields

$$\begin{aligned}
&\|\nabla v\varphi\|_{L^\infty}\\
&\leqslant c+\tilde{C}\|\Omega\varphi\|_{L^\infty}\left(\log(\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c)+\int_{t_0}^{t}(\tilde{C}\|\nabla v(\tau)\|_{L^\infty(B(\frac{5}{6}\rho))}+c(\tilde{c}))\mathrm{d}\tau\right).
\end{aligned}$$

This together with (6.2) yields

$$\begin{aligned}
&-t\|\Omega(t)\varphi\|_{L^\infty}+\int_{t_0}^{t}\|\Omega(t)\varphi\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant -t_0\|\Omega(t_0)\varphi\|_{L^\infty}+c+c(\tilde{c})\int_{t_0}^{t}(-\tau)\|\Omega(\tau)\varphi\|_{L^\infty}\log(\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c)\mathrm{d}\tau\\
&\quad+c(\tilde{c})\int_{t_0}^{t}(-\tau)\|\Omega(\tau)\varphi\|_{L^\infty}\left(\int_{t_0}^{\tau}\|\nabla v(s)\|_{L^\infty(B(\frac{5}{6}\rho))}\mathrm{d}s\right)\mathrm{d}\tau
\end{aligned}$$

by using Proposition 6.1. Let $t_0<t_1<t$. By the assumption (1.9), we choose t_0 so that $sup_{t_0<t<0}(-t)^2\|\nabla u(t)\|_{L^\infty(B(\rho))}<\varepsilon$, then

$$
\begin{aligned}
&-t\|\Omega(t)\varphi\|_{L^\infty}+\int_{t_0}^{t_1}\|\Omega(t)\varphi\|_{L^\infty}\mathrm{d}\tau+\int_{t_1}^{t}\|\Omega(t)\varphi\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant -t_0\|\Omega(t_0)\varphi\|_{L^\infty}+c+c(\tilde{c})\int_{t_0}^{t_1}(-\tau)\|\Omega(\tau)\varphi\|_{L^\infty}\log(\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c)\mathrm{d}\tau\\
&\quad +c(\tilde{c})\int_{t_1}^{t}(-\tau)\|\Omega(\tau)\varphi\|_{L^\infty}\log(\|\nabla\Omega(t_0)\varphi\|_{L^\kappa}+c)\mathrm{d}\tau+c(\tilde{c})\varepsilon\int_{t_0}^{t}\|\Omega(\tau)\varphi\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant c(t_0,t_1)+c(\tilde{c})c(t_0)\int_{t_1}^{t}(-\tau)\|\Omega(\tau)\varphi\|_{L^\infty}\mathrm{d}\tau+\frac{1}{4}\int_{t_0}^{t}\|\Omega(\tau)\varphi\|_{L^\infty}\mathrm{d}\tau
\end{aligned}
$$

by taking advantage of $u\in C([-\rho,0);W^{2,\kappa}(B(\rho)))$ for some $\kappa>2$. Here $c(t_0)$ is dependent on $\|u(t_0)\|_{W^{2,\kappa}(B(\rho))}$ and c and $c(t_0,t_1)$ is dependent on $u\in C([t_0,t_1];W^{2,\kappa}(B(\rho)))$ and c. Choosing t_1 close enough to 0 so that $c(\tilde{c})c(t_0)(-\tau)<\frac{1}{4}$ for $t_1<\tau<0$, then

$$
-t\|\Omega(t)\varphi\|_{L^\infty}+\int_{t_0}^{t}\|\Omega(t)\varphi\|_{L^\infty}\mathrm{d}\tau\leqslant c.
$$

Then by the expressions of ω and Ω we complete the proof of Theorem 1.3. □

7. Proof of Theorem 1.4

In this section we will prove Theorem 1.4 by using special structure of 2D resistive MHD equations which improves partially the nonlocal non blow-up criterion in [2].

Due to $H^1(\mathbb{R}^2)\hookrightarrow L^2(\mathbb{R}^2)\cap W^{2,\kappa}(\mathbb{R}^2)$, similar to the a priori estimates in [36] (or see [2,34]), we have the following a priori estimates.

Lemma 7.1. *If $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap W^{2,\kappa}(\mathbb{R}^2)$ for $\kappa>2$, then for any $2<q<\infty$ we have*

$$
\begin{aligned}
&(u,b)\in L^\infty(0,T;H^1(\mathbb{R}^2)),\ \nabla b\in L^2(0,T;H^1(\mathbb{R}^2)),\\
&b\in L^\infty(0,T;L^\infty(\mathbb{R}^2)),\ \nabla b\in L^\infty(0,T;L^q(\mathbb{R}^2)).
\end{aligned}
$$

Similar to Lemma 3.6, we borrow some arguments used in [19] to prove the following lemma.

