



Hearing pseudoconvexity in Lipschitz domains with holes via $\bar{\partial}$

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Abstract Let $\Omega = \tilde{\Omega} \setminus \bar{D}$ where $\tilde{\Omega}$ is a bounded domain with connected complement in \mathbb{C}^n (or more generally in a Stein manifold) and D is relatively compact open subset of $\tilde{\Omega}$ with connected complement in $\tilde{\Omega}$. We obtain characterizations of pseudoconvexity of $\tilde{\Omega}$ and D through the vanishing or Hausdorff property of the Dolbeault cohomology groups of Ω on various function spaces. In particular, we show that if the boundaries of $\tilde{\Omega}$ and D are Lipschitz and C^2 -smooth respectively, then both $\tilde{\Omega}$ and D are pseudoconvex if and only if 0 is not in the spectrum of the $\bar{\partial}$ -Neumann Laplacian of Ω on $(0, q)$ -forms for $1 \leq q \leq n - 2$ when $n \geq 3$; or 0 is not a limit point of the spectrum of the $\bar{\partial}$ -Neumann Laplacian on $(0, 1)$ -forms when $n = 2$.

Keywords Dolbeault cohomology · L^2 -Dolbeault cohomology · Serre duality · $\bar{\partial}$ -Neumann Laplacian · Pseudoconvexity

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1 Introduction

The classical theorem B of H. Cartan states that for any coherent analytic sheaf \mathcal{F} over a Stein manifold X , the sheaf cohomology groups $H^q(X, \mathcal{F})$ vanish for all $q \geq 1$ (cf. [12, Theorem 7.4.3]). The converse is also true ([24, p. 65]; see [12, pp. 86–89] for a proof of this equivalence). The $\bar{\partial}$ -complexes of the smooth forms, the L^2_{loc} forms, and the currents on an open set Ω in a complex manifold are all resolutions of the sheaf \mathcal{O} of germs of holomorphic functions. As a consequence, the Dolbeault cohomology groups obtained through these resolutions are all isomorphic to $H^q(\Omega, \mathcal{O})$ and will all be denoted by $H^{0,q}(\Omega)$. However, this isomorphism does not hold in general if one considers the Dolbeault cohomology groups for the $\bar{\partial}$ -complexes acting on function spaces with some growth or regularity properties on the boundary of the open set.

When Ω is a relatively compact pseudoconvex domain in a Stein manifold, it follows from Hörmander's L^2 -existence theorem for the $\bar{\partial}$ -operator that the L^2 -Dolbeault cohomology groups $H^{0,q}_{L^2}(\Omega)$ vanish for $q \geq 1$. The converse of Hörmander's theorem also holds, under the assumption that the interior of the closure of Ω is the domain Ω itself. Sheaf theoretic arguments for the Dolbeault cohomology groups [2, 16, 23, 24, 29] can be modified to give a proof of this fact (see, e.g., [8, 9]). We remark that some regularity of the boundary is necessary in order to characterize pseudoconvexity by the L^2 -Dolbeault cohomology. The Dolbeault isomorphism also fails to hold between the usual Dolbeault cohomology groups and the L^2 -Dolbeault cohomology groups. For example, on the unit ball in \mathbb{C}^2 minus the center, the L^2 -Dolbeault cohomology group $H^{0,1}_{L^2}(\mathbb{B}^2 \setminus \{0\})$ is trivial but the classical Dolbeault cohomology group $H^{0,1}(\mathbb{B}^2 \setminus \{0\})$ is infinite dimensional.

It follows from the work of Kohn [15] that, if Ω is a bounded pseudoconvex domain in \mathbb{C}^n with smooth boundary, then the Dolbeault cohomology groups $H^{0,q}_{\infty}(\bar{\Omega})$ with forms smooth up to the boundary also vanish for all $q > 0$ and the converse is also true. But this does not hold in general if the boundary of the domain is not smooth. For example, on the Hartogs' triangle $\mathbb{T} = \{(z, w) \in \mathbb{C}^2; |z| < |w| < 1\}$, by Sibony [28], we have that $H^{0,1}(\mathbb{T})$ is trivial but $H^{0,1}_{\infty}(\bar{\mathbb{T}})$, the Dolbeault cohomology of forms with coefficients smooth up to the boundary, is infinite dimensional. In fact, it is not even Hausdorff (see [21]). In general, the Dolbeault cohomology groups with some growth or regularity properties up to the boundary are not the same as the usual Dolbeault cohomology.

Andreotti and Grauert [1] showed that, for a relatively compact domain Ω with smooth boundary in a complex manifold that is strictly pseudoconvex (or more generally, satisfies Condition a_q)¹, the Dolbeault cohomology group $H^{p,q}(\Omega)$ is finite dimensional. Further-

¹ Recall that the boundary $b\Omega$ satisfies condition a_q if the Levi form of a defining function has either at least $q + 1$ negative eigenvalues or at least $n - q$ positive eigenvalues at every boundary point.

more, by Grauert's bumping methods, the usual Dolbeault cohomology group is isomorphic to the Dolbeault cohomology group with regularity up to the boundary (see, e.g., [11]). It follows from Hörmander's work [13, § 3.4] that on a bounded domain Ω with C^3 -smooth boundary in a complex manifold, if the boundary $b\Omega$ satisfies the a_q and a_{q+1} conditions, then $H_{L^2}^{p,q}(\Omega)$ is isomorphic to $H^{p,q}(\Omega)$. (See [22] for relevant results.) In particular, when Ω is an annulus between two C^3 -smooth strictly pseudoconvex domains in a complex manifold, the L^2 cohomology group $H_{L^2}^{0,q}(\Omega)$ is finite dimensional when $0 < q < n - 1$. When the domain Ω is an annulus between two weakly pseudoconvex domains with C^3 smooth boundary in \mathbb{C}^n , similar results are proved in Shaw [26]. The regularity of the boundary can be relaxed by assuming only that Ω has C^2 boundary (see [27]). In this case, $H_{L^2}^{p,q}(\Omega) = 0$ for all $q \neq 0$ and $q \neq n - 1$.

In this paper, we study the Dolbeault cohomology groups on various function spaces for a bounded domain Ω in \mathbb{C}^n (or more generally in a Stein manifold) in the form of $\Omega = \tilde{\Omega} \setminus \bar{D}$ where $\tilde{\Omega}$ is a bounded domain with connected complement in \mathbb{C}^n and D is relatively compact open subset of $\tilde{\Omega}$ with connected complement and with finitely many connected components. We obtain characterizations of pseudoconvexity of $\tilde{\Omega}$ and D through vanishing or Hausdorff property of the Dolbeault cohomology groups on various function spaces, including those of forms with coefficients smooth up to the boundary and of extendable currents. In particular, we consider L^2 -Dolbeault cohomology groups and spectral theory for $\bar{\partial}$ -Neumann Laplacian on the domain Ω . We show that if the outer boundary $b\tilde{\Omega}$ is Lipschitz and the inner boundary bD is C^2 -smooth, then both $\tilde{\Omega}$ and D are pseudoconvex if and only if 0 is not in the spectrum of the $\bar{\partial}$ -Neumann Laplacian on $(0, q)$ -forms for $1 \leq q \leq n - 2$ when $n \geq 3$; or 0 is not a limit point for the spectrum of the $\bar{\partial}$ -Neumann Laplacian on $(0, 1)$ -forms when $n = 2$ (see Corollary 5.4, Theorems 5.6, and 5.7 below).

An earlier result in this spirit is due to Trapani [32]. In his case, vanishing and Hausdorff properties of the classical Dolbeault cohomology groups characterize the holomorphic convexity of \bar{D} ; not just the pseudoconvexity of D . Our results and methods are different from the results in Trapani [32] (see the remark at the end of Sect. 3).

The plan of the paper is as follows: In Sect. 2, we first recall the classical Serre duality and the definitions of various Dolbeault cohomology groups and their duals. In Sect. 3, we study the Dolbeault cohomology groups on forms that are smooth up to the boundary and on extendable currents. We obtain necessary and sufficient conditions for the vanishing or Hausdorff properties of these Dolbeault cohomology groups. Sections 4 and 5 are devoted to the L^2 -theory for $\bar{\partial}$ on such a domain Ω . In Sect. 4, we study the relationship between the Dolbeault cohomology groups of Ω and those of $\tilde{\Omega}$ and D on L^2 or W^1 spaces. In Sect. 5, we establish necessary and sufficient conditions for such domains Ω that have vanishing L^2 or W^1 -Dolbeault cohomology groups. In particular, we show that when the boundary of D is Lipschitz, the Dolbeault cohomology groups with W_{loc}^1 -coefficients are vanishing for $1 \leq q < n - 1$ and Hausdorff for $q = n - 1$ if and only if both $\tilde{\Omega}$ and D are pseudoconvex (see Corollary 5.3). When the inner boundary bD is C^2 and the outer boundary $b\tilde{\Omega}$ is Lipschitz, we obtain a characterization of pseudoconvexity of $\tilde{\Omega}$ and D by the vanishing and Hausdorff properties of the L^2 -Dolbeault cohomology groups, which implies the above-mentioned statement that one can determine pseudoconvexity of $\tilde{\Omega}$ and D from spectral properties of the $\bar{\partial}$ -Neumann Laplacian on Ω .

2 Dolbeault cohomology and the Serre duality

Let X be an n -dimensional complex manifold. If $D \subset\subset X$ is a relatively compact subset of X , we consider the $\bar{\partial}$ -complexes and their dual complexes associated with several spaces of forms attached to D (see section 2 in [20] for more details).

We first recall definitions of several function spaces and their duals. Let $\mathcal{E}(D)$ be the space of C^∞ -smooth functions on D with its classical Fréchet topology. It is well known that its dual can be identified with the space $\mathcal{E}'(X)$ of distributions with compact support in D . We will also consider the space $C^\infty(\bar{D})$ of smooth functions on the closure of D ; this is the space of the restrictions to \bar{D} of C^∞ -smooth functions on X . It can also be identified with the quotient of the space of C^∞ -smooth functions on X by the ideal of functions that vanish with all their derivatives on \bar{D} . We endow $C^\infty(\bar{D})$ with the Fréchet topology induced by $\mathcal{E}(X)$. If D has Lipschitz boundary, the dual space of $C^\infty(\bar{D})$ is the space $\mathcal{E}'_D(X)$ of distributions on X with support contained in \bar{D} . In general, when the boundary of D is not necessarily Lipschitz, then we only have that the dual space of $C^\infty(\bar{D})$ is always a subspace of $\mathcal{E}'_D(X)$. We refer the reader to Laurent-Thiébaud and Shaw [20] for details.

Denote by $\mathcal{D}_{\bar{D}}(X)$ the subspace of $\mathcal{D}(X)$ consisting of functions with support in \bar{D} , endowed with the natural Fréchet topology. If D has Lipschitz boundary, the dual of $\mathcal{D}_{\bar{D}}(X)$ coincides with the space of restrictions to D of distributions on X . This dual is called the space of extendable distribution on D and will be denoted by $\check{\mathcal{D}}'(\bar{D})$. Moreover $\mathcal{D}_{\bar{D}}(X)$ is a Montel space as a closed subspace of the Montel space $\mathcal{E}(X)$ and hence reflexive, which implies that the dual space of $\check{\mathcal{D}}'(\bar{D})$, endowed with the strong dual topology, is the space $\mathcal{D}'_{\bar{D}}(X)$. As before, when the boundary of D is not Lipschitz, we only have that the dual space of $\check{\mathcal{D}}'(\bar{D})$ is a subspace of $\mathcal{D}'_{\bar{D}}(X)$.

Recall that a *cohomological complex of topological vector spaces* is a pair (E^\bullet, d) , where $E^\bullet = (E^q)_{q \in \mathbb{Z}}$ is a sequence of topological vector spaces and $d = (d^q)_{q \in \mathbb{Z}}$ is a sequence of densely defined closed linear maps d^q from E^q into E^{q+1} that satisfy $d^{q+1} \circ d^q = 0$. To any cohomological complex (E^\bullet, d) we associate cohomology groups $(H^q(E^\bullet))_{q \in \mathbb{Z}}$ defined by

$$H^q(E^\bullet) = \ker d^q / \text{Im } d^{q-1}$$

and endowed with the quotient topology. We fix $0 \leq p \leq n$ and set $E^q = 0$ and $d^q \equiv 0$, if $q < 0$ and $0 \leq p \leq n$. We consider the $\bar{\partial}$ -complex (E^\bullet, d) with $d^q = \bar{\partial}$ if $0 \leq q \leq n$ and $d^q \equiv 0$ if $q > n$ acting on:

- (1) $E^q = \mathcal{E}^{p,q}(D)$, the space of C^∞ -smooth (p, q) -forms on D .
- (2) $E^q = C^\infty_{p,q}(\bar{D})$, the space of C^∞ -smooth (p, q) -forms on \bar{D} .
- (3) $E^q = \check{\mathcal{D}}'^{p,q}(\bar{D})$, the space of extendable (p, q) -currents on D .

The associated cohomology groups with (1)–(3) are denoted respectively by $H^{p,q}(D)$, $H^{p,q}(\bar{D})$, and $\check{H}^{p,q}(\bar{D})$.

The dual complex of a cohomological complex (E_\bullet, d) of topological vector spaces is the homological complex (E'_\bullet, d') , where $E_\bullet = (E^q)_{q \in \mathbb{Z}}$ with E'_q the strong dual of E^q and $d' = (d'_q)_{q \in \mathbb{Z}}$ with d'_q the transpose of the map d^q .

