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A History of Existence Theorems for the Cauchy–Riemann Complex in L^2 Spaces

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Dedicated to the memory of D.C. Spencer

ABSTRACT. The purpose of this paper is to give a historical survey of the development of methods in the theory of partial differential equations for the study of the Levi and Cousin problems in complex analysis. Success was achieved by the mid 1960's but we begin further back, with the background in Hodge theory and with early unsuccessful attempts to exploit the Bergman kernel. Some examples of later date illustrating the usefulness of such methods are also given.

1. Introduction

In the theory of analytic functions of one complex variable the study of the Laplace operator and the Cauchy–Riemann operator $\partial/\partial\bar{z}$ has always played a central role. However, a theory of functions of several complex variables was first developed by means of inductive procedures starting from the one-dimensional case. It was not until the 1960's that an alternative and supplementary approach became possible using methods from the theory of partial differential equations. The purpose of this article is to present a historical survey of this development.

We begin in Section 2 by recalling the Hodge theory of harmonic forms and the theorem of de Rham which was the starting point for the general theory. For a while the analytic foundations of Hodge theory were somewhat shaky but around 1940 it was well established on compact manifolds without boundary. An extension to compact manifolds with boundary was developed in the early 1950's. In retrospect the analysis only required fairly classical tools from the study of boundary problems for the Laplacian, but it stimulated the systematic codification of the theory of elliptic boundary problems which was achieved in the 1950's.

Already in 1922 Stefan Bergman had introduced the reproducing kernel for holomorphic functions which now carries his name. It is as easy to define for functions of several variables as in the one-dimensional case. In view of its relation to conformal mapping and other applications in that case it was perhaps natural to take it as a starting point for an analytic attack on the Cauchy–Riemann equations in several variables. We shall discuss some such attempts in Section 3, for

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although they failed, they seem to have led Spencer to formulate the basic $\bar{\partial}$-Neumann problem around the middle of the 1950's.

The solution of the $\bar{\partial}$-Neumann problem required the development of new tools in the theory of partial differential equations. Section 4 is devoted to the progress made from 1958 to 1964, when the basic results were well established after some setbacks along the way. In the examples of the developments after 1964 given in Section 5 we shall essentially limit ourselves to the *existence* theorems which follow from the study of the $\bar{\partial}$-Neumann problem, and the selection of the examples is also strongly biased by the author's personal research experience. (Much more related material can be found in two surveys [11, 12] by Demailly.) For the extensive work which has been done concerning the relations between the regularity at the boundary of solutions of the $\bar{\partial}$-Neumann problem and the geometry of the boundary we refer the reader to the survey articles [10] by D'Angelo and Kohn and [8] by Boas and Straube.

Finally there are three appendices. The first deals with some technical points concerning the operators in L^2 spaces defined by first order differential operators. The second recalls some basic elementary facts from functional analysis. We have put this material in appendices in order to avoid breaking up the presentation with technicalities. The third appendix reproduces a short history of the $\bar{\partial}$-Neumann problem which I found in my files together with reprints by Spencer. This history is probably written by him and is included as a second opinion.

The early work on the $\bar{\partial}$-Neumann problem owes much more to D.C. Spencer than is documented in print. His insight and enthusiasm inspired much of that activity. When the first version of this article was finished, in January 2002, I had hoped to be able to get his comments on my presentation but learned that he had died a month before. Instead I take this opportunity to dedicate the article to his memory.

I would also like to thank Ragnar Sigurdsson who caused this article to be written by encouraging me to give a historical survey talk at the Nordan meeting in Reykjavík March 8-10, 2002. I am also grateful to J.J. Kohn for commenting on the first version of this manuscript, and to Mei-Chi Shaw for her remarks on another appendix in that draft which in extended form has now become a separate article [32].

2. The theorem of de Rham and Hodge theory

In his thesis de Rham [51] proved that on any compact C^∞ manifold Ω (no boundary) there always exists a closed p form with given periods, and that it is unique modulo derived forms. In modern language, the cohomology with real (or complex) coefficients is isomorphic to the quotient of the space of closed forms by the space of derived ones. Introducing a smooth Riemannian metric in Ω gives a Euclidean metric on the p forms at a point, and by integration of the square with respect to the Riemannian volume measure one defines an L^2 norm for a smooth p form f on Ω. If $df = 0$, where d is the exterior differential operator, and the norm of f is minimal in its residue class modulo derived forms, then $(f, dg) = 0$ when g is a $p-1$ form. Hence $d^*f = 0$ where d^* is the adjoint of d, and then