Lemma 7.2. *If $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap W^{2,\kappa}(\mathbb{R}^2)$ for $\kappa>2$, then for any $1<p_1<2$ and $2\leqslant p_2,q<\infty$ we have*

$$
\omega\in L^\infty(0,T;L^q(\mathbb{R}^2)),\ \nabla^2 b\in L^{p_1}(0,T;L^q(\mathbb{R}^2))\cap L^{p_2}(0,T;L^\kappa(\mathbb{R}^2)).
$$

Proof. By virtue of Lemma 2.5 and $(1.1)_2$, we have

$$\int_0^t \|\Delta b(\tau)\|_{L^q}^{p_1} \mathrm{d}\tau \leqslant c\|\nabla b_0\|_{L^q}^{p_1} + c\int_0^t (\|(b\cdot\nabla u)(\tau)\|_{L^q}^{p_1} + \|(u\cdot\nabla b)(\tau)\|_{L^q}^{p_1})\mathrm{d}\tau. \tag{7.1}$$

Multiplying equation $(3.3)_1$ by $\omega|\omega|^{q-2}$, and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{q}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^q}^q \leqslant \|b\cdot\nabla j\|_{L^q}\|\omega\|_{L^q}^{q-1}.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^q}^{p_1} \leqslant c\|b\cdot\nabla j\|_{L^q}\|\omega\|_{L^q}^{p_1-1}.$$

Thanks to (7.1), integrating in time on $[0,t)$ and applying Young's inequality, we have

$$\begin{aligned}
\|\omega(t)\|_{L^q}^{p_1} &\leqslant \|\omega_0\|_{L^q}^{p_1} + c\int_0^t (\|b\|_{L^\infty}^{p_1}\|\nabla j\|_{L^q}^{p_1} + \|\omega\|_{L^q}^{p_1})\mathrm{d}\tau \\
&\leqslant c + c\int_0^t (\|b\|_{L^\infty}^{2p_1}\|\nabla u\|_{L^q}^{p_1} + \|b\|_{L^\infty}^{p_1}\|\nabla b\|_{L^q}^{p_1}\|u\|_{L^\infty}^{p_1} + \|\omega\|_{L^q}^{p_1})\mathrm{d}\tau \\
&\leqslant c + c\int_0^t \|\omega\|_{L^q}^{p_1}\mathrm{d}\tau,
\end{aligned}$$

where we use Lemma 7.1 and the following Gagliardo-Nirenberg inequality

$$\|u\|_{L^\infty} \leqslant c\|u\|_{L^2} + c\|\nabla u\|_{L^q}.$$

By Gronwall's inequality, we obtain $\omega \in L^\infty(0,T;L^q(\mathbb{R}^2))$. Then, by virtue of (7.1), we have $\nabla^2 b \in L^{p_1}(0,T;L^q(\mathbb{R}^2))$.

Furthermore, using Lemma 2.5 and $(1.1)_2$, we see that

$$\begin{aligned}
\int_0^t \|\Delta b(\tau)\|_{L^\kappa}^{p_2}\mathrm{d}\tau &\leqslant c\|\Delta b_0\|_{L^\kappa}^{p_2} + c\int_0^t (\|(b\cdot\nabla u)(\tau)\|_{L^\kappa}^{p_2} + \|(u\cdot\nabla b)(\tau)\|_{L^\kappa}^{p_2})\mathrm{d}\tau \\
&\leqslant c.
\end{aligned}$$

This completes the proof of this lemma. □

Exploiting the special structure of 2D resistive MHD equations, we have the following a priori estimates.

Lemma 7.3. *If* $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2)$ *for* $\kappa > 2$, *we have* $\Delta b \in L^\infty(0,T; L^\kappa(\mathbb{R}^2))$, $\Delta b + b\cdot\nabla u \in L^p(0,T; L^\infty(\mathbb{R}^2))$ *and* $\nabla(\Delta b + b\cdot\nabla u) \in L^p(0,T; L^\kappa(\mathbb{R}^2))$ *for* $1 < p < 2$.

Proof. We write (3.19) in its integral form, for $t \in [0,T)$ we have

$$\begin{aligned}
&(\Delta b + b\cdot\nabla u)(t)\\
&= e^{t\Delta}((\Delta b_0 + b_0\cdot\nabla u_0)) + \int_0^t e^{(t-\tau)\Delta}\Big(- b(\tau)\cdot\nabla\big((u\cdot\nabla u)(\tau)\big)\\
&\quad + b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau) - (u\cdot\nabla b)(\tau)\cdot\nabla u(\tau) + (b\cdot\nabla u)(\tau)\cdot\nabla u(\tau)\\
&\quad + \Delta b(\tau)\cdot\nabla u(\tau) - b(\tau)\cdot\nabla(\nabla p(\tau)) - \nabla\cdot\nabla(u\cdot\nabla b)(\tau)\Big)\mathrm{d}\tau. \qquad (7.2)
\end{aligned}$$