Set $E'_q = 0$ and $d'_q \equiv 0$ if $q < 0$. If the domain D has Lipschitz boundary, then the dual complexes of the previous cohomological complexes are (E'_\bullet, d') with $d'_q = \bar{\partial}$ for $0 \leq q \leq n$ and:

- (1) $E'_q = \mathcal{E}^{n-p, n-q}(D)$, the space of currents with compact support in D .
- (2) $E'_q = \mathcal{E}^{n-p, n-q}_D(X)$, the space of currents with compact support in X whose support is contained in \overline{D} .
- (3) $E'_q = \mathcal{D}^{n-p, n-q}_D(X)$, the space of C^∞ -smooth forms with compact support in \overline{D} .

The associated cohomology groups are denoted respectively by $H_c^{n-p, n-q}(D)$, $H_{D, cur}^{n-p, n-q}(X)$ and $H_{D, \infty}^{n-p, n-q}(X)$.

The next proposition is a direct consequence of the Hahn–Banach theorem.

Proposition 2.1 *Let (E^\bullet, d) and (E'_\bullet, d') be two dual complexes, then*

$$\overline{\text{Im } d^q} = \left\{ g \in E^{q+1} \mid \langle g, f \rangle = 0, \forall f \in \text{Ker } d'_q \right\}.$$

Let us recall the main result of the Serre duality (see [3, 19, 25]).

Theorem 2.2 *Let (E^\bullet, d) and (E'_\bullet, d') be two dual complexes. Assume $H_{q+1}(E'_\bullet) = 0$, then either $H^{q+1}(E^\bullet) = 0$ or $H^{q+1}(E^\bullet)$ is not Hausdorff.*

If (E^\bullet, d) is a complex of Fréchet-Schwartz spaces or of dual of Fréchet-Schwartz spaces, then, for any $q \in \mathbb{Z}$, $H^{q+1}(E^\bullet)$ is Hausdorff if and only if $H_q(E'_\bullet)$ is Hausdorff.

As a consequence of the previous theorem and of the solvability of the Cauchy–Riemann equation with prescribed support in the closure of a bounded domain with connected complement in a Stein manifold of dimension $n \geq 2$, in bidegree $(p, 1)$, $0 \leq p \leq n$, we obtain

Theorem 2.3 *Let X be a Stein manifold of complex dimension $n \geq 2$ and $D \subset\subset X$ a relatively compact subset of X such that $X \setminus D$ is connected. Then*

- (i) *Either $H^{0, n-1}(D) = 0$ or $H^{0, n-1}(D)$ is not Hausdorff;*

if moreover D has Lipschitz boundary,

- (ii) *Either $H_\infty^{0, n-1}(\overline{D}) = 0$ or $H_\infty^{0, n-1}(\overline{D})$ is not Hausdorff;*
- (iii) *Either $\dot{H}^{0, n-1}(\overline{D}) = 0$ or $\dot{H}^{0, n-1}(\overline{D})$ is not Hausdorff.*

For a more general result and the proof of this theorem, see Theorem 3.2 in Laurent-Thiébaud and Shaw [20].

3 Characterization of pseudoconvexity by extendable Dolbeault cohomology

3.1 Necessary condition for the outside boundary

Let X be a Stein manifold of complex dimension $n \geq 2$ and Ω a relatively compact domain in X . Let us denote by D the union of all the relatively compact connected components of $X \setminus \overline{\Omega}$ and set $\tilde{\Omega} = \Omega \cup \overline{D}$. Note that $\tilde{\Omega} \setminus D$ is a closed subset of $\tilde{\Omega}$ and we use $H_\infty^{0, q}(\tilde{\Omega} \setminus D)$ to denote the Dolbeault cohomology groups for forms smooth up to the boundary of D .

In his paper [32], Trapani proved that if $H^{0, q}(\Omega) = 0$ for all $1 \leq q \leq n - 2$ and $H^{0, n-1}(\Omega)$ is Hausdorff, then $\tilde{\Omega}$ has to be pseudoconvex. We now prove that this result extends to cohomology with growth or regularity conditions up to the boundary.

Theorem 3.1 *Let X , Ω and $\tilde{\Omega}$ be as above. Then*

- (i) for each $q \in \mathbb{N}$ with $1 \leq q \leq n - 2$, $H_{\infty}^{0,q}(\tilde{\Omega} \setminus D) = 0$ implies $H^{0,q}(\tilde{\Omega}) = 0$;
- (ii) $H_{\infty}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff implies $H^{0,n-1}(\tilde{\Omega}) = 0$

Proof Let us first prove (i). Let $f \in C_{0,q}^{\infty}(\tilde{\Omega})$ be a $\bar{\partial}$ -closed form. Then its restriction to $\tilde{\Omega} \setminus D$ is a $\bar{\partial}$ -closed smooth form on $\tilde{\Omega} \setminus D$. Since $H_{\infty}^{0,q}(\tilde{\Omega} \setminus D) = 0$, there exists a form $g \in C_{0,q-1}^{\infty}(\tilde{\Omega} \setminus D)$ such that $f = \bar{\partial}g$ on Ω . Let \tilde{g} be a smooth extension of g to $\tilde{\Omega}$. Then $f - \bar{\partial}\tilde{g}$ is smooth and $\bar{\partial}$ -closed on $\tilde{\Omega}$ and $f - \bar{\partial}\tilde{g}$ vanishes on $\tilde{\Omega} \setminus D$. Therefore we can extend $f - \bar{\partial}\tilde{g}$ by 0 to a $\bar{\partial}$ -closed form on X . Since X is Stein, there exists a smooth form u on X such that $f = \bar{\partial}(\tilde{g} + u)$ on $\tilde{\Omega}$.

Let us now consider the assertion (ii). We first prove that $H^{0,n-1}(\tilde{\Omega})$ is Hausdorff. Let $f \in C_{0,n-1}^{\infty}(\tilde{\Omega})$ be such that, for any $\bar{\partial}$ -closed $(n, 1)$ -current T with compact support in $\tilde{\Omega}$, $\langle T, f \rangle = 0$. Then for any $(n, 1)$ -current S with compact support in $\tilde{\Omega} \setminus D$, we have

$$\langle S, f \rangle = 0.$$

In particular, f is orthogonal to any current with coefficients in the dual space of $C^{\infty}(\tilde{\Omega} \setminus D)$.

Using the assumptions that $H_{\infty}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff, we get from Proposition 2.1 the existence of a form $g \in C_{0,n-2}^{\infty}(\tilde{\Omega} \setminus D)$ such that $f = \bar{\partial}g$ on Ω . Then following the proof of assertion (i) we obtain that $H^{0,n-1}(\tilde{\Omega})$ is Hausdorff. Since $X \setminus \tilde{\Omega}$ is connected, it follows from Theorem 2.3 (see also Theorem 3.2 in [20] and Theorem 2 in [31]) that $H^{0,n-1}(\tilde{\Omega}) = 0$. \square

It is well known (see, for example, Corollary 4.2.6 and Theorem 4.2.9 in [12]) that a domain U in \mathbb{C}^n is pseudoconvex if and only if we have $H^{0,q}(U) = 0$ for all $1 \leq q \leq n - 1$. So from Theorem 3.1 we can deduce.

Corollary 3.2 *Let $n \geq 2$ and $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume $H_{\infty}^{0,q}(\tilde{\Omega} \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{\infty}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff, then $\tilde{\Omega}$ is pseudoconvex.*

Note that, replacing smooth forms up to the boundary by extendable currents in the proof of Theorem 3.1, we get:

Proposition 3.3 *Let $n \geq 2$ and $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume that $\check{H}^{0,q}(\tilde{\Omega} \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $\check{H}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff. Then $\tilde{\Omega}$ is pseudoconvex.*

Let us notice that the vanishing or the Hausdorff property of the Dolbeault cohomology groups of the annulus $\Omega = \tilde{\Omega} \setminus D$ are in fact independent of the larger domain $\tilde{\Omega}$ as soon as it satisfies some cohomological conditions.

Proposition 3.4 *Let $D \subset\subset \tilde{\Omega}_1 \subset\subset \tilde{\Omega}_2$ be bounded domains in \mathbb{C}^n such that $H^{0,q}(\tilde{\Omega}_2) = 0$ for some q , $1 \leq q \leq n - 1$. Then we have the following:*

- (i) $H_{\infty}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$ implies $H_{\infty}^{0,q}(\tilde{\Omega}_2 \setminus D) = 0$,
- (i') $\check{H}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$ implies $\check{H}^{0,q}(\tilde{\Omega}_2 \setminus D) = 0$,

and

- (ii) $H_{\infty}^{0,q}(\tilde{\Omega}_1 \setminus D)$ is Hausdorff implies $H_{\infty}^{0,q}(\tilde{\Omega}_2 \setminus D)$ is Hausdorff,
- (ii') $\check{H}^{0,q}(\tilde{\Omega}_1 \setminus D)$ is Hausdorff implies $\check{H}^{0,q}(\tilde{\Omega}_2 \setminus D)$ is Hausdorff.

Proof Let us first prove (i). Let $f \in C_{0,q}^{\infty}(\tilde{\Omega}_2 \setminus D)$ such that $\bar{\partial}f = 0$. Using the assumption $H_{\infty}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$, we get a form $u \in C_{0,q-1}^{\infty}(\tilde{\Omega}_1 \setminus D)$, which satisfies $\bar{\partial}u = f$ on $\tilde{\Omega}_1 \setminus D$. Let

χ be a smooth function on \mathbb{C}^n with support in $\tilde{\Omega}_1$ and identically equal to 1 on a neighborhood of \bar{D} . Consider $\bar{\partial}(\chi u)$, it belongs to $C_{0,q}^\infty(\tilde{\Omega}_2 \setminus D)$ and is such that $f - \bar{\partial}(\chi u)$ vanishes near the boundary of D . After extending by zero in D , we get a $\bar{\partial}$ -closed form on $\tilde{\Omega}_2$, still denoted by $f - \bar{\partial}(\chi u)$. Since $H_{0,q}^{0,q}(\tilde{\Omega}_2) = 0$, there exists $v \in C_{0,q-1}^\infty(\tilde{\Omega}_2)$ such that $\bar{\partial}v = f - \bar{\partial}(\chi u)$. After restriction to $\tilde{\Omega}_2 \setminus D$, we get $f = \bar{\partial}(\chi u + v)$ on $\tilde{\Omega}_2 \setminus \bar{D}$.

For assertion (ii), note that if $f \in C_{0,q}^\infty(\tilde{\Omega}_2 \setminus \bar{D})$ is orthogonal to $\bar{\partial}$ -closed currents with compact support in $\tilde{\Omega}_2 \setminus D$, it is orthogonal to $\bar{\partial}$ -closed currents with compact support in $\tilde{\Omega}_1 \setminus D$, then the proof follows the same argument used to prove (i). The proofs of (i') and (ii') are similar by substituting smooth forms with currents. \square

Corollary 3.5 *Let $D \subset\subset \tilde{\Omega}_1 \subset\subset \tilde{\Omega}_2$ be bounded domains in \mathbb{C}^n such that $\tilde{\Omega}_2$ is pseudoconvex.*

(i) *Assume that $H_{0,q}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{0,q}^{0,n-1}(\tilde{\Omega}_1 \setminus D)$ is Hausdorff.*

Then $H_{0,q}^{0,q}(\tilde{\Omega}_2 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{0,q}^{0,n-1}(\tilde{\Omega}_2 \setminus D)$ is Hausdorff.

(ii) *Assume that $\check{H}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $\check{H}^{0,n-1}(\tilde{\Omega}_1 \setminus D)$ is Hausdorff.*

Then $\check{H}^{0,q}(\tilde{\Omega}_2 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $\check{H}^{0,n-1}(\tilde{\Omega}_2 \setminus D)$ is Hausdorff.

Note that by Corollary 3.2 and Proposition 3.3, if $\Omega_1 = \tilde{\Omega}_1 \setminus D$ is connected, each of the hypothesis in (i) or (ii) of Corollary 3.5 forces $\tilde{\Omega}_1$ to be pseudoconvex. Note also that if $\Omega_2 = \tilde{\Omega}_2 \setminus D$ is connected, the condition $\tilde{\Omega}_2$ pseudoconvex is necessary for the conclusion in (i) or (ii) of Corollary 3.5 to hold.

3.2 Necessary condition for the inside boundary

Let X be a connected, complex manifold and D a relatively compact open subset of X with Lipschitz boundary. If D is not a domain (i.e., a connected open set), then the Lipschitz boundary assumption implies that D is a finite union of domains. The Dolbeault cohomology groups for D are the direct sum of the corresponding cohomology groups on each connected components. In this section we will prove that in any dimension $n \geq 2$ and for some Dolbeault cohomology groups on $X \setminus D$, such as forms smooth up to the boundary, vanishing and Hausdorff properties of these groups implies pseudoconvexity for the domain D , provided its boundary is sufficiently smooth. A related result for $n = 2$ was proved in Trapani [30].

We will first relate the Hausdorff property or the vanishing of the cohomology groups with prescribed support in \bar{D} with the same property for the cohomology groups of $X \setminus D$.