$$\|f + dg\|^2 = \|f\|^2 + \|dg\|^2$$

which means that f is the only minimizer. Hodge, who was an algebraic geometer, had difficulties with proving the *existence* of a smooth minimizer, in analogy with the classical difficulties in Riemann's use of Dirichlet's principle. Hodge's first version in [23] was in his own words 'crude in the extreme' (Atiyah [4, p. 178]) and a later version in his book [24] also contained a serious error [4, p. 179]. A complete justification was given by Hermann Weyl in the classical article [55]. The proof has become so integrated in the theory that today the problem looks quite trivial. The equations $df = 0$ and $d^*f = 0$ can be summed up in $\Delta f = 0$ where $\Delta = d^*d + dd^*$, for $(\Delta f, f) = (df, df) + (d^*f, d^*f) = 0$ if and only if $df = 0$ and $d^*f = 0$; then f is called a *harmonic form.* The Hodge Laplacian Δ is a second-order selfadjoint *elliptic* differential operator on p forms; the principal symbol is the Riemannian square $|\xi|^2$ of the cotangent vectors times the identity. Hence every smooth p form f has a unique decomposition with smooth p forms g and h,

$$f = \Delta g + h = d^*\, dg + dd^* g + h\ , \tag{2.1}$$

where $\Delta h = 0$. If $df = 0$ then $dd^*\, dg = 0$, hence $\|d^*\, dg\|^2 = (dd^*\, dg, dg) = 0$, so $f = d(d^*g) + h$, which means that h is the unique harmonic form in the residue class of f.

The aim of Hodge was to study complex projective algebraic manifolds. Using the natural metric there he refined the preceding Hodge theory by decomposing the harmonic forms according to type. Recall that a differential form on a complex manifold is of type (p, q) if it is of degree p in dz and q in $d\bar{z}$. The exterior differential operator d has a unique decomposition $d = \partial + \bar\partial$ where ∂ (resp. $\bar\partial$) maps forms of type (p, q) to forms of type $(p+1, q)$ (resp. $(p, q+1)$). The fact that $d^2 = 0$ implies that $\partial^2 = 0$, that $\bar\partial^2 = 0$, and that $\bar\partial\partial + \partial\bar\partial = 0$. To have the adjoint d^* split nicely requires an important condition on the metric. A complex manifold with a Riemannian metric ds^2 which is hermitian, thus

$$ds^2 = \sum_{j,k=1}^{n} h_{jk}(z)\, dz_j\, d\bar{z}_k, \quad h_{jk} = \overline{h_{kj}}\ ,$$

in local complex coordinates, is called a *Kähler manifold* if for every point one can choose local complex coordinates vanishing there such that $h_{jk}(z) = \delta_{jk} + O(|z|^2)$. (The geometrical significance is that parallel transport is complex linear.) This implies that the invariantly associated exterior differential form of type (1,1)

$$\sum_{j,k=1}^{n} h_{jk}(z)\, dz_j \wedge d\bar{z}_k$$

is closed, for it is obvious that the differential vanishes at a point where the first order derivatives of all h_{jk} vanish. The condition is also sufficient, for given local complex coordinates vanishing at the chosen point we can by a linear diagonalization achieve that $h_{jk}(0) = \delta_{jk}$. The vanishing of the differential implies that $\partial h_{jk}(0)/\partial z_l = \partial h_{lk}(0)/\partial z_j$. If we set $w_j = z_j + \frac{1}{2}\sum_{k,l=1}^{n} \partial h_{kj}(0)/\partial z_l\, z_l z_k$, it follows from this symmetry that

$$dw_j = dz_j + \sum_{k,l=1}^{n} \partial h_{kj}(0)/\partial z_l z_l\, dz_k, \quad \text{hence}$$

$$|dw_j|^2 = |dz_j|^2 + \sum_{k,l=1}^{n} \partial h_{kj}(0)/\partial z_l z_l\, dz_k\, d\bar{z}_j + \sum_{k,l=1}^{n} \partial h_{jk}(0)/\partial \bar{z}_l \bar{z}_l\, dz_j\, d\bar{z}_k + O\big(|z|^2\big)|dz|^2,$$

where we have used the hermitian symmetry. Hence the coefficients of $ds^2 - \sum_1^n |dw_j|^2$ vanish to second order at the origin which proves that the w_j coordinates have the required property. Using such coordinates it is clear that the Hodge Laplacian maps a form of type (p, q) to another form of type (p, q) and that the adjoint of ∂ (resp. $\bar\partial$) maps it to a form of type $(p-1, q)$ (resp. $(p, q-1)$.) If f is a harmonic form of degree r and we write $f = \sum_{p+q=r} f_{p,q}$ where $f_{p,q}$ is of type (p, q), then $0 = \Delta f = \sum \Delta f_{p,q}$ implies $\Delta f_{p,q} = 0$ since $\Delta f_{p,q}$ is of type (p, q).