We estimate

$$\begin{aligned}
&\|(\Delta b + b\cdot\nabla u)(t)\|_{L^\kappa}\\
&\leqslant \|(\Delta b_0 + b_0\cdot\nabla u_0)\|_{L^\kappa} + \int_0^t (t-\tau)^{-\frac{1}{2}}\Big(\|b(\tau)(u\cdot\nabla u)(\tau)\|_{L^\kappa} + \|\nabla(u\cdot\nabla b)(\tau)\|_{L^\kappa}\\
&\quad + \|b(\tau)\nabla p(\tau)\|_{L^\kappa}\Big)\mathrm{d}\tau + \int_0^t \|b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau) - (u\cdot\nabla b)(\tau)\cdot\nabla u(\tau)\\
&\quad + (b\cdot\nabla u)(\tau)\cdot\nabla u(\tau) + \Delta b(\tau)\cdot\nabla u(\tau)\|_{L^\kappa}\mathrm{d}\tau\\
&\leqslant c.
\end{aligned}$$

As a consequence, $\Delta b + b\cdot\nabla u \in L^\infty(0,T; L^\kappa(\mathbb{R}^2))$. By Lemma 7.2, we have $\Delta b \in L^\infty(0,T; L^\kappa(\mathbb{R}^2))$.

Furthermore, using Lemma 2.5, Lemma 7.1 and Lemma 7.2, we estimate

$$\begin{aligned}
&\|\nabla((\Delta b + b\cdot\nabla u))\|_{L^p(0,T;L^\kappa)}\\
&\leqslant c\|(\Delta b_0 + b_0\cdot\nabla u_0)\|_{L^\kappa} + \|b(u\cdot\nabla u)\|_{L^p(0,T;L^\kappa)} + \|\nabla(u\cdot\nabla b)\|_{L^p(0,T;L^\kappa)}\\
&\quad + \|b\nabla p\|_{L^p(0,T;L^\kappa)} + \|(b\cdot\nabla(b\cdot\nabla b) - (u\cdot\nabla b)\cdot\nabla u\\
&\quad + (b\cdot\nabla u)\cdot\nabla u + \Delta b\cdot\nabla u\|_{L^p(0,T;L^\kappa)}\\
&\leqslant c.
\end{aligned}$$

Consequently, we have $\nabla(\Delta b + b\cdot\nabla u) \in L^p(0,T; L^\kappa(\mathbb{R}^2))$ for $1 < p < 2$. By Sobolev's embedding theorem, we arrive at $\Delta b + b\cdot\nabla u \in L^p(0,T; L^\infty(\mathbb{R}^2))$ for $1 < p < 2$. □

Lemma 7.4. *If* $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap W^{2,\kappa}(\mathbb{R}^2)$ *for* $\kappa > 2$, *for any* $0 < t < T$ *we have*

$$\|\nabla\omega(t)\|_{L^\kappa} \leqslant ce^{c\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau}.$$

Proof. Noting that $\partial_1 j = \Delta b_2$ and $\partial_2 j = -\Delta b_1$, equation (3.3) is changed into the following form

$$\begin{aligned}\partial_t \omega + u\cdot\nabla\omega &= b_1(\Delta b_2 + b\cdot\nabla u_2) - b_2(\Delta b_1 + b\cdot\nabla u_1) - b_1 b\cdot\nabla u_2 + b_2 b\cdot\nabla u_1\\ &= L - b_1 b\cdot\nabla u_2 + b_2 b\cdot\nabla u_1,\end{aligned} \tag{7.3}$$

where $L = b_1(\Delta b_2 + b\cdot\nabla u_2) - b_2(\Delta b_1 + b\cdot\nabla u_1)$. Applying ∇ to (7.3), it yields

$$\begin{aligned}\partial_t \nabla\omega + u\cdot\nabla(\nabla\omega) = &-\nabla u\cdot\nabla\omega + \nabla L - \nabla(b_1 b)\cdot\nabla u_2 + \nabla(b_2 b)\cdot\nabla u_1\\ &-b_1 b\cdot\nabla(\nabla u_2) + b_2 b\cdot\nabla(\nabla u_1).\end{aligned}$$

Multiplying the above equation by $\nabla\omega|\nabla\omega|^{\kappa-2}$, and integrating over $\mathbb{R}^2$, we have

$$\begin{aligned}\frac{1}{\kappa}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^\kappa}^\kappa \leqslant\ & c\|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^\kappa}^\kappa + c\|\nabla L\|_{L^\kappa}\|\nabla\omega\|_{L^\kappa}^{\kappa-1}\\ &+c\|b\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\nabla u\|_{L^\kappa}\|\nabla\omega\|_{L^\kappa}^{\kappa-1} + c\|b\|_{L^\infty}^2\|\nabla\omega\|_{L^\kappa}^\kappa.\end{aligned}$$