Proposition 3.6 *Assume $H^{0,q-1}(X) = 0$, for some $2 \leq q \leq n$. We have*

(i) *if $H_{D,\infty}^{0,q}(X)$ is Hausdorff, then $H_{0,q-1}^{0,q-1}(X \setminus D)$ is Hausdorff.*

(ii) *if $H_{D,cur}^{0,q}(X)$ is Hausdorff, then $\check{H}^{0,q-1}(X \setminus D)$ is Hausdorff.*

Proof Let $f \in C_{0,q-1}^\infty(X \setminus D)$ be a $\bar{\partial}$ -closed form such that for any $\bar{\partial}$ -closed $(n, n - q + 1)$ -current T on X with compact support in $X \setminus D$, we have $\langle T, f \rangle = 0$. Let \tilde{f} be a smooth extension of f to X , then $\bar{\partial}\tilde{f}$ is a $\bar{\partial}$ -closed smooth $(0, q)$ -form on X with support in \bar{D} . Let us prove that for any $\bar{\partial}$ -closed $(n, n - q)$ -current S on D extendable to a current in X , we have $\langle S, \bar{\partial}\tilde{f} \rangle = 0$. Let \tilde{S} be an extension of S to X with compact support, then, we get

$$\langle S, \bar{\partial}\tilde{f} \rangle = \langle \tilde{S}, \bar{\partial}\tilde{f} \rangle = \langle \bar{\partial}\tilde{S}, \tilde{f} \rangle$$

and since $T = \bar{\partial}\tilde{S}$ is a $\bar{\partial}$ -closed $(n, n - q + 1)$ -current on X with compact support in $X \setminus D$, the orthogonality property of f implies

$$\langle \bar{\partial}\tilde{S}, \tilde{f} \rangle = \langle \bar{\partial}\tilde{S}, f \rangle = 0.$$

By hypothesis $H_{D,\infty}^{0,q}(X)$ is Hausdorff, therefore there is a smooth $(0, q - 1)$ -form g in X with support in \bar{D} such that $\bar{\partial}\tilde{f} = \bar{\partial}g$ on X . Hence $\tilde{f} - g$ is a $\bar{\partial}$ -closed smooth $(0, q - 1)$ -form on X , whose restriction to $X \setminus D$ is equal to f . Therefore, $H^{0,q-1}(X) = 0$ implies that there exists a smooth $(0, q - 2)$ -form h on X such that $\bar{\partial}h = \tilde{f} - g$ and by restriction to $X \setminus D$ we get $\bar{\partial}h = f$ on $X \setminus \bar{D}$. This proves that $H_{\infty}^{0,q-1}(X \setminus D)$ is Hausdorff. Assertion (ii) can be proved in the same way. □

Proposition 3.7 *Assume $H^{0,q}(X) = 0$ and either $H^{n,n-q+1}(X) = 0$ or $H_c^{n,n-q+1}(X) = 0$, for some $2 \leq q \leq n$. We have*

- (i) *if $H_{\infty}^{0,q-1}(X \setminus D)$ is Hausdorff, then $H_{D,\infty}^{0,q}(X)$ is Hausdorff.*
- (ii) *if $\check{H}^{0,q-1}(X \setminus \bar{D})$ is Hausdorff, then $H_{D,cur}^{0,q}(X)$ is Hausdorff.*

Proof Let f be a $\bar{\partial}$ -closed $(0, q)$ -form on X with support contained in \bar{D} such that for any $\bar{\partial}$ -closed $(n, n - q)$ -current T on D extendable as a current to X , we have $\langle T, f \rangle = 0$. Since $H^{0,q}(X) = 0$, there exists a smooth $(0, q - 1)$ -form g on X such that $\bar{\partial}g = f$ on X . In particular, $\bar{\partial}g = 0$ on $X \setminus \bar{D}$.

Let S be a $\bar{\partial}$ -closed $(n, n - q + 1)$ -current on X with compact support in $X \setminus D$. Since $H^{n,n-q+1}(X) = 0$ or $H_c^{n,n-q+1}(X) = 0$, there exists a $(n, n - q)$ -current U on X such that $\bar{\partial}U = S$. (Here and hereafter we use $H_c^{n,n-q}(X)$ to denote the Dolbeault cohomology groups with compact support in X .) Hence $\bar{\partial}U = 0$ on D . Thus

$$\langle S, g \rangle = \langle \bar{\partial}U, g \rangle = \langle U, \bar{\partial}g \rangle = \langle U, f \rangle = 0,$$

by hypothesis on f . Therefore, the Hausdorff property of $H_{\infty}^{0,q-1}(X \setminus D)$ implies that there exists a smooth $(0, q - 2)$ -form h on $X \setminus D$ such that $\bar{\partial}h = g$. Let \tilde{h} be a smooth extension of h to X . Then $u = g - \bar{\partial}\tilde{h}$ is a smooth form with support in \bar{D} and

$$\bar{\partial}u = \bar{\partial}(g - \bar{\partial}\tilde{h}) = \bar{\partial}g = f.$$

Assertion (ii) is proved in the same way. □

Corollary 3.8 *Let X be a complex manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X with Lipschitz boundary. Assume $H^{0,q}(X) = 0$ and $H^{0,q+1}(X) = 0$, for some $1 \leq q \leq n - 1$, then*

$$\begin{aligned} H_{D,\infty}^{0,q+1}(X) \text{ is Hausdorff} &\Leftrightarrow H_{\infty}^{0,q}(X \setminus D) \text{ is Hausdorff,} \\ H_{D,cur}^{0,q+1}(X) \text{ is Hausdorff} &\Leftrightarrow \check{H}^{0,q}(X \setminus \bar{D}) \text{ is Hausdorff.} \end{aligned}$$

Proof By Theorem 2.2, the assumption $H^{0,q+1}(X) = 0$ implies that $H_c^{n,n-q}(X)$ is Hausdorff. Since $H^{0,q}(X) = 0$, we have $H_c^{n,n-q}(X) = 0$. The corollary then follows by applying Propositions 3.6 and 3.7. □

By applying the Serre duality for the complexes $(\mathcal{E}^{m,\bullet}(X), \bar{\partial})$ and $(\mathcal{D}^{n,\bullet}(X), \bar{\partial})$, we obtain.

Corollary 3.9 *Assume that X is a Stein manifold and D a relatively compact open subset of X with Lipschitz boundary. Then for any $1 \leq q \leq n - 1$,*

- (i) $H_{\infty}^{0,q}(X \setminus D)$ is Hausdorff if and only if $\check{H}^{n,n-q}(\overline{D})$ is Hausdorff;
- (ii) $\check{H}^{0,q}(X \setminus D)$ is Hausdorff if and only if $H_{\infty}^{n,n-q}(\overline{D})$ is Hausdorff.

In the special case $q = 1$, the following theorem is a direct consequence of Corollary 3.9 and Theorem 2.3.

Theorem 3.10 *Let X be a Stein manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X with Lipschitz boundary such that $X \setminus D$ is connected. Then*

- (i) $H_{\infty}^{0,1}(X \setminus D)$ is Hausdorff if and only if $\check{H}^{n,n-1}(\overline{D}) = 0$,
- (ii) $\check{H}^{0,1}(X \setminus D)$ is Hausdorff if and only if $H_{\infty}^{n,n-1}(\overline{D}) = 0$.

Theorem 3.11 *Let X be a complex manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X . Assume $H^{0,q}(X) = 0$, $H^{0,q+1}(X) = 0$, for some $1 \leq q \leq n - 1$, then*

$$H_{D,\infty}^{0,q+1}(X) = 0 \iff H_{\infty}^{0,q}(X \setminus D) = 0;$$

$$H_{D,cur}^{0,q+1}(X) = 0 \iff \check{H}^{0,q}(X \setminus D) = 0.$$

Proof The proof is analogous to the proof of Corollary 3.8, Propositions 3.6, and 3.7. □

Corollary 3.12 *Let X be a Stein manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X with Lipschitz boundary such that $X \setminus D$ is connected. Consider the assertions*

- (i) For all $1 \leq q \leq n - 2$, $\check{H}^{0,q}(X \setminus D) = 0$ and $\check{H}^{0,n-1}(X \setminus D)$ is Hausdorff;
- (ii) For all $2 \leq q \leq n - 1$, $H_{D,cur}^{0,q}(X) = 0$ and $H_{D,cur}^{0,n}(X)$ is Hausdorff;
- (iii) for all $1 \leq q \leq n - 1$, $H_{\infty}^{n,q}(\overline{D}) = 0$.
- (i') For all $1 \leq q \leq n - 2$, $H_{\infty}^{0,q}(X \setminus D) = 0$ and $H_{\infty}^{0,n-1}(X \setminus D)$ is Hausdorff;
- (ii') For all $2 \leq q \leq n - 1$, $H_{D,\infty}^{0,q}(X) = 0$ and $H_{D,\infty}^{0,n}(X)$ is Hausdorff;
- (iii') for all $1 \leq q \leq n - 1$, $\check{H}^{n,q}(\overline{D}) = 0$.

Then the triplets of assertions (i), (ii) and (iii), respectively (i'), (ii') and (iii'), are equivalent.

Proof Using the Serre duality the corollary follows from Corollary 3.9, Theorem 3.10 and Theorem 3.11. □

Note that, under the hypotheses of Corollary 3.12, the assertions (ii) and (ii') still hold for $q = 0$ and $q = 1$ by analytic continuation and the fact that $H_c^{0,1}(X) = 0$ and $X \setminus D$ is connected.

As mentioned above in Sect. 3.1, a domain D in \mathbb{C}^n is pseudoconvex if and only if we have $H^{0,q}(D) = 0$ for all $1 \leq q \leq n - 1$. Analogous results also hold for the Dolbeault cohomology of forms smooth up to the boundary for any n and of extendable currents for $n = 2$.

Theorem 3.13 *Let $D \subset \mathbb{C}^n$ be a domain such that $\text{interior}(\overline{D}) = D$. If $H_{\infty}^{0,q}(\overline{D})$ is finite dimensional for any $1 \leq q \leq n - 1$, then D is pseudoconvex. Moreover, when $n = 2$, D is pseudoconvex provided $\check{H}^{0,1}(\overline{D})$ is finite dimensional.*

Proof Following Laufer’s argument ([17]; see also Theorem 5.1 [18]), we obtain that if $H_\infty^{0,q}(\bar{D})$ (respectively $\check{H}^{0,q}(\bar{D})$) is finite dimensional, then $H_\infty^{0,q}(\bar{D}) = 0$ (respectively $\check{H}^{0,q}(\bar{D}) = 0$). The Laufer’s argument can be applied here because the spaces $\mathcal{E}(\bar{D})$ and $\check{D}'(\bar{D})$ are invariant under differentiation and multiplication by polynomials. Thus we can assume that $H_\infty^{0,q}(\bar{D}) = 0$ for all $1 \leq q \leq n - 1$ or in the case when $n = 2$, $\check{H}^{0,1}(\bar{D}) = 0$.

The proof uses the forms introduced in Laufer [16] (see also [8]). We will prove by contradiction. Suppose that D is not pseudoconvex. Then there exists a domain \tilde{D} strictly containing D such that any holomorphic function on D extends holomorphically to \tilde{D} . Since $\text{interior}(\bar{D}) = D$, after a translation and a rotation we may assume that $0 \in \tilde{D} \setminus \bar{D}$ and there exists a point z_0 in the intersection of the plane $\{(z_1, \dots, z_n) \in \mathbb{C}^2 \mid z_1 = \dots = z_{n-1} = 0\}$ with D that belongs to the same connected components (as the origin) of the intersection of that plane with \tilde{D} .

For any integer q such that $1 \leq q \leq n$ and $\{k_1, \dots, k_q\} \subset \{1, \dots, n\}$, we set

$$u(k_1, \dots, k_q) = \frac{(q - 1)!}{|z|^{2q}} \sum_{j=1}^q (-1)^j \bar{z}_{k_j} \widehat{d\bar{z}_{k_j}},$$

where $\widehat{d\bar{z}_{k_j}} = d\bar{z}_{k_1} \wedge \dots \wedge \widehat{d\bar{z}_{k_j}} \wedge \dots \wedge d\bar{z}_{k_q}$. (Here, as usual, $\widehat{d\bar{z}_{k_j}}$ indicates the deletion of $d\bar{z}_{k_j}$ from the wedge product.) Note that $u(k_1, \dots, k_q)$ is a smooth form on $\mathbb{C}^n \setminus \{0\}$. Since $0 \notin \bar{D}$, $u(k_1, \dots, k_q) \in C^\infty_{(0,q-1)}(\bar{D})$. Moreover, $u(k_1, \dots, k_q)$ is skew-symmetric with respect to the indexes in the tuple (k_1, \dots, k_q) . In particular, $u(k_1, \dots, k_q) = 0$ when two k_j ’s are identical. A direct calculation yields that

$$\bar{\partial}u(k_1, \dots, k_q) = \sum_{l=1}^n z_l u(l, k_1, \dots, k_q). \tag{3.1}$$

For any $1 \leq q \leq n - 1$, we consider the following assertion.

H(q): For all integer $r < q$ and all multi-index $K = (k_1, \dots, k_r)$, setting $K = \emptyset$ if $r = 0$, there exists a smooth $(0, n - r - 2)$ -form $v(K)$ on \bar{D} such that

$$\bar{\partial}v(K) = \sum_{j=1}^r (-1)^j z_{k_j} v(K \setminus k_j) + (-1)^{r+|K|} u((1, \dots, n) \setminus K),$$

where $|K| = k_1 + \dots + k_r$ and $(K \setminus J)$ denotes the tuple of remaining indexes after deleting those in J from K .

Note that $u(1, \dots, n)$ is a $\bar{\partial}$ -closed smooth $(0, n - 1)$ -form on $\mathbb{C}^n \setminus \{0\}$ which contains \bar{D} . Since $H_\infty^{0,n-1}(\bar{D}) = 0$, there exists a smooth $(0, n - 2)$ -form $v(\emptyset)$ on \bar{D} such that $\bar{\partial}v(\emptyset) = u(1, \dots, n)$. Therefore **H(1)** is satisfied.

Let us prove now that if $1 \leq q \leq n - 2$ and **H(q)** is satisfied, then **H(q+1)** is satisfied. It is sufficient to prove the existence of the $v(K)$ ’s satisfying the assertion **H(q+1)** for any multi-index of length q , the other ones for $r < q$ already exist by **H(q)**. Let $K = (k_1, \dots, k_q)$, we set

$$w(K) = \sum_{j=1}^q (-1)^j z_j v(K \setminus k_j) + (-1)^{q+|K|} u((1, \dots, n) \setminus K).$$

The $(0, n - q - 1)$ -form $w(K)$ is smooth on \bar{D} and moreover, using (3.1) and the hypothesis **H(q)**, a straightforward calculation proves that $\bar{\partial}w(K) = 0$ on D . Since $H_\infty^{0,n-q-1}(\bar{D}) = 0$, there exists a smooth $(0, n - q - 2)$ -form $v(K)$ on \bar{D} such that

$$\bar{\partial}v(K) = w(K) = \sum_{j=1}^q (-1)^j z_j v(K \setminus k_j) + (-1)^{q+|K|} u((1, \dots, n) \setminus K).$$

A finite induction process implies that $\mathbf{H}(n - 1)$ is satisfied and for $K = (1, \dots, n - 1)$, we can consider the function

$$F = w(1, \dots, n - 1) = \sum_{j=1}^{n-1} (-1)^j z_j v((1, \dots, n - 1) \setminus j) - (-1)^{n+\frac{n(n-1)}{2}} u(n).$$

It is smooth on \bar{D} and satisfies $\bar{\partial}F = 0$ on D . As F is a holomorphic function on D , it can be extended holomorphically to \tilde{D} . However, we have $F(0, \dots, z_n) = (-1)^{1+\frac{n(n-1)}{2}} \frac{1}{z_n}$ on $D \cap \{z_1 = \dots = z_{n-1} = 0\}$, which is holomorphic and singular at $z_n = 0$, which gives the contradiction since $0 \in \tilde{D} \setminus D$.

When $n = 2$, let us denote by $B(z_1, z_2)$ the $(0, 1)$ -form $\frac{\bar{z}_1 d\bar{z}_2 - \bar{z}_2 d\bar{z}_1}{|z|^4}$ derived from the Bochner–Martinelli kernel in \mathbb{C}^2 , it is a $\bar{\partial}$ -closed form on $\mathbb{C}^2 \setminus \{0\}$. Then the L^1 -function $\frac{\bar{z}_2}{|z|^2}$ defines a distribution in \mathbb{C}^2 which satisfies $\bar{\partial} \left(\frac{-\bar{z}_2}{|z|^2} \right) = z_1 B(z_1, z_2)$ on $\mathbb{C}^2 \setminus \{0\}$. On the other hand, if $\check{H}^{0,1}(\bar{D}) = 0$, there exists an extendable distribution v such that $\bar{\partial}v = B$ on D and by regularity of the $\bar{\partial}$ in bidegree $(0, 1)$, v is smooth on D , since B is smooth on $\mathbb{C}^2 \setminus \{0\}$. Set $F = z_1 v + \frac{\bar{z}_2}{|z|^2}$, then F is a holomorphic function on D , so it extends holomorphically to \tilde{D} , but we have $F(0, z_2) = \frac{1}{z_2}$ on $D \cap \{z_1 = 0\}$, which is holomorphic and singular at $z_2 = 0$. This gives the contradiction since $0 \in \tilde{D} \setminus D$. □

Returning to the case when $\Omega = \tilde{\Omega} \setminus \bar{D}$ is a bounded domain in \mathbb{C}^n , where D is the union of all relatively compact connected components of $\mathbb{C}^n \setminus \bar{\Omega}$ as in Sect. 3.1. We can easily derive the following corollary from Corollary 3.12 and the results of Sect. 3.1.

Corollary 3.14 *Let $D \subset\subset \tilde{\Omega}$ be bounded open subsets of \mathbb{C}^n , $n \geq 2$, such that $\mathbb{C}^n \setminus \tilde{\Omega}$ is connected. Assume D has Lipschitz boundary and $\Omega = \tilde{\Omega} \setminus \bar{D}$ is connected. Consider the assertions*

- (i) *For all $1 \leq q \leq n - 2$, $\check{H}^{0,q}(\tilde{\Omega} \setminus D) = 0$ and $\check{H}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff;*
- (ii) *for all $1 \leq q \leq n - 1$, $H_\infty^{n,q}(\bar{D}) = 0$ and $\tilde{\Omega}$ is pseudoconvex.*
- (i') *For all $1 \leq q \leq n - 2$, $H_\infty^{0,q}(\tilde{\Omega} \setminus D) = 0$ and $H_\infty^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff;*
- (ii') *for all $1 \leq q \leq n - 1$, $\check{H}^{n,q}(\bar{D}) = 0$ and $\tilde{\Omega}$ is pseudoconvex.*

Then the pairs of assertions (i) and (ii), respectively (i') and (ii'), are equivalent.

We are now in position to give characterizations of pseudoconvexity of the inside and outside boundaries for domains with holes in terms of their Dolbeault cohomology on various spaces, depending on the regularity of the boundary of the holes.

Corollary 3.15 *Let $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n , $n \geq 2$, such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume D has smooth boundary. Then $\check{H}^{0,q}(\tilde{\Omega} \setminus D) = 0$, for all $1 \leq q \leq n - 2$, and $\check{H}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff if and only if $\tilde{\Omega}$ and D are pseudoconvex.*

Proof The necessary condition is a direct consequence of Corollary 3.14 and Theorem 3.13. To get the sufficient condition, we use Kohn’s result [15], which asserts that if D is a pseudoconvex domain with smooth boundary then $H_\infty^{n,q}(\bar{D}) = 0$, for all $1 \leq q \leq n - 1$ and apply Corollary 3.14. □

The following corollary is an analogous of Corollary 3.15 for smooth forms in \mathbb{C}^2 under the assumption that D has Lipschitz boundary. The reason for the restriction to dimension 2 comes from the proof of Theorem 3.13. Note that in the cases of the extendable current cohomology, since the restriction of a distribution to a complex hyperplane does not exist in general (see also the remark before Lemma 5.2), we need the regularity property for the $\bar{\partial}$ -operator which in this case holds only in bidegree $(0, 1)$.

Corollary 3.16 *Let $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^2 such that both $\mathbb{C}^2 \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume D has Lipschitz boundary. Then $H_{\infty}^{0,1}(\tilde{\Omega} \setminus D)$ is Hausdorff if and only if $\tilde{\Omega}$ and D are pseudoconvex.*

Proof The necessary condition is a direct consequence of Corollary 3.14 and Theorem 3.13. The sufficient condition follows from Theorem 5 in Chakrabarti and Shaw [4]. □

Remark 3.17 One can only characterize pseudoconvexity for such domain Ω with holes by using the right cohomology groups. In an earlier paper by Trapani [30], he proved that if Ω is the annulus between a pseudoconvex domain and some Diederich–Fornaess worm domain $D \Subset \mathbb{C}^2$ with smooth boundary, the classical Dolbeault cohomology group $H^{0,1}(\Omega)$ is not Hausdorff. Thus if we replace $H_{\infty}^{0,1}(\tilde{\Omega} \setminus D)$ by $H^{0,1}(\tilde{\Omega} \setminus \bar{D})$, Corollary 3.16 does not hold.

4 Characterization of pseudoconvexity by L^2 and W^1 Dolbeault cohomology

Let X be a Stein manifold of dimension $n \geq 2$, equipped with a hermitian metric. Let Ω be a bounded domain in X . Let $L^2_{p,q}(\Omega)$ be the space of (p, q) -forms with L^2 -coefficients. Let $\bar{\partial}_{p,q} : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q+1}(\Omega)$ be the densely defined closed operator such that its domain consists of all $f \in L^2_{p,q}(\Omega)$ such that $\bar{\partial}_{p,q}f$, defined in the sense of distribution, is in $L^2_{p,q+1}(\Omega)$. Let $\bar{\partial}^*_{p,q}$ be its Hilbert space adjoint. We drop the subscript p, q when there is no danger of confusion.

Let $\bar{\partial}_c : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q+1}(\Omega)$ be the minimal (strong) closure of $\bar{\partial}$. By this we mean that $f \in \text{Dom}(\bar{\partial}_c)$ if and only if that there exists a sequence of smooth forms f_v in $C^\infty_{p,q}(\Omega)$ compactly supported in Ω such that $f_v \rightarrow f$ and $\bar{\partial}f_v \rightarrow \bar{\partial}f$ in L^2 . Let $\vartheta = \bar{\partial}^*_c$ be the dual of $\bar{\partial}_c$. Then ϑ is equal to the maximal (weak) L^2 closure of the operator $\vartheta : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q-1}(\Omega)$. We also define an operator $\bar{\partial}_{\bar{c}} : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q+1}(\Omega)$ to be the closure of $\bar{\partial}$ such that f is in the domain of $\bar{\partial}_{\bar{c}}$ if and only if we extend f to be zero outside $\bar{\Omega}$, there exists a $L^2_{p,q+1}(X)$ form g supported in $\bar{\Omega}$ such that $\bar{\partial}f = g$ in X . The operator $\bar{\partial}_{\bar{c}}$ corresponds to solving $\bar{\partial}$ with prescribed support on $\bar{\Omega}$ in the L^2 sense.

We denote the cohomology groups in $L^2_{p,q}(\Omega)$ with respect to $\bar{\partial}$ and $\bar{\partial}_c$ by $H_{L^2}^{p,q}(\Omega)$ and $H_{c,L^2}^{p,q}(\Omega)$. The cohomology group for $\bar{\partial}_{\bar{c}}$ is denoted by $H_{\bar{c},L^2}^{p,q}(\Omega)$. Note that

$$H_{\bar{c},L^2}^{p,q}(\Omega) = H_{\bar{c},L^2}^{p,q}(X).$$

Suppose that Ω is a bounded domain in X with Lipschitz boundary. Then the weak and strong maximal (or minimal) L^2 extensions are the same by Friedrichs’ lemma (see [13], [6]). When Ω has Lipschitz boundary, we have $\bar{\partial}_c = \bar{\partial}_{\bar{c}}$ and

$$H_{c,L^2}^{p,q}(\Omega) = H_{\bar{c},L^2}^{p,q}(X). \tag{4.1}$$

(For a proof of this fact, see, e.g., Lemma 2.4 in [20]).

The cohomology groups for the $\bar{\partial}$ -complexes on forms with W^1 or W^1_{loc} coefficients are defined similarly and denoted by $H_{W^1}^{p,q}(\Omega)$ and $H_{W^1_{loc}}^{p,q}(\Omega)$ respectively. We will use $W^1_{loc}(X \setminus \Omega)$ to denote the space of functions with W^1 coefficients on compact subsets of $X \setminus \Omega$ and $H_{W^1_{loc}}^{p,q}(X \setminus \Omega)$ to denote the Dolbeault cohomology groups for forms with coefficients in $W^1_{loc}(X \setminus \Omega)$. Thus the space $H_{W^1_{loc}}^{p,q}(X \setminus \Omega)$ is different from the space of $H_{W^1_{loc}}^{p,q}(X \setminus \bar{\Omega})$, which is isomorphic to the usual Dolbeault cohomology group $H^{p,q}(X \setminus \bar{\Omega})$. The dual space of $W^1_{loc}(X \setminus \Omega)$ is denoted by $W^{-1}(X \setminus \Omega)$. The dual complex is denoted by $\bar{\partial}_c : W_{p,q-1}^{-1}(X \setminus \Omega) \rightarrow W_{p,q}^{-1}(X \setminus \Omega)$ and its corresponding cohomology groups are denoted by $H_{c,W^{-1}}^{p,q}(X \setminus \Omega)$. Again, when the boundary of Ω is Lipschitz, this corresponds to solving $\bar{\partial}$ in $W^{-1}(X)$ spaces with compact support in $X \setminus \Omega$.

We first establish the following version of the Hartogs phenomenon. Without loss of generality, we will deal with only $(0, q)$ -forms.

Lemma 4.1 *Let $\tilde{\Omega}$ be a domain in a Stein manifold X equipped with a hermitian metric and let K be a compact subset of $\tilde{\Omega}$. Let $\Omega = \tilde{\Omega} \setminus K$. Then $\bar{\partial}_{q-1}^{\tilde{\Omega}}$ has closed range provided $\bar{\partial}_q^{\Omega}$ has closed range.*

Proof It suffices to show that there exists a constant $C > 0$ such that for any $f \in \text{dom} \left(\bar{\partial}_{q-1}^{\tilde{\Omega}} \right)$, one can find $u \in \text{dom} \left(\bar{\partial}_{q-1}^{\tilde{\Omega}} \right)$ such that

$$\begin{cases} \bar{\partial}u = \bar{\partial}f & \text{on } \tilde{\Omega}, \\ \|u\|_{\tilde{\Omega}} \leq C \|\bar{\partial}f\|_{\tilde{\Omega}}. \end{cases} \tag{4.2}$$

(see, e.g., [14, Appendix 1]). Evidently, the restriction $f|_{\Omega}$ of f to Ω is in $\text{dom} \left(\bar{\partial}_{q-1}^{\Omega} \right)$.