By Lemma 7.1, Lemma 7.2 and Lemma 7.3, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^\kappa} \leqslant c(\|\nabla u\|_{L^\infty} + 1)\|\nabla\omega\|_{L^\kappa} + c\|\nabla L\|_{L^\kappa} + c.$$

Using Gronwall's inequality, we arrive at

$$\|\nabla\omega(t)\|_{L^\kappa} \leqslant c\Big(\|\nabla\omega_0\|_{L^\kappa} + \int_0^t (\|\nabla L(\tau)\|_{L^\kappa} + 1)\mathrm{d}\tau\Big)e^{c\int_0^t(\|\nabla u(\tau)\|_{L^\infty}+1)\mathrm{d}\tau}.$$

This combined with Lemma 7.3 completes the proof of this lemma. □

Now we start to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\varphi \in C_c^\infty(B(2))$ be a cut off function such that $0 \leqslant \varphi \leqslant 1$, $\varphi \equiv 1$ on $B(1)$ and $|\nabla\varphi| \leqslant 1$. Let $\varphi_\sigma(x) = \varphi(\frac{x}{\sigma})$. Multiplying $(1.1)_2$ by $h(x)\varphi_\sigma(x)$ for $\sigma > 1$, then

$$\begin{aligned}&(h(x)\varphi_\sigma(x)b)_t - \Delta(h(x)\varphi_\sigma(x)b)\\ &= -2\nabla(h(x)\varphi_\sigma(x))\nabla b - \Delta(h(x)\varphi_\sigma(x))b - u\cdot\nabla b h(x)\varphi_\sigma(x) + b\cdot\nabla u h(x)\varphi_\sigma(x).\end{aligned} \tag{7.4}$$

We can write (7.4) under its following integral form,

$$\begin{aligned}h(x)\varphi_\sigma(x)b(x,t) = e^{t\Delta}(h\varphi_\sigma b_0) + \int_0^t e^{(t-\tau)\Delta}(&-2\nabla(h\varphi_\sigma)\nabla b(\tau) - \Delta(h\varphi_\sigma)b(\tau)\\ &-(u\cdot\nabla b)(\tau)h\varphi_\sigma + (b\cdot\nabla u)(\tau)h\varphi_\sigma)\mathrm{d}\tau.\end{aligned}$$

Taking the L^∞-norm, it follows that

$$\|h\varphi_\sigma b(t)\|_{L^\infty(\mathbb{R}^2)} \leqslant \|h\varphi_\sigma b_0\|_{L^\infty(\mathbb{R}^2)} + c\int_0^t \|e^{(t-\tau)\Delta}(\Delta(h\varphi_\sigma)b(\tau))\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau$$
$$+c\int_0^t \|\nabla e^{(t-\tau)\Delta}(\nabla(h\varphi_\sigma)b(\tau))\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau + c\int_0^t \|\nabla e^{(t-\tau)\Delta}(h\varphi_\sigma(ub)(\tau))\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau$$
$$+c\int_0^t \|e^{(t-\tau)\Delta}(\nabla(h\varphi_\sigma)(ub)(\tau))\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau.$$

By virtue of assumption for h in Theorem 1.4, we obtain

$$\|h\varphi_\sigma b(t)\|_{L^\infty(\mathbb{R}^2)} \leqslant \|hb_0\|_{L^\infty(\mathbb{R}^2)} + c\int_0^t (1+(t-\tau)^{-\frac{1}{2}})\|hb(\tau)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau$$
$$+c\int_0^t (1+(t-\tau)^{-\frac{1}{2}})\|u(\tau)\|_{L^\infty(\mathbb{R}^2)}\|hb(\tau)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau$$
$$\leqslant c + c\Big(\int_0^t \|hb(\tau)\|^3_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau\Big)^{\frac{1}{3}}.$$

Let $\sigma \to +\infty$, this yields

$$\|hb(t)\|^3_{L^\infty(\mathbb{R}^2)} \leqslant c + c\int_0^t \|hb(\tau)\|^3_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau.$$

Using Gronwall's inequality, we see that

$$\|hb(t)\|_{L^\infty(\mathbb{R}^2)} \leqslant c. \tag{7.5}$$

In virtue of (7.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^\infty} \leqslant c\|b^2\nabla u\|_{L^\infty} + \|L\|_{L^\infty}.$$

Thanks to Lemma 7.1 and Lemma 7.3, we deduce that

$$\|\omega(t)\|_{L^\infty} \leqslant \|\omega_0\|_{L^\infty} + c\int_0^t (\|(b^2\nabla u)(\tau)\|_{L^\infty} + \|L(\tau)\|_{L^\infty})\mathrm{d}\tau$$

$$\leqslant c + c\int_0^t \|(b^2\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau. \tag{7.6}$$