Thus under the assumption, there exists a $g \in \text{dom} \left(\bar{\partial}_{q-1}^{\Omega} \right)$ such that

$$\begin{cases} \bar{\partial}g = \bar{\partial}f & \text{on } \Omega, \\ \|g\|_{\Omega} \leq C_1 \|\bar{\partial}f\|_{\Omega}, \end{cases}$$

where C_1 is independent of f . Let χ be a smooth cut-off function such that $0 \leq \chi \leq 1, \chi = 1$ in a neighborhood of K , and $\text{supp } \chi \subset \subset \tilde{\Omega}$. Consider $\alpha = \bar{\partial}f - \bar{\partial}(1 - \chi)g = \chi \bar{\partial}f + \bar{\partial}\chi \wedge g$. Then α is a $\bar{\partial}$ -closed form in $L^2_{0,q}(\tilde{\Omega})$ and $\text{supp } \alpha \subset \subset \tilde{\Omega}$. Extend α to 0 outside $\tilde{\Omega}$ and apply Hörmander’s L^2 -estimates to a bounded pseudoconvex domain $\hat{\Omega} \supset \supset \tilde{\Omega}$, we then obtain $v \in \text{dom} \left(\bar{\partial}_{q-1}^{\hat{\Omega}} \right)$ such that

$$\bar{\partial}v = \alpha \text{ on } \hat{\Omega}, \quad \|v\|_{\hat{\Omega}} \leq C_2 \|\alpha\|_{\hat{\Omega}}$$

for some constant $C_2 > 0$. Let $u = v + (1 - \chi)g$. Then u satisfies the desired property (4.2). □

Lemma 4.2 *Let $\tilde{\Omega}$ be a bounded domain with Lipschitz boundary in a Stein manifold X of dimension n such that $X \setminus \bar{\tilde{\Omega}}$ is connected. Then $\mathcal{R} \left(\bar{\partial}_{n-1}^{\tilde{\Omega}} \right) = \mathcal{N} \left(\bar{\partial}_{n-2}^{*\tilde{\Omega}} \right)$ where \mathcal{R} and \mathcal{N} denote the range and the null spaces of the relevant operators.*

Proof Let $f \in \mathcal{N} \left(\overline{\partial_{n-2}^{*\tilde{\Omega}}} \right)$. Let f^0 be the extension of f to X such that $f^0 = 0$ on $X \setminus \tilde{\Omega}$.

Then f^0 is compactly supported and $\vartheta f^0 = 0$ in the sense of distribution on X where ϑ denotes the formal adjoint of $\bar{\partial}$. Let $*$ be the Hodge $*$ -operator given by $\langle u, v \rangle dV = u \wedge *v$ where dV is the volume form. Then $\vartheta = - * \bar{\partial} *$ (see, e.g., [6, § 9.1]). Thus $\bar{\partial}_{n,1}^X (*f^0) = 0$.

Since $\tilde{\Omega}$ has Lipschitz boundary, we have from (4.1) that

$$H_{c,L^2}^{n,1}(\tilde{\Omega}) = H_{\infty,L^2}^{n,1}(X) = H_{c,L^2}^{n,1}(X) = \{0\}.$$

Note that in the last two identities we use the assumptions that X is Stein and $X \setminus \tilde{\Omega}$ is connected. Thus there exists $u \in \text{dom} \left(\bar{\partial}_c^{\tilde{\Omega}} \right)$ such that $\bar{\partial}_c^{\tilde{\Omega}} u = *f$ in X . Since $*u$ is in the domain of $\bar{\partial}^{*\tilde{\Omega}}$, we have $\bar{\partial}^{*\tilde{\Omega}}(*u) = f$ on $\tilde{\Omega}$. The lemma is proved. □

Let Ω be a relatively compact domain in X . We denote by D the union of all the relatively compact connected components of $X \setminus \tilde{\Omega}$ and set $\tilde{\tilde{\Omega}} = \Omega \cup \bar{D}$.

Theorem 4.3 *Let X, Ω and $\tilde{\tilde{\Omega}}$ be as above. Then*

- (i) *for each $1 \leq q \leq n - 2, H_{L^2}^{0,q}(\Omega) = 0$ implies $H_{L^2}^{0,q}(\tilde{\tilde{\Omega}}) = 0$;*
- (ii) *if $\tilde{\tilde{\Omega}}$ has Lipschitz boundary, $H_{L^2}^{0,n-1}(\Omega)$ is Hausdorff implies $H_{L^2}^{0,n-1}(\tilde{\tilde{\Omega}}) = 0$.*

Similarly, we have

- (iii) *for each $1 \leq q \leq n - 2, H_{W_{loc}^1}^{0,q}(\tilde{\tilde{\Omega}} \setminus D) = 0$ implies $H^{0,q}(\tilde{\tilde{\Omega}}) = 0$;*
- (iv) *if D has Lipschitz boundary, $H_{W_{loc}^1}^{0,n-1}(\tilde{\tilde{\Omega}} \setminus D)$ is Hausdorff implies $H^{0,n-1}(\tilde{\tilde{\Omega}}) = 0$.*

Proof The proof of (i) exactly the same as in Theorem 3.1 or Lemma 4.1.

The proof of (ii) follows from the L^2 Serre duality (see [4,20]). We provide a simple proof here for the benefit of the reader.

If $\mathcal{R} \left(\overline{\partial_{n-2}^{\tilde{\tilde{\Omega}}}} \right)$ is closed, then $H_{L^2}^{0,n-1}(\tilde{\tilde{\Omega}})$ is trivial. This is a direct consequence of Lemma 4.2 since

$$\mathcal{R} \left(\overline{\partial_{n-2}^{\tilde{\tilde{\Omega}}}} \right) = \mathcal{N} \left(\overline{\partial_{n-2}^{*\tilde{\tilde{\Omega}}}} \right)^\perp = \mathcal{R} \left(\overline{\partial_{n-1}^{*\tilde{\tilde{\Omega}}}} \right)^\perp = \mathcal{N} \left(\overline{\partial_{n-1}^{\tilde{\tilde{\Omega}}}} \right). \tag{4.3}$$

Note that in the first equality, we use the fact that $\mathcal{R} \left(\overline{\partial_{n-2}^{\tilde{\tilde{\Omega}}}} \right)$ is closed.

The proof of (iii) is analogous to the proof of (i), using interior regularity we can choose g with W^1 coefficients on a neighborhood of the support of $\bar{\partial}\chi$, if $f \in (W_{loc}^1)_{0,q}(\tilde{\tilde{\Omega}})$. We get $H_{W_{loc}^1}^{0,q}(\tilde{\tilde{\Omega}}) = 0$ and by the Dolbeault isomorphism $H^{0,q}(\tilde{\tilde{\Omega}}) = 0$. The proof of (iv) is similar to that of (ii) in Theorem 3.1. □

As in the smooth case, we have:

Corollary 4.4 *Let $n \geq 2$ and $D \subset\subset \tilde{\tilde{\Omega}}$ be two relatively compact open subsets of \mathbb{C}^n such that both $\mathbb{C}^n \setminus \tilde{\tilde{\Omega}}$ and $\tilde{\tilde{\Omega}} \setminus D$ are connected and D has Lipschitz boundary. Assume $H_{W_{loc}^1}^{0,q}(\tilde{\tilde{\Omega}} \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{W_{loc}^1}^{0,n-1}(\tilde{\tilde{\Omega}} \setminus D)$ is Hausdorff, then $\tilde{\tilde{\Omega}}$ is pseudoconvex.*

From Theorem 4.3, we deduce:

Corollary 4.5 *Let $n \geq 2$ and $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected, and $\tilde{\Omega}$ has Lipschitz boundary. Assume $H_{L^2}^{0,q}(\tilde{\Omega} \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{L^2}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff, then $\tilde{\Omega}$ is pseudoconvex.*

Proof We will postpone the proof to the corollary in Theorem 5.1. □

As in the smooth case, the vanishing or the Hausdorff property of the Dolbeault cohomology groups of the annulus $\Omega = \tilde{\Omega} \setminus D$ are in fact independent of the larger domain $\tilde{\Omega}$ as soon as it satisfies some cohomological conditions. The following was proved in Chakrabarti et al. [5].

Corollary 4.6 *Let $D \subset\subset \tilde{\Omega}_1 \subset\subset \tilde{\Omega}_2$ be bounded domains in \mathbb{C}^n such that $\tilde{\Omega}_2$ is pseudoconvex. Assume $H_{L^2}^{0,q}(\tilde{\Omega}_1 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{L^2}^{0,n-1}(\tilde{\Omega}_1 \setminus D)$ is Hausdorff, then $H_{L^2}^{0,q}(\tilde{\Omega}_2 \setminus D) = 0$, if $1 \leq q \leq n - 2$, and $H_{L^2}^{0,n-1}(\tilde{\Omega}_2 \setminus D)$ is Hausdorff.*

Note that by Corollary 4.5, if $\Omega_1 = \tilde{\Omega}_1 \setminus D$ is connected and Ω_1 has Lipschitz boundary, the hypothesis of Corollary 4.6 forces $\tilde{\Omega}_1$ to be pseudoconvex. Note also that if $\Omega_2 = \tilde{\Omega}_2 \setminus D$ is connected and Ω_2 has Lipschitz boundary, the condition $\tilde{\Omega}_2$ pseudoconvex is necessary for the conclusion of Corollary 4.6 to hold.

Next we will develop in the L^2 and W^1 settings what is done in Sect. 3.2 for forms smooth up to the boundary and extendable currents.

Theorem 4.7 *Let X be a Stein manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X with Lipschitz boundary such that $X \setminus D$ is connected. The following assertions are equivalent:*

- (i) *For all $1 \leq q \leq n - 2$, $H_{W^1_{loc}}^{0,q}(X \setminus D) = 0$ and $H_{W^1_{loc}}^{0,n-1}(X \setminus D)$ is Hausdorff;*
- (ii) *For all $2 \leq q \leq n - 1$, $H_{c,W^{-1}}^{n,q}(X \setminus D) = 0$ and $H_{c,W^{-1}}^{n,n}(X \setminus D)$ is Hausdorff;*
- (iii) *For all $2 \leq q \leq n - 1$, $H_{D,L^2_{loc}}^{0,q}(X) = 0$, $H_{D,L^2_{loc}}^{0,n}(X)$ is Hausdorff;*
- (iv) *for all $1 \leq q \leq n - 1$, $H_{L^2}^{n,q}(D) = 0$.*

Proof The Serre duality for the complexes $((W^1_{loc})^{0,\bullet}(X \setminus D), \bar{\partial})$, respectively $((L^2_D)^{n,\bullet}(X), \bar{\partial})$, implies the equivalence between (i) and (ii), respectively (iii) and (iv). Thus it is sufficient to prove that (i) implies (iii) and (iv) implies (ii).

Let us prove now that, for any $2 \leq q \leq n$, if $H_{W^1_{loc}}^{0,q-1}(X \setminus D)$ is Hausdorff, then $H_{D,L^2_{loc}}^{0,q}(X)$ is Hausdorff. Let f be a $\bar{\partial}$ -closed $(0, q)$ -form on X with L^2 coefficients and support contained in \bar{D} such that for any $\bar{\partial}$ -closed L^2 -form u of bidegree $(n, n - q)$ on D , we have $\langle u, f \rangle = 0$. Since $H^{0,q}(X) = 0$ and by interior regularity, there exists a form g in $(W^1_{loc})_{0,q-1}(X)$ such that $\bar{\partial}g = f$ on X , in particular $\bar{\partial}g = 0$ on $X \setminus \bar{D}$.

Let S be a $\bar{\partial}$ -closed $(n, n - q + 1)$ -current on X with compact support in $X \setminus D$ and coefficients in $W^{-1}(X)$, then, since $H_c^{n,n-q+1}(X) = 0$, there exists an $(n, n - q)$ -current U with compact support on X such that $\bar{\partial}U = S$ and in particular $\bar{\partial}U = 0$ on D . Moreover U can be chosen with L^2_{loc} coefficients. Thus

$$\langle S, g \rangle = \langle \bar{\partial}U, g \rangle = \langle U, \bar{\partial}g \rangle = \langle U, f \rangle = 0,$$

by hypothesis on f .

Therefore the Hausdorff property of $H_{W_{loc}^1}^{0,q-1}(X \setminus D)$ implies there exists a $(0, q - 2)$ -form h on $X \setminus D$ coefficients in $W_{loc}^1(X \setminus D)$ such that $\bar{\partial}h = g$. Let \tilde{h} be a W_{loc}^1 -extension of h to X then $u = g - \bar{\partial}\tilde{h}$ is a L^2 -form with support in \bar{D} and

$$\bar{\partial}u = \bar{\partial}(g - \bar{\partial}\tilde{h}) = \bar{\partial}g = f.$$

In the same way we can prove that, for any $2 \leq q \leq n - 1$, if $H_{W_{loc}^1}^{0,q-1}(X \setminus D) = 0$, then $H_{\bar{D}, L_{loc}^2}^{0,q}(X) = 0$.

Assume now that for some $1 \leq q \leq n - 2$, $H_{L^2}^{n,q}(D) = 0$. Let f be a $\bar{\partial}$ -closed form in $W_{n,q+1}^{-1}(X)$ with compact support in $X \setminus D$. Since X is a Stein manifold, there exists a (n, q) -form g with compact support in X such that $\bar{\partial}g = f$ and by interior regularity g can be chosen with L^2 coefficients on D . Since the support of f is contained in $X \setminus D$, g is $\bar{\partial}$ -closed in D and as $H_{L^2}^{n,q}(D) = 0$, we get $g = \bar{\partial}h$ for some $(n, q - 1)$ -form h in $L_{n,q-1}^2(D)$. Let \tilde{h} be the extension of h by 0 outside of D . Then $g - \bar{\partial}\tilde{h}$ vanishes on D and satisfies $\bar{\partial}(g - \bar{\partial}\tilde{h}) = f$. This shows $H_{c, W^{-1}}^{n,q+1}(X \setminus D) = 0$.

To end the proof, assume $H_{L^2}^{n,n-1}(D) = 0$. Let f be a $\bar{\partial}$ -closed form in $W_{n,n}^{-1}(X)$ with compact support in $X \setminus D$ orthogonal to the $\bar{\partial}$ -closed functions, which are W^1 in $X \setminus D$ and in particular to the holomorphic functions in X . The Hausdorff property of $H_{c, W^{-1}}^{n,n}(X)$ implies that there exists an $(n, n - 1)$ -form g with compact support in X such that $\bar{\partial}g = f$ and by interior regularity, g can be chosen with L^2 coefficients on D . As for $1 \leq q \leq n - 2$, we can conclude that $f = \bar{\partial}u$, where u is in $W_{n,n-1}^{-1}(X)$ with compact support in $X \setminus D$. □

Theorem 4.8 *Let X be a Stein manifold of complex dimension $n \geq 2$ and D be a relatively compact open subset of X with Lipschitz boundary such that $X \setminus D$ is connected. The following assertions are equivalent:*

- (i) *For all $1 \leq q \leq n - 2$, $H_{L^2}^{0,q}(X \setminus D) = 0$ and $H_{L^2}^{0,n-1}(X \setminus D)$ is Hausdorff*
- (ii) *For all $2 \leq q \leq n - 1$, $H_{c, L^2}^{n,q}(X \setminus D) = 0$ and $H_{c, L^2}^{n,n}(X \setminus D)$ is Hausdorff*
- (iii) *for all $1 \leq q \leq n - 1$, $H_{W^1}^{n,q}(D) = 0$.*

Proof The equivalence between (i) and (ii) is a direct consequence of the Serre duality (see [4] or Theorem 2.2). From Proposition 4.7 in Laurent-Thiébaud and Shaw [20], we know that $H_{c, L^2}^{n,n}(X \setminus D)$ is Hausdorff if and only if $H_{W^1}^{n,n-1}(D) = 0$. Let us now prove the equivalence between (ii) and (iii) for the other degrees.

Assume (ii) is satisfied. Let $f \in W_{n,q}^1(D)$, $1 \leq q \leq n - 2$, be a $\bar{\partial}$ -closed form on D and let \tilde{f} be a W^1 extension with compact support of f to X . The $(0, q + 1)$ -form $\bar{\partial}\tilde{f}$ has L^2 coefficients and compact support in $X \setminus D$. By (ii), $H_{c, L^2}^{n,q+1}(X \setminus D) = 0$ and therefore there exists a form $g \in L_{n,q}^2(X)$ with compact support in $X \setminus D$ such that $\bar{\partial}\tilde{f} = \bar{\partial}g$. So the form $\tilde{f} - g$ is a $\bar{\partial}$ -closed form on X whose restriction to D is equal to f . As X is a Stein manifold, $\tilde{f} - g = \bar{\partial}h$ for some L_{loc}^2 -form h on X . Then we have $f = \bar{\partial}h$ on D and it follows from interior regularity that we can choose h to belong to $W_{n,q-1}^1(D)$, which proves $H_{W^1}^{n,q}(D) = 0$.

Let us prove the converse. Let $f \in L_{n,q}^2(X)$, $2 \leq q \leq n - 1$, be a $\bar{\partial}$ -closed form with compact support in $X \setminus D$. Since X is a Stein manifold, there exists a $(n, q - 1)$ -form g with compact support in X such that $\bar{\partial}g = f$ and by interior regularity g can be chosen with W^1 coefficients on D . Since the support of f is contained in $X \setminus D$, g is $\bar{\partial}$ -closed in D and as $H_{W^1}^{n,q-1}(D) = 0$, we get $g = \bar{\partial}h$ for some $(n, q - 2)$ -form h in $W_{n,q-2}^1(D)$. Let \tilde{h} be a

W^1 extension of h with compact support in X , which exists since D is a relatively compact domain with Lipschitz boundary. Then $g - \bar{\partial}h$ vanishes on D and satisfies $\bar{\partial}(g - \bar{\partial}h) = f$. This shows $H_{c,L^2}^{n,q}(X \setminus D) = 0$. \square

Corollary 4.9 *Let $D \subset\subset \tilde{\Omega}$ be bounded open subsets of \mathbb{C}^n , $n \geq 2$, such that $\mathbb{C}^n \setminus \tilde{\Omega}$ is connected. Assume D has Lipschitz boundary and $\Omega = \tilde{\Omega} \setminus \bar{D}$ is connected. Consider the assertions:*

- (i) *For all $1 \leq q \leq n - 2$, $H_{W_{loc}^1}^{0,q}(\tilde{\Omega} \setminus D) = 0$ and $H_{W_{loc}^1}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff;*
- (ii) *For all $1 \leq q \leq n - 1$, $H_{L^2}^{n,q}(D) = 0$ and $\tilde{\Omega}$ is pseudoconvex.*

and if $\tilde{\Omega}$ has Lipschitz boundary

- (i') *For all $1 \leq q \leq n - 2$, $H_{L^2}^{0,q}(\Omega) = 0$ and $H_{L^2}^{0,n-1}(\Omega)$ is Hausdorff;*
- (ii') *For all $1 \leq q \leq n - 1$, $H_{W^1}^{n,q}(D) = 0$ and $\tilde{\Omega}$ is pseudoconvex.*

Then the pairs of assertions (i) and (ii) are equivalent, and if moreover $\tilde{\Omega}$ has Lipschitz boundary, (i') and (ii'), are equivalent.

5 Hearing pseudoconvexity with L^2 Dolbeault cohomology

It is well known (see e.g. Corollary 4.2.6 and Theorem 4.2.9 in [12]) that a domain D in \mathbb{C}^n is pseudoconvex if and only if we have $H^{0,q}(D) = 0$ for all $1 \leq q \leq n - 1$. This result also holds for the L^2 -cohomology, provided D satisfies $\text{interior}(\bar{D}) = D$ (see [8] and references therein). We will extend the results to the case when the Dolbeault cohomology groups with $W^{s,p}$ -forms are finite dimensional for any given $s \geq 0$ and $p \geq 1$. (Here $W^{s,p}(D)$ is the L^p -Sobolev space of order s .)

Theorem 5.1 *Let $D \subset\subset \mathbb{C}^n$ be a bounded domain such that $\text{interior}(\bar{D}) = D$. Let $s \geq 0$ and $p \geq 1$. If $H_{W^{s,p}}^{0,q}(D)$ is finite dimensional for all $1 \leq q \leq n - 1$, then D is pseudoconvex.*

We will present a proof using an idea of Laufer [16] as in the proof of Theorem 3.13. The subtle difference is that while it makes sense to restrict a smooth form on a domain to the intersection of the domain with a complex hyperplane, restriction of an L^p -form to a complex hyperplane is not well-defined. This difficulty was overcome by appropriately modifying the construction of Laufer so that the factor z_l in (3.1) is replaced by z_l^m for a positive integer m . By choosing m sufficiently large, we are able to make this restriction work. We now provide the detail, following Fu [8]. The following simple lemma illustrates the idea behind the construction of the forms $u_{\alpha,m}(k_1, k_2, \dots, k_q)$ given by (5.2) below.

Lemma 5.2 *Let $v_1, \dots, v_{n-1} \in L^p(D)$, $p \geq 1$, and let m be a positive integer. Assume that G is a continuous function on D such that*

$$G(z) = \sum_{j=1}^{n-1} z_j^m v_j(z). \tag{5.1}$$

If $m \geq 2(n - 1)/p$, then $G(0, \dots, 0, z_n) = 0$ for all $(0, \dots, 0, z_n) \in D$.

Proof Let $(0, \dots, 0, z_n^0) \in D$. Write $z' = (z_1, \dots, z_{n-1})$. Then for a sufficiently small positive numbers a_1 and a_2 , we have

$$D(a_1, a_2) := \{|z'| < a_1\} \times \{|z_n - z_n^0| < a_2\} \subset D.$$

For any $\delta \in (0, 1)$, we have

$$\begin{aligned} \left(\int_{D(a_1, a_2)} |G(\delta z', z_n)|^p dV \right)^{1/p} &\leq a_1^m \delta^m \sum_{j=1}^{n-1} \left(\int_{D(a_1, a_2)} |v_j(\delta z', z_n)|^p dV \right)^{1/p} \\ &\leq a_1^m \delta^{m-2(n-1)/p} \sum_{j=1}^{n-1} \left(\int_{D(a_1 \delta, a_2)} |v_j(z', z_n)|^p dV \right)^{1/p} \\ &\leq a_1^m \delta^{m-2(n-1)/p} \sum_{j=1}^{n-1} \left(\int_D |v_j(z)|^p \chi_{D(a_1 \delta, a_2)}(z) dV \right)^{1/p}. \end{aligned}$$

Since $m \geq 2(n - 1)/p$, letting $\delta \rightarrow 0$, we obtain from the Lebesgue dominated convergence theorem that

$$\int_{D(a_1, a_2)} |G(0', z_n)|^p dV = 0.$$

Thus $G(0', z_n) = 0$ for $|z_n - z_n^0| < a_2$. □

Proof of Theorem 5.1 The proof for $W^{s,p}$ -cohomology is the same as for L^p -cohomology. For economy of notation, we will only provide the proof for L^p -cohomology. Proving by contradiction, we assume that D is not pseudoconvex. Then there exists a domain $\tilde{D} \supsetneq D$ such that every holomorphic function on D extends to \tilde{D} . After a translation and a unitary transformation, we may assume that the origin is in $\tilde{D} \setminus \bar{D}$ and there is a point z^0 in the intersection of z_n -plane with D that is in the same connected component of $\tilde{D} \cap \{z_1 = 0\}$ as the origin.

For any integers $\alpha \geq 0, m \geq 1, q \geq 1, \{k_1, \dots, k_{q-1}\} \subset \{1, 2, \dots, n - 1\}$ and $k_q = n$, let

$$u_{\alpha, m}(k_1, \dots, k_q) = \frac{(\alpha + q - 1)! \bar{z}_n^{\alpha} (\bar{z}_{k_1} \dots \bar{z}_{k_q})^{m-1}}{r_m^{\alpha+q}} \sum_{j=1}^q (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}} \tag{5.2}$$

where $r_m = |z_1|^{2m} + \dots + |z_n|^{2m}$. Evidently, $u(k_1, \dots, k_q)$ is a smooth form on $\mathbb{C}^n \setminus \{0\}$. Since $0 \notin \bar{D}$, $u(k_1, \dots, k_q) \in L^p_{(0, q-1)}(D)$. Moreover, $u(k_1, \dots, k_q)$ is skew-symmetric with respect to the indices (k_1, \dots, k_{q-1}) . In particular, $u(k_1, \dots, k_q) = 0$ when two k_j 's are identical.

For $K = (k_1, \dots, k_q)$, write $d\bar{z}_K = d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q}$, $\bar{z}_K^{m-1} = (\bar{z}_{k_1} \dots \bar{z}_{k_q})^{m-1}$. Denoted by $(k_1, \dots, k_q \setminus J)$ the tuple of remaining indices after deleting those in J from (k_1, \dots, k_q) . It follows from straightforward computations that

$$\begin{aligned} \bar{\partial} u_{\alpha, m}(k_1, \dots, k_q) &= - \frac{(\alpha + q)! m \bar{z}_n^{\alpha} \bar{z}_K^{m-1}}{r_m^{\alpha+q+1}} \left(r_m d\bar{z}_K \right. \\ &\quad \left. + \left(\sum_{\ell=1}^n \bar{z}_\ell^{m-1} z_\ell^m d\bar{z}_\ell \right) \wedge \left(\sum_{j=1}^q (-1)^j \bar{z}_{k_j} \widetilde{d\bar{z}_{k_j}} \right) \right) \\ &= m \sum_{\ell=1}^{n-1} z_\ell^m u_{\alpha, m}(\ell, k_1, \dots, k_q). \end{aligned}$$

In particular, $u_{\alpha, m}(1, \dots, n)$ is $\bar{\partial}$ -closed. Our next goal is to solve the $\bar{\partial}$ -equation in L^p -spaces inductively with the $(0, n - 1)$ -forms $u_{\alpha, m}(1, \dots, n)$ as the initial data, and eventually

produce an L^p -holomorphic function on D . This holomorphic function has a holomorphic extension to \tilde{D} . By way of the construction, the extension has singularity at the origin, which leads to a contradiction. We now provide the details.

We fix $m \geq 2(n - 1)/p$. Let M be an integer such that $M > \dim H_{L^p}^{0,q}(D)$ for all $1 \leq q \leq n - 1$. Let \mathcal{F}_0 be the linear span of $\{u_{\alpha,m}(1, \dots, n); \alpha = 1, \dots, M^{n-1}\}$. For any $u \in \mathcal{F}_0$ and for any $\{k_1, \dots, k_{q-1}\} \subset \{1, \dots, n - 1\}$, we set

$$u(k_1, \dots, k_{q-1}, n) = \sum_{j=1}^k c_j u_{\alpha_j,m}(k_1, \dots, k_{q-1}, n)$$

if $u = \sum_{j=1}^k c_j u_{\alpha_j,m}(1, \dots, n)$. We decompose \mathcal{F}_0 into a direct sum of M^{n-2} subspaces, each of which is M -dimensional. Since $\dim H_{L^p}^{0,n-1}(D) < M$ and $u_{\alpha,m}(1, \dots, n) \in \mathcal{N}(\bar{\partial}_{n-1})$, there exists a non-zero form u in each of the subspaces such that $\bar{\partial}v_u(\emptyset) = u$ for some $v_u(\emptyset) \in L_{(0,n-2)}^p(D)$. Let \mathcal{F}_1 be the M^{n-2} -dimensional linear span of all such u 's. We extend $u \mapsto v_u(\emptyset)$ linearly to all $u \in \mathcal{F}_1$.