Taking advantage of the following Sobolev's embedding theorem

$$\|\nabla u\|_{L^\infty} \leqslant c + c\|u\|_{L^2} + c\|\omega\|_{L^\infty}\log(e + \|\nabla\omega\|_{L^\kappa}),$$

and using Lemma 7.1 and Lemma 7.4, we infer that

$$\|\nabla u(t)\|_{L^\infty} \leqslant c + c\|\omega(t)\|_{L^\infty}(1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau).$$

This combined with (7.6) yields

$$\begin{aligned}
\|\nabla u(t)\|_{L^\infty} &\leqslant c + c(1 + \int_0^t \|(b^2\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau)\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau \\
&\leqslant c + c(1 + \int_0^t \|b(\tau)h\|_{L^\infty}^2\|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau)\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau \\
&\leqslant c + c(1 + \int_0^t \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau)\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau
\end{aligned} \tag{7.7}$$

by making use of Lemma 7.1 and (7.5).

If $\epsilon = 0$, thanks to the following assumption (1.10) in Theorem 1.4

$$\int_0^T \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau < +\infty,$$

we can obtain $\nabla u \in L^\infty(0, T; L^\infty(\mathbb{R}^2))$ by Gronwall's inequality.

Consider $0 < \epsilon \leqslant \frac{1}{2}$. Let $T_0 < t < T$, where T_0 is a constant determined later. Then

$$\begin{aligned}
&\|\nabla u(t)\|_{L^\infty} \\
&\leqslant c + c(1 + \int_0^{T_0} \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau + \int_{T_0}^t \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau) \times \\
&(\int_0^{T_0} \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau + \int_{T_0}^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau)
\end{aligned}$$

$$\leqslant c + c(\int_0^{T_0} \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau)^2 + c(1 + \int_0^{T_0} \|(\nabla u)(\tau)\|_{L^\infty} \mathrm{d}\tau) \int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau$$

$$+ c \int_{T_0}^{t} \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty} \mathrm{d}\tau \int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty} \mathrm{d}\tau. \tag{7.8}$$

Owing to Sobolev's embedding theorem and the assumption $u \in C([0, T); W^{2,\kappa}(\mathbb{R}^2))$ in Theorem 1.4, we deduce that

$$\|\nabla u(t)\|_{L^\infty}$$

$$\leqslant c(T_0) + c(T_0) \sup_{s\in(T_0,T)} \|\nabla u(s)\|_{L^\infty}^{1-\epsilon} \int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty}^{\epsilon} \mathrm{d}\tau$$

$$+ \sup_{s\in(T_0,T)} \|\nabla u(s)\|_{L^\infty} \int_{T_0}^{t} \|\nabla u(\tau)\|_{L^\infty}^{\epsilon} \mathrm{d}\tau \int_{T_0}^{t} \|\frac{1}{h^2}(\nabla u)(\tau)\|_{L^\infty}^{1-\epsilon} \mathrm{d}\tau,$$

where $c(T_0)$ is a constant depending on T_0. Noting that the assumption (1.10) in Theorem 1.4, we can choose T_0 such that

$$c \int_{T_0}^{T} \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{\epsilon} \mathrm{d}\tau \int_{T_0}^{T} \|\frac{1}{h^2}\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1-\epsilon} \mathrm{d}\tau < \frac{1}{4}.$$

Applying Young's inequality, we see that

$$\sup_{s\in(T_0,T)} \|\nabla u(s)\|_{L^\infty}$$

$$\leqslant \frac{1}{2} \sup_{s\in(T_0,T)} \|\nabla u(s)\|_{L^\infty} + c(T_0) + c(T_0)(\int_{T_0}^{T} \|\nabla u(\tau)\|_{L^\infty}^{\epsilon} \mathrm{d}\tau)^{\frac{1}{\epsilon}}.$$

Then, it is easy to obtain $\nabla u \in L^\infty(0, T; L^\infty(\mathbb{R}^2))$.

As a consequence, we prove $(u, b) \in L^\infty(0, T; W^{2,\kappa}(\mathbb{R}^2))$ by using Lemma 7.3 and Lemma 7.4.

If $b_0 \in C_c^\infty(\mathbb{R}^2)$, replacing $h(x)$ by $e^{c(1+|x-y|^2)^{\frac{1}{2}}}$ in (7.4), then we have

$$\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}} \varphi_\sigma b(t)\|_{L^\infty(\mathbb{R}^2)}$$

$$\leqslant \|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}} b_0\|_{L^\infty(\mathbb{R}^2)} + c \int_0^{t} (1 + (t-\tau)^{-\frac{1}{2}}) \|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}} b(\tau)\|_{L^\infty(\mathbb{R}^2)} \mathrm{d}\tau$$

$$+c\int_0^t (1+(t-\tau)^{-\frac{1}{2}})\|u(\tau)\|_{L^\infty(\mathbb{R}^2)}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(\tau)\|_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau$$

$$\leqslant c(R,b_0)+c\Big(\int_0^t \sup_{y\in\overline{B(R)}}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(\tau)\|^3_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau\Big)^{\frac{1}{3}}$$

for $y\in\overline{B(R)}$, where $c(R,b_0)$ is a constant depends only on R and b_0. Let $\sigma\to+\infty$, we arrive at

$$\sup_{y\in\overline{B(R)}}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(t)\|^3_{L^\infty(\mathbb{R}^2)}\leqslant c(R,b_0)+c\int_0^t \sup_{y\in\overline{B(R)}}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(\tau)\|^3_{L^\infty(\mathbb{R}^2)}\mathrm{d}\tau.$$