For $0 \leq q \leq n - 1$, we use induction on q to construct an M^{n-q-2} -dimensional subspace \mathcal{F}_{q+1} of \mathcal{F}_q with the properties that for any $u \in \mathcal{F}_{q+1}$, there exists $v_u(k_1, \dots, k_q) \in L_{(0,n-q-2)}^p(D)$ for all $\{k_1, \dots, k_q\} \subset \{1, \dots, n - 1\}$ such that $v_u(k_1, \dots, k_q)$ depends linearly on u ; $v_u(k_1, \dots, k_q)$ is skew-symmetric with respect to indices $K = (k_1, \dots, k_q)$; and

$$\bar{\partial}v_u(K) = m \sum_{j=1}^q (-1)^j z_{k_j}^m v_u(K \setminus k_j) + (-1)^{q+|K|} u(1, \dots, n \setminus K),$$

where $|K| = k_1 + \dots + k_q$.

We now show how to construct \mathcal{F}_{q+1} and $v_u(k_1, \dots, k_q)$ for $u \in \mathcal{F}_{q+1}$ and $\{k_1, \dots, k_q\} \subset \{1, \dots, n - 1\}$ once \mathcal{F}_q has been constructed. For any $u \in \mathcal{F}_q$ and any $\{k_1, \dots, k_q\} \subset \{1, \dots, n - 1\}$, write $K = (k_1, \dots, k_q)$, and let

$$w_u(K) = m \sum_{j=1}^q (-1)^j z_{k_j}^m v_u(K \setminus k_j) + (-1)^{q+|K|} u(1, \dots, n \setminus K).$$

Then as in the previous case,

$$\bar{\partial}w_u(K) = (-1)^{q+|K|} \left(-m \sum_{j=1}^q z_{k_j}^m u(k_j, (1, \dots, n \setminus K)) + \bar{\partial}u(1, \dots, n \setminus K) \right) = 0.$$

We again decompose \mathcal{F}_q into a direct sum of M^{n-q-2} linear subspaces, each of which is M -dimensional. Since $\dim(H_{L^p}^{0,n-q-2}(D)) < M$ and $\bar{\partial}w_u(K) = 0$, there exists a non-zero form u in each of these subspaces such that $\bar{\partial}v_u(K) = w_u(K)$ for some $v_u(K) \in L_{(0,n-q-2)}^p(D)$. Since $w_u(K)$ is skew-symmetric with respect to indices K , we may choose $v_u(K)$ to be skew-symmetric with respect to K as well. The subspace \mathcal{F}_{q+1} of \mathcal{F}_q is then the linear span of all such u 's.

Note that $\dim(\mathcal{F}_{n-1}) = 1$. Let u be any non-zero form in \mathcal{F}_{n-1} and let

$$F(z) = w_u(1, \dots, n - 1) = m \sum_{j=1}^{n-1} z_j^m v_u(1, \dots, n - 1 \setminus j) - (-1)^{n+\frac{n(n-1)}{2}} u(n).$$

Then $F \in L^p(D)$ and $\bar{\partial}F = 0$. Therefore, F is holomorphic on D and hence has a holomorphic extension to \tilde{D} . Restricting to z_n -plane, by Lemma 5.2, we have

$$F(O', z_n) = -(-1)^{n+\frac{n(n-1)}{2}} u(n)(O', z_n) = (-1)^{n+\frac{n(n-1)}{2}} \sum_{\alpha=1}^M c_\alpha \frac{\alpha!}{z_n^{m(\alpha+1)}}$$

where the c_α 's are constants, not all zeros. This contradicts the analyticity of F near the origin. We therefore conclude the proof of Theorem 5.1. \square

Corollary 5.3 *Let $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n , $n \geq 2$, such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume D has Lipschitz boundary. Then $H_{W_{loc}^1}^{0,q}(\tilde{\Omega} \setminus D) = 0$, for all $1 \leq q \leq n - 2$, and $H_{W_{loc}^1}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff if and only if $\tilde{\Omega}$ and D are pseudoconvex.*

Proof The necessary condition is a direct consequence of Corollary 4.9 and Theorem 5.1. The sufficient condition follows from Hörmander vanishing L^2 -theory and Corollary 4.9. \square

Next we give a characterisation of the annulus domain by its L^2 Dolbeault cohomology when the inner hole D has C^2 boundary.

Corollary 5.4 *Let $D \subset\subset \tilde{\Omega}$ be two relatively compact open subsets of \mathbb{C}^n , $n \geq 2$, such that both $\mathbb{C}^n \setminus \tilde{\Omega}$ and $\tilde{\Omega} \setminus D$ are connected. Assume D has C^2 boundary and $\tilde{\Omega}$ has Lipschitz boundary. Then $H_{L^2}^{0,q}(\tilde{\Omega} \setminus D) = 0$, for all $1 \leq q \leq n - 2$, and $H_{L^2}^{0,n-1}(\tilde{\Omega} \setminus D)$ is Hausdorff if and only if $\tilde{\Omega}$ and D are pseudoconvex.*

Proof We first prove sufficiency. If D has C^3 boundary and $n \geq 3$, this follows directly from [26]. If the boundary is only C^2 , it follows from Theorem 3 in Harrington [10] that $H_{W^1}^{n,q}(D) = 0$, for all $1 \leq q \leq n - 1$. (When the boundary is C^∞ , this follows from the work of Kohn [15].) The sufficiency then follows from Corollary 4.9.

The necessary condition is a direct consequence of Corollary 4.9 and Theorem 5.1. \square

Next we will set up the spectral theory for the $\bar{\partial}$ -Neumann operator. Let X be a complex manifold of dimension n equipped with a hermitian metric. Let Ω be a bounded domain in X . Let

$$Q_{p,q}^\Omega(u, v) = \langle \bar{\partial}_{p,q} u, \bar{\partial}_{p,q} v \rangle_\Omega + \langle \bar{\partial}_{p,q-1}^* u, \bar{\partial}_{p,q-1}^* v \rangle_\Omega$$

be the sesquilinear form on $L^2_{p,q}(\Omega)$ with domain of definition $\text{dom}(Q_{p,q}^\Omega) = \text{dom}(\bar{\partial}_{p,q}) \cap \text{dom}(\bar{\partial}_{p,q-1}^*)$. Then $Q_{p,q}^\Omega$ is densely defined and closed. It then follows from general operator theory (see [7]) that $Q_{p,q}^\Omega$ uniquely determines a densely defined, non-negative, self-adjoint operator $\square_{p,q}^\Omega : L^2_{p,q}(\Omega) \rightarrow L^2_{p,q}(\Omega)$ such that $\text{dom}((\square_{p,q}^\Omega)^{1/2}) = \text{dom}(Q_{p,q}^\Omega)$ and

$$Q_{p,q}^\Omega(u, v) = \left\langle (\square_{p,q}^\Omega)^{1/2} u, (\square_{p,q}^\Omega)^{1/2} v \right\rangle, \quad \text{for } u, v \in \text{dom}(Q_{p,q}^\Omega).$$

Moreover,

$$\text{dom}(\square_{p,q}^\Omega) = \left\{ u \in \text{dom}(Q_{p,q}^\Omega) \mid \bar{\partial}_{p,q} u \in \text{dom}(\bar{\partial}_{p,q}^*), \bar{\partial}_{p,q-1}^* u \in \text{dom}(\bar{\partial}_{p,q-1}) \right\}.$$

The operator $\square_{p,q}^\Omega$ is the $\bar{\partial}$ -Neumann Laplacian on $L^2_{p,q}(\Omega)$. (We refer the reader to [9, §2] for a spectral theoretic setup for the $\bar{\partial}$ -Neumann Laplacian.) We will drop the superscript

and/or subscript from $\square_{p,q}^\Omega$ when their appearances are either inconsequential or clear from the context.

Let $\sigma(\square_{p,q})$ be the spectrum of $\square_{p,q}$. Recall that $\sigma(\square_{p,q})$ is the complement in \mathbb{C} of the resolvent set which consists of all $\lambda \in \mathbb{C}$ such that $\lambda I - \square_{p,q} : \text{dom}(\square_{p,q}) \rightarrow L^2_{p,q}(\Omega)$ is one-to-one, onto, and has bounded inverse. (See [7] for relevant material on spectral theory of differential operators.) Since $\square_{p,q}$ is a non-negative self-adjoint operator on a Hilbert space, $\sigma(\square_{p,q})$ is a non-empty closed subset of the interval $[0, \infty)$. Let $\sigma_e(\square_{p,q})$ be the essential spectrum of $\square_{p,q}$; namely, points in $\sigma(\square_{p,q})$ that are either isolated points of the spectrum but eigenvalues of infinity multiplicity; or limit points of the spectrum. By definition, the essential spectrum $\sigma_e(\square_{p,q})$ is also a closed subset and the set of limit points of $\sigma_e(\square_{p,q})$ is the same as that of $\sigma(\square_{p,q})$. We summarize the following spectral theoretic interpretations for positivity of the $\bar{\partial}$ -Neumann Laplacian $\square_{p,q}$ in the following proposition:

Proposition 5.5 *Let $\square_{p,q}$ be the $\bar{\partial}$ -Neumann Laplacian on (p, q) -forms.*

- (1) *0 is not a limit point of $\sigma(\square_{p,q})$ if and only if both $\mathcal{R}(\bar{\partial}_{p,q-1})$ and $\mathcal{R}(\bar{\partial}_{p,q})$ are closed.*
- (2) *$0 \notin \sigma_e(\square_{p,q})$ if and only if $\mathcal{R}(\bar{\partial}_{p,q-1})$ and $\mathcal{R}(\bar{\partial}_{p,q})$ are closed, and $H^{p,q}_{L^2}(\Omega)$ is finite dimensional.*
- (3) *$0 \notin \sigma(\square_{p,q})$ if and only if $\mathcal{R}(\bar{\partial}_{p,q-1})$ and $\mathcal{R}(\bar{\partial}_{p,q})$ are closed, and $H^{p,q}_{L^2}(\Omega)$ is trivial.*

We refer the reader to [13, § 1.1] (see also [14, Appendix A]) for proofs of (1) and (3) and to [8, § 2] (see also [9, § 2]) for a proof of (2).

Recall that, for a bounded domain Ω in a complex, hermitian, n -dimensional manifold, the top degree L^2 -cohomology groups $H^{p,n}_{L^2}(\Omega)$, $0 \leq p \leq n$, always vanish, hence $\mathcal{R}(\bar{\partial}_{p,n-1})$ is closed in this case. So, in top degree, the first assertion of Proposition 5.5 becomes: 0 is not a limit point of $\sigma(\square_{p,n-1})$ if and only if $\mathcal{R}(\bar{\partial}_{p,n-2})$ is closed. Combining Proposition 5.5 with Theorems 4.8 and 5.1, we then have:

Theorem 5.6 *Let X be a Stein manifold of dimension $n \geq 3$, equipped with a hermitian metric. Let $\Omega = \tilde{\Omega} \setminus \bar{D}$ where $\tilde{\Omega}$ is a relatively compact domain with connected complement in X and $D \subset\subset \tilde{\Omega}$ is an open set with connected complement in $\tilde{\Omega}$ and with C^2 -boundary. If both $\tilde{\Omega}$ and D are pseudoconvex, then there exists a constant $C > 0$ such that*

$$\inf \sigma(\square_{p,q}^\Omega) \geq C \tag{5.3}$$

for all $0 \leq p \leq n$ and $1 \leq q \leq n - 2$ and

$$\sigma(\square_{p,n-1}^\Omega) \cap (0, C) = \emptyset. \tag{5.4}$$

The converse of the above theorem also holds. We summarize the results in a slight more general form as follows:

Theorem 5.7 *Let $\tilde{\Omega}$ be a bounded domain with connected complement in a hermitian Stein manifold and let D be a relatively compact open subset of $\tilde{\Omega}$ with connected complement. Let $\Omega = \tilde{\Omega} \setminus \bar{D}$. Suppose $\tilde{\Omega}$ and D have Lipschitz boundary. Fix $0 \leq p \leq n$. If $0 \notin \sigma_e(\square_{p,q})$ for $1 \leq q \leq n - 2$ when $n \geq 3$ or if 0 is not a limit point for $\sigma_e(\square_{p,1})$ when $n = 2$, then both $\tilde{\Omega}$ and D are pseudoconvex.*

The above theorem is a consequence of Proposition 5.5 and the characterization of pseudoconvexity by L^2 -cohomology groups in Sect. 4. Note that when $n \geq 3$, one only need to assume the positivity of $\sigma_e(\square_{p,q})$ for $1 \leq q \leq n - 2$. For $n = 2$, we use, as noted above,

that $\mathcal{R}(\bar{\partial}_{p,n-1})$ is always closed. For completeness, we provide the proof of Theorem 5.7. We first establish the following spectral theoretic version of the Hartogs phenomenon, as in Theorem 4.3. Without loss of generality, we will deal only with $(0, q)$ -forms.

Lemma 5.8 *Let $\tilde{\Omega}$ be a domain in a Stein manifold X equipped with a hermitian metric and let K be a compact subset of $\tilde{\Omega}$. Let $\Omega = \tilde{\Omega} \setminus K$. Then*

- (1) $\inf \sigma_e \left(\square_{\tilde{\Omega}}^q \right) > 0$ provided $\inf \sigma_e \left(\square_{\Omega}^q \right) > 0$.
- (2) $\inf \sigma \left(\square_{\tilde{\Omega}}^q \right) > 0$ provided $\inf \sigma \left(\square_{\Omega}^q \right) > 0$

Proof To prove (1), from Lemma 4.1 and Proposition 5.5, it suffices to show that $H_{L^2}^{0,q}(\tilde{\Omega})$ is finite dimensional provided $H_{L^2}^{0,q}(\Omega)$ is finite dimensional. Let $R: \mathcal{N} \left(\bar{\partial}_{\tilde{\Omega}}^q \right) \rightarrow \mathcal{N} \left(\bar{\partial}_{\Omega}^q \right)$ be the restriction map $\beta \mapsto \beta|_{\Omega}$. Repeating arguments used in Lemma 4.1 (with $\bar{\partial}f$ replaced by β) yields that R induces an injective homomorphism from $H_{L^2}^{0,q}(\tilde{\Omega})$ into $H_{L^2}^{0,q}(\Omega)$. Therefore, $\dim H_{L^2}^{0,q}(\tilde{\Omega}) \leq \dim H_{L^2}^{0,q}(\Omega)$. This concludes the proof of (1) and hence that of (2). \square

The following lemma is a spectral theoretic interpretation of Theorem 4.8 (in a slightly more general form).