Applying for Gronwall's inequality, we see that

$$\sup_{y\in\overline{B(R)}}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(t)\|_{L^\infty(\mathbb{R}^2)}\leqslant c(R,b_0)e^{ct}.$$

Similar to (7.7), we obtain

$$\begin{aligned}
&\|\nabla u(t)\|_{L^\infty}\\
&\leqslant c+c\Big(1+\int_0^t \sup_{y\in\overline{B(R)}}\|e^{c(1+|\bullet-y|^2)^{\frac{1}{2}}}b(t)\|^2_{L^\infty(\mathbb{R}^2)}\\
&\quad\times\inf_{y\in\overline{B(R)}}\|e^{-c(1+|\bullet-y|^2)^{\frac{1}{2}}}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau\Big)\int_0^t\|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau\\
&\leqslant c+c(1+\int_0^t \inf_{y\in\overline{B(R)}}\|e^{-c(1+|\bullet-y|^2)^{\frac{1}{2}}}(\nabla u)(\tau)\|_{L^\infty}\mathrm{d}\tau)\int_0^t\|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau.
\end{aligned}$$

Furthermore, by virtue of the assumption (1.11), we can also prove $(u,b)\in L^\infty(0,T;W^{2,\kappa}(\mathbb{R}^2))$, which concludes the proof of Theorem 1.4. □

8. Proof of Theorem 1.5

In this section we will prove Theorem 1.5 by using special structure of 2D resistive MHD equations.

Due to $H^1(\mathbb{R}^2)\hookrightarrow L^2(\mathbb{R}^2)\cap C^{1,\alpha}(\mathbb{R}^2)$, similar to Lemma 7.1, we have the following a priori estimates.

Lemma 8.1. *If $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap C^{1,\alpha}(\mathbb{R}^2)$, then for any $2<q<\infty$ we have*

$$(u,b)\in L^\infty(0,T;H^1(\mathbb{R}^2)),\ \ \nabla b\in L^2(0,T;H^1(\mathbb{R}^2)),$$

$$b \in L^\infty(0,T;L^\infty(\mathbb{R}^2)),\ \ \nabla b \in L^\infty(0,T;L^q(\mathbb{R}^2)).$$

Thanks to $L^q(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \cap C^{1,\alpha}(\mathbb{R}^2)$ for any $2 < q < \infty$, similar to Lemma 7.2, we have the following lemma.

Lemma 8.2. *If $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap C^{1,\alpha}(\mathbb{R}^2)$, then for any $1 < p < 2$ and $2 \leqslant q < \infty$ we have*

$$\omega \in L^\infty(0,T;L^q(\mathbb{R}^2)),\ \ \nabla^2 b \in L^p(0,T;L^q(\mathbb{R}^2)).$$

Exploiting the special structure of 2D resistive MHD equations, we have the following a priori estimates.

Lemma 8.3. *If $(u_0, b_0) \in L^2(\mathbb{R}^2) \cap C^{1,\alpha}(\mathbb{R}^2)$, we have $b \in L^\infty(0,T;C^{1,\alpha}(\mathbb{R}^2))$ and $\Delta b + b \cdot \nabla u \in L^1(0,T;C^{0,\alpha}(\mathbb{R}^2))$.*

Proof. Here we often use the equivalent norms between $C^{0,\alpha}$ and $B^\alpha_{\infty,\infty}$, namely $C^{0,\alpha} = B^\alpha_{\infty,\infty}$.

Choosing q large enough such that $\alpha + \frac{2}{q} < 1$. Thanks to (7.2), we have

$$\begin{aligned}
&\|(\Delta b + b\cdot\nabla u)\|_{L^1(0,T;C^{0,\alpha}(\mathbb{R}^2))}\\
\leqslant\ & c\|e^{t\Delta}((\Delta b_0 + b_0\cdot\nabla u_0))\|_{L^1(0,T;B^\alpha_{\infty,\infty}(\mathbb{R}^2))} + c\Big\|\int_0^t e^{(t-\tau)\Delta}\Big(-b(\tau)\cdot\nabla\big((u\cdot\nabla u)(\tau)\big)\\
&+b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau) - (u\cdot\nabla b)(\tau)\cdot\nabla u(\tau) + (b\cdot\nabla u)(\tau)\cdot\nabla u(\tau)\\
&+\Delta b(\tau)\cdot\nabla u(\tau) - b(\tau)\cdot\nabla(\nabla p(\tau)) - \nabla\cdot\nabla(u\cdot\nabla b)(\tau)\Big)\mathrm{d}\tau\Big\|_{L^1(0,T;B^\alpha_{\infty,\infty}(\mathbb{R}^2))}\\
\leqslant\ & c\int_0^T(\|\Delta_{-1}\nabla b_0\|_{L^\infty} + \sup_{k\geqslant 0}2^{k(1+\alpha)}e^{-c2^{2k}t}\|\Delta_k\nabla b_0\|_{L^\infty})\mathrm{d}\tau + c\|b_0\cdot\nabla u_0\|_{L^1(0,T;C^{0,\alpha}(\mathbb{R}^2))}\\
&+\Big\|\int_0^t\Big(\|\Delta_{-1}(b(\tau)(u\cdot\nabla u)(\tau))\|_{L^2} + \|\Delta_{-1}(\nabla(u\cdot\nabla b)(\tau))\|_{L^2}\\
&+\|\Delta_{-1}(b(\tau)\nabla p(\tau))\|_{L^2}\Big)\mathrm{d}\tau\Big\|_{L^1(0,T)}\\
&+\Big\|\int_0^t\Big\|\Delta_{-1}\Big(b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau) - (u\cdot\nabla b)(\tau)\cdot\nabla u(\tau)\\
&+(b\cdot\nabla u)(\tau)\cdot\nabla u(\tau) + \Delta b(\tau)\cdot\nabla u(\tau)\Big)\Big\|_{L^2}\mathrm{d}\tau\Big\|_{L^1(0,T)}\\
&+\Big\|\int_0^t\sup_{k\geqslant 0}2^{k(1+\alpha+\frac{2}{q})}e^{-c2^{2k}(t-\tau)}\Big(\|\Delta_k(b(\tau)(u\cdot\nabla u)(\tau))\|_{L^q} + \|\Delta_k(\nabla(u\cdot\nabla b)(\tau))\|_{L^q}
\end{aligned}$$

$$+\|\Delta_k(b(\tau)\nabla p(\tau))\|_{L^q}\Big)\mathrm{d}\tau\Big\|_{L^1(0,T)} + \Big\|\int_0^t \sup_{k\geqslant 0} 2^{k(\alpha+\frac{2}{q})}e^{-c2^{2k}(t-\tau)}\Big\|\Delta_k\Big(b(\tau)\cdot\nabla(b\cdot\nabla b)(\tau)$$

$$-(u\cdot\nabla b)(\tau)\cdot\nabla u(\tau)+(b\cdot\nabla u)(\tau)\cdot\nabla u(\tau)+\Delta b(\tau)\cdot\nabla u(\tau)\Big)\Big\|_{L^q}\mathrm{d}\tau\Big\|_{L^1(0,T)}$$

$$\leqslant c$$

by taking advantage of Lemma 2.2, Lemma 2.3, Lemma 8.1 and Lemma 8.2. Here we also use the following inequality

$$\|\Delta_k f\|_{L^q}\leqslant c\|f\|_{L^q}$$

for any $k\geqslant -1$.

Recall $C^{1,\alpha}=B^{1+\alpha}_{\infty,\infty}$. Similarly, from equation $(1.1)_2$, we have

$$\begin{aligned}
&\|b(t)\|_{C^{1,\alpha}}\\
&\leqslant \|e^{t\Delta}b_0\|_{C^{1,\alpha}}+c\int_0^t\|e^{(t-\tau)\Delta}(u\cdot\nabla b-b\cdot\nabla u)(\tau)\|_{B^{1+\alpha}_{\infty,\infty}}\mathrm{d}\tau\\
&\leqslant \|b_0\|_{C^{1,\alpha}}+c\int_0^t(\|\Delta_{-1}(u\cdot\nabla b-b\cdot\nabla u)(\tau))\|_{L^2}\\
&\quad+\sup_{k\geqslant 0}2^{k(1+\alpha+\frac{2}{q})}e^{-c2^{2k}\tau}\|\Delta_k(u\cdot\nabla b-b\cdot\nabla u)(\tau))\|_{L^q})\mathrm{d}\tau\\
&\leqslant c. \quad \square
\end{aligned}$$

Lemma 8.4. *If $(u_0,b_0)\in L^2(\mathbb{R}^2)\cap C^{1,\alpha}(\mathbb{R}^2)$, for any $0<t<T$ we have*

$$\|\omega(t)\|_{C^{0,\alpha}}\leqslant ce^{c\int_0^t\|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau}.$$