Lemma 5.9 *Let D be a relatively compact open set in a Stein manifold X of dimension $n \geq 2$ with connected complement and Lipschitz boundary. Let $\Omega = X \setminus \bar{D}$. Then $0 \notin \sigma_e \left(\square_{\Omega}^q \right)$ for $1 \leq q \leq n - 2$ and 0 is not a limit point for $\sigma_e \left(\square_{\Omega}^q \right)$ if and only if $\dim H_{W^1}^{n,q}(D) < \infty$, $1 \leq q \leq n - 2$, and $\dim H_{W^1}^{n,n-1}(D) = 0$.*

Proof Note that by Proposition 5.5, as already mentioned above, 0 is not a limit point of $\sigma_e \left(\square_{\Omega}^q \right)$ is equivalent to $\mathcal{R}(\bar{\partial}_{n-2}^{\Omega})$ is closed, which is equivalent to $H_{L^2}^{0,n-1}(\Omega)$ is Hausdorff. By [20, Proposition 4.7], this is also equivalent to $H_{W^1}^{n,n-1}(D) = \{0\}$. In light of Proposition 5.5 and by L^2 -Serre duality, $\dim H_{L^2}^{0,q}(\Omega) = \dim H_{\Omega, L^2}^{n,n-q}(X)$ for $1 \leq q \leq n - 1$. So it remains to show that for $1 \leq q \leq n - 2$, $\dim H_{W^1}^{n,q}(D) = \dim H_{\Omega, L^2}^{n,q+1}(X)$, provided one of these quantities is finite.

Suppose $\dim H_{\Omega, L^2}^{n,q+1}(X) = N < \infty$. Let $g_j, 1 \leq j \leq N$, be $\bar{\partial}$ -closed $(n, q + 1)$ -forms supported on $\bar{\Omega}$ such that $\{[g_j]\}_{j=1}^N$ spans $H_{\Omega, L^2}^{n,q+1}(X)$. Since X is Stein, there exists (n, q) -forms h_j with L_{loc}^2 -coefficients such that $\bar{\partial}h_j = g_j$. (Here, we identify g_j with its extension to X by setting $g_j = 0$ outside \bar{D} .) Since g_j is supported on $X \setminus D$, $\bar{\partial}h_j = 0$ on D . Now let $f \in W_{n,q}^1(D)$ be any $\bar{\partial}$ -closed form. Since D has Lipschitz boundary, there exists an extension $\tilde{f} \in W_{n,q}^1(X)$ of f to X . Since $\bar{\partial}\tilde{f} = 0$ on D , under the assumption, there exists a $g \in L_{n,q}^2(X)$, supported on $X \setminus D$, such that

$$\bar{\partial}\tilde{f} = \bar{\partial}g + \sum_{j=1}^N c_j g_j = \bar{\partial}g + \sum_{j=1}^N c_j \bar{\partial}h_j,$$

for some constants $c_j \in \mathbb{C}, 1 \leq j \leq N$. Therefore, there exists $(n, q - 1)$ -form u with L_{loc}^2 -coefficients such that

$$\tilde{f} = g + \sum_{j=1}^N c_j h_j + \bar{\partial}u.$$

Note that using the interior ellipticity of $\bar{\partial} \oplus \bar{\partial}^*$, we may choose the forms h_j and u to have W^1 -coefficients on D . Restricting to D , we then have $f = \sum_{j=1}^N c_j h_j + \bar{\partial}u$, which implies that $\dim H_{W^1}^{n,q}(D) \leq N$.

Conversely, suppose $\dim H_{W^1}^{n,q}(D) = N < \infty$. Let $\{g_j\}_{j=1}^N \subset W_{n,q}^1(D)$ be $\bar{\partial}$ -closed forms such that $\{[g_j]\}_{j=1}^N$ spans $H_{W^1}^{n,q}(D)$. Let $f \in L_{n,q+1}^2(X)$ be a $\bar{\partial}$ -closed form with compact support in $X \setminus D$. Since X is a Stein manifold, there exists an (n, q) -form u with L_{loc}^2 -coefficients such that $\bar{\partial}u = f$. Using the interior ellipticity of $\bar{\partial} \oplus \bar{\partial}^*$, we may choose u so that $u \in W_{n,q}^1(D)$. Since f is supported on $X \setminus D$, $\bar{\partial}u = 0$ on D . Under the assumption, there exists $h \in W_{n,q-1}^1(D)$ such that

$$u = \bar{\partial}h + \sum_{j=1}^N b_j g_j,$$

for some constants $b_j \in \mathbb{C}$, $1 \leq j \leq N$. Let $\tilde{g}_j, \tilde{h} \in W_{n,q}^1(X)$ be any extensions of g_j and h respectively from D to X with compact supports in X . Let $g = u - \bar{\partial}\tilde{h} - \sum_{j=1}^N b_j \tilde{g}_j$. Then g is compactly supported on $X \setminus D$ and

$$\bar{\partial}g = f - \sum_{j=1}^N b_j \bar{\partial}\tilde{g}_j,$$

which implies that $\dim H_{\Omega, L^2}^{n,q+1}(X) \leq N$. □

We are now in a position to prove Theorem 5.7. By Lemma 5.8, Lemma 4.1 and Theorem 4.3(ii), we know that $H_{L^2}^{0,q}(\tilde{\Omega})$, $1 \leq q \leq n - 2$, are finite dimensional and $H_{L^2}^{0,n-1}(\tilde{\Omega})$ is trivial. By Theorem 5.1, $\tilde{\Omega}$ is pseudoconvex.

Applying Lemma 5.9 with $X = \tilde{\Omega}$, we then have $\dim H_{W^1}^{0,q}(D) < \infty$, $1 \leq q \leq n - 2$, and $H_{W^1}^{0,n-1}(D) = \{0\}$. Applying Theorem 5.1 again, we then conclude that D is pseudoconvex.

Remark 5.10 In Corollary 5.3, we only need to assume that the boundary is Lipschitz when we use Dolbeault cohomology with W^1 coefficients. But in Corollary 5.4, the boundary D needs to be C^2 smooth. Note that the necessary condition in Corollary 5.4 still holds if the boundary of D is only Lipschitz. We do not know if we can replace the C^2 assumption by the Lipschitz condition in Corollary 5.4 or Theorem 5.6. We conjecture that they still hold if the boundary of D is only Lipschitz. This has been verified when the inner domain D is a product domain or piecewise smooth pseudoconvex domain (see the results in [5]). In particular, when the domain Ω is the annulus between a ball and a bidisc in \mathbb{C}^2 , one has that $H_{L^2}^{0,1}(\Omega)$ is Hausdorff. This yields the W^1 estimates for $\bar{\partial}$ on bidisc using Corollary 4.9. The general case with Lipschitz holes is still an open problem.

Remark 5.11 Suppose that the number of holes in $\tilde{\Omega} \setminus \Omega = \bar{D}$ is infinite and each component is pseudoconvex, the boundary Ω is not Lipschitz. But one still can have $interior \bar{\Omega} = \Omega$. We do not know if the $\bar{\partial}$ -Neumann operator $\square_{p,q}$ has closed range for $1 \leq q \leq n - 1$. In fact, one does not even know if the classical Neumann operator has closed range.

References

1. Andreotti, A., Grauert, H.: Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. Fr.* **90**, 193–259 (1962)
2. Broecker, T.: Zur L^2 -Kohomologie beschränkter Gebiete, *Bonner Mathematische Schriften*, vol. 145. Universitt Bonn, Bonn (1983)
3. Cassa, A.: Coomologia separata sulle varietà analitiche complesse. *Ann. Sc. Norm. Sup. Pisa* **25**, 290–323 (1971)
4. Chakrabarti, D., Shaw, M.-C.: L^2 Serre duality on domains in complex manifolds and applications. *Trans. Am. Math. Soc.* **364**, 3529–3554 (2012)
5. Chakrabarti, D., Laurent-Thiébaud, C., Shaw, M.-C.: The L^2 -Dolbeault cohomology of annuli, Preprint
6. Chen, S.-C., Shaw, M.-C.: *Partial Differential Equations in Several Complex Variables*, AMS/IP Studies in Advanced Mathematics, vol. 19. American Mathematical Society, Providence and International Press, Boston (2001)
7. Davies, E.B.: *Spectral Theory and Differential Operators*, Cambridge Studies in Advanced Mathematics, vol. 42. Cambridge University Press, Cambridge (1995)
8. Fu, S.: Hearing pseudoconvexity with the Kohn Laplacian. *Math. Ann.* **331**, 475–485 (2005)
9. Fu, S.: Positivity of the $\bar{\partial}$ -Neumann Laplacian. In: Ebenfelt, P., Hungerbühler, N., Kohn, J., Mok, N., Straube, E. (eds.) *Complex Analysis: Several Complex Variables and Connections with PDEs and Geometry* (Fribourg 2008), Trends in Mathematics, pp. 145–158. Springer, New York (2010)
10. Harrington, P.: Sobolev estimates for the Cauchy–Riemann complex on C^1 pseudoconvex domains. *Math. Z.* **262**, 199–217 (2009)
11. Henkin, G.M., Leiterer, J.: *Andreotti–Grauert Theory by Integral Formulas*, Progress in Mathematics, vol. 74. Birkhäuser, Berlin (1988)
12. Hörmander, L.: *An Introduction to Complex Analysis in Several Complex Variables*. Van Nostrand, Princeton (1990)
13. Hörmander, L.: L^2 estimates and existence theorems for the $\bar{\partial}$ operator. *Acta Math.* **113**, 89–152 (1965)
14. Hörmander, L.: The null space of the $\bar{\partial}$ -Neumann operator. *Ann. Inst. Fourier (Grenoble)* **54**, 1305–1369 (2004)
15. Kohn, J.J.: Global regularity for $\bar{\partial}$ on weakly pseudoconvex manifolds. *Trans. Am. Math. Soc.* **181**, 273–292 (1973)
16. Laufer, H.B.: On sheaf cohomology and envelopes of holomorphy. *Ann. Math.* **84**, 102–118 (1966)
17. Laufer, H.B.: On the infinite dimensionality of the Dolbeault cohomology groups. *Proc. Am. Math. Soc.* **52**, 293–296 (1975)
18. Laurent-Thiébaud, C.: *Théorie des fonctions holomorphes de plusieurs variables*. Savoirs actuels, InterEditions/CNRS Editions, Paris (1997)
19. Laurent-Thiébaud, C., Leiterer, J.: On Serre duality. *Bull. Sci. Math.* **124**, 93–106 (2000)
20. Laurent-Thiébaud, C., Shaw, M.-C.: On the Hausdorff property of some Dolbeault cohomology groups. *Math. Z.* **274**, 1165–1176 (2013)
21. Laurent-Thiébaud, C., Shaw, M.-C.: Non-closed range property for the Cauchy–Riemann operator. In: *Analysis and Geometry*, Springer Proceedings of the Conference held in Tunisia in the Memory of Salah Baouendi, vol. 127, pp. 207–218 (2015)
22. Ohsawa, T.: Isomorphism theorems for cohomology groups of weakly 1-complete manifolds. *Publ. Res. Inst. Math. Sci.* **18**(1), 191–232 (1982)
23. Ohsawa, T.: Complete Kähler manifolds and function theory of several complex variables. *Sugaku Expos.* **1**, 75–93 (1988)
24. Serre, J.-P.: Quelques problèmes globaux relatifs aux variétés de Stein, *Colloque sur les Fonctions de Plusieurs Variables*, pp. 57–68, Brussels (1953)
25. Serre, J.-P.: Un théorème de dualité. *Comment. Math. Helv.* **29**, 9–26 (1955)
26. Shaw, M.-C.: Global solvability and regularity for $\bar{\partial}$ on an annulus between two weakly pseudoconvex domains. *Trans. Am. Math. Soc.* **291**, 255–267 (1985)
27. Shaw, M.-C.: The closed range property for $\bar{\partial}$ on domains with pseudoconcave boundary. In: Ebenfelt, P., Hungerbühler, N., Kohn, J., Mok, N., Straube, E. (eds.) *Complex Analysis: Several Complex Variables and Connections with PDEs and Geometry* (Fribourg 2008), Trends in Mathematics, pp. 307–320. Springer, New York (2010)
28. Sibony, N.: Prolongement des fonctions holomorphes bornées et métrique de Carathéodory. *Invent. Math.* **29**, 205–230 (1975)
29. Siu, Y.-T.: Non-countable dimensions of cohomology groups of analytic sheaves and domains of holomorphy. *Math. Z.* **102**, 17–29 (1967)
30. Trapani, S.: Coomologia di Hausdorff e forma di levi. *Ann. Mat. Pura Appl.* **144**, 391–401 (1986)

31. Trapani, S.: Involuppi di olomorfia et gruppi di coomologia di hausdorff. *Rend. Semin. Mat. Univ. Padova* **75**, 25–37 (1986)
32. Trapani, S.: Holomorphically convex compact sets and cohomology. *Pac. J. Math.* **134**, 179–196 (1988)