Proof. Applying Δ_k to equation (7.3), yields

$$\begin{aligned}
\partial_t\Delta_k\omega+(S_{k-1}u\cdot\nabla)\Delta_k\omega&=(S_{k-1}u\cdot\nabla)\Delta_k\omega-\Delta_k(u\cdot\nabla\omega)+\Delta_k L\\
&\quad-\Delta_k(b_1b\cdot\nabla u_2)+\Delta_k(b_2b\cdot\nabla u_1),
\end{aligned}\tag{8.1}$$

where $L=b_1(\Delta b_2+b\cdot\nabla u_2)-b_2(\Delta b_1+b\cdot\nabla u_1)$. Taking the L^∞-norm of ω in (8.1), we have

$$\|\Delta_k\omega\|_{L^\infty}\leqslant\|\Delta_k\omega_0\|_{L^\infty}+\int_0^t\|(S_{k-1}u\cdot\nabla)\Delta_k\omega-\Delta_k(u\cdot\nabla\omega)\|_{L^\infty}\mathrm{d}\tau$$

$$+\int_0^t\|\Delta_k L\|_{L^\infty}\mathrm{d}\tau+\int_0^t\|\Delta_k(b_1b\cdot\nabla u_2)-\Delta_k(b_2b\cdot\nabla u_1)\|_{L^\infty}\mathrm{d}\tau.$$

By Lemma 2.4 and Lemma 8.3, we obtain

$$\begin{aligned}
&\|\omega\|_{C^{0,\alpha}} \leqslant c\|\omega\|_{B^{\alpha}_{\infty,\infty}} \\
&\leqslant c\|\omega_0\|_{B^{\alpha}_{\infty,\infty}} + c\int_0^t \|\nabla u\|_{L^\infty}\|\omega\|_{B^{\alpha}_{\infty,\infty}}\mathrm{d}\tau + c\int_0^t \|L\|_{C^{0,\alpha}}\mathrm{d}\tau + c\int_0^t \|b^2\nabla u\|_{C^{0,\alpha}}\mathrm{d}\tau \\
&\leqslant c + c\int_0^t (\|\nabla u\|_{L^\infty}+1)\|\omega\|_{C^{0,\alpha}}\mathrm{d}\tau.
\end{aligned}$$

By Gronwall's inequality, we complete the proof of this lemma. □

Now we start to prove Theorem 1.5.

Proof of Theorem 1.5. Thanks to Lemma 8.1 and Lemma 8.3, from equation (7.3) it is easy to obtain

$$\begin{aligned}
\|\omega(t)\|_{L^\infty} &\leqslant \|\omega_0\|_{L^\infty} + c\int_0^t (\|(b^2\nabla u)(\tau)\|_{L^\infty} + \|L(\tau)\|_{L^\infty})\mathrm{d}\tau \\
&\leqslant c + c\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau. \qquad (8.2)
\end{aligned}$$

By taking advantage of the following Sobolev's embedding theorem

$$\|\nabla u\|_{L^\infty} \leqslant c + c\|u\|_{L^2} + c\|\omega\|_{L^\infty}\log(e+\|\omega\|_{C^{0,\alpha}}),$$

and using Lemma 8.1 and Lemma 8.4, we deduce that

$$\|\nabla u(t)\|_{L^\infty} \leqslant c + c\|\omega(t)\|_{L^\infty}(1+\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau).$$

By (8.2), we get

$$\begin{aligned}
\|\nabla u(t)\|^{\frac{1}{2}}_{L^\infty} &\leqslant c + c\int_0^t \|\nabla u(\tau)\|_{L^\infty}\mathrm{d}\tau \\
&\leqslant c + c\int_0^t \|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty}\log(e+\|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty})\frac{\|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty}}{\log(e+\|\nabla u(\tau)\|^{\frac{1}{2}}_{L^\infty})}\mathrm{d}\tau.
\end{aligned}$$

Therefore, by Gronwall's inequality and (1.12), we prove $\|\nabla u(t)\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} < +\infty$. Thus, we obtain $\|\omega(t)\|_{L^\infty(0,T;C^{0,\alpha}(\mathbb{R}^2))} < +\infty$ by (8.2). This combined with Lemma 8.3 yields $(u, b) \in L^\infty(0, T; L^2(\mathbb{R}^2)) \cap L^\infty(0, T; C^{1,\alpha}(\mathbb{R}^2))$. □

Acknowledgments

Jiahong Wu was partially supported by NSF grant DMS 2104682 and DMS 2309748. Jiefeng Zhao was partially supported by the National Natural Science Foundation of China (No. 11901165), Natural Science Foundation of Henan Province (No. 242300420245) and the Fundamental Research Funds for the Universities of Henan Province (NSFRF240813). Daoguo Zhou was partially supported by the National Natural Science Foundation of China (No. 12071113) and Research Start-up Funding Program of Hangzhou Normal University (No. 4235C50223204065).

Data availability

All relevant data are within the paper.

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