Hence the space $\mathcal{H}^r$ of harmonic forms of degree r is the direct sum of the spaces of harmonic forms $\mathcal{H}^{p,q}$ of type (p,q) with $p+q=r$, so the Betti numbers $h^r = \dim \mathcal{H}^r$ can be split into $h^r = \sum_{p+q=r} h^{p,q}$ where $h^{p,q} = \dim \mathcal{H}^{p,q}$. These numbers are independent of the choice of Kähler metric, for a positive linear combination of Kähler metrics is Kähler, and the splitting of h^r into a sum of integers must vary continuously with the metric.

The closed form $\partial\bar\partial \log(|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2)$ in $\mathbf{C}^{n+1} \setminus \{0\}$ is a lifting of a closed form in the projective space $P^n_{\mathbf{C}}$ which when $z_0 = 1$ corresponds to the Mannoury metric

$$|dz|^2/(1+|z|^2) - |\langle \bar z, dz\rangle|^2/(1+|z|^2)^2$$

which is obviously Kähler at the origin, hence everywhere in view of the invariance. An analytic submanifold of $P^n_{\mathbf{C}}$ with the induced metric is therefore also Kähler, and it is in this context that Hodge originally introduced the invariants $h^{p,q}$. A classical introduction to Kähler manifolds is Weil's book [54]. The preceding brief history is taken from Atiyah [4] where in addition to biographical information there is much more on the mathematical work of Atiyah's teacher Hodge.

The de Rham theorem is also applicable to a manifold with boundary. Let Ω be an open relatively compact subset of a C^∞ manifold and assume that the boundary $\partial\Omega$ is smooth. The cohomology of $\overline{\Omega}$ in degree p with real (or complex) coefficients is still isomorphic to the quotient of the space of smooth p forms in $\overline{\Omega}$ by the space of derived ones. If f is a p form with $df = 0$ and the L^2 norm is minimal in the class of f, then $(f, dg) = 0$ for all smooth $p-1$ forms g. When g has compact support in Ω this means precisely that $d^*f = 0$ where d^* is the formal adjoint of d, mapping p forms to $p-1$ forms. However, for general g we must also have $d\varrho \lrcorner f = 0$ on $\partial\Omega$ if ϱ is a *defining function* for Ω, that is, $\varrho < 0$ in Ω, $\varrho = 0$ and $d\varrho \neq 0$ on $\partial\Omega$. (Here $\lrcorner$ denotes inner multiplication which is adjoint to exterior multiplication.) Together these conditions mean that f is in the kernel of the minimal differential operator d^*_c defined by d^*, that is, the closure in L^2 of the operator defined at first only for smooth forms with compact support in Ω. (See Appendix A.) It is the adjoint of the maximal operator defined in L^2 by d, which we shall simply denote by d. The appropriate Laplace operator to consider is now $\Delta = d^*_c d + dd^*_c$. An equation $\Delta u = f$ for p forms u and f means as before an *elliptic differential equation* in Ω, with principal symbol $|\xi|^2$, but in addition there are now two boundary conditions

$$d\varrho \lrcorner u = 0, \quad d\varrho \lrcorner du = 0, \quad \text{on } \partial\Omega\,.$$

In general the interior product $d\varrho \lrcorner$ maps p forms at the boundary to $p-1$ forms on the boundary which are $p-1$ forms in the boundary lifted by the projection $\overline{\Omega} \mapsto \partial\Omega$ along the normal. Hence the boundary operators $u \mapsto d\varrho \lrcorner du|_{\partial\Omega}$ and $u \mapsto d\varrho \lrcorner u|_{\partial\Omega}$ take values in fiber bundles of dimension $\binom{n-1}{p}$ and $\binom{n-1}{p-1}$, respectively. The sum is equal to the fiber dimension $\binom{n}{p}$ of the bundle of p forms in $\overline{\Omega}$. (This is also a consequence of the fact that $d\varrho \lrcorner$ defines an exact sequence in the exterior algebra.) *The boundary problem is elliptic.* To verify this it suffices to consider a boundary point with geodesic local coordinates such that $d\varrho = -dx_n$. Then ellipticity means that bounded p form solutions of the constant-coefficient Laplacian $\Delta u = 0$ in $\{x \in \mathbf{R}^n; x_n > 0\}$ which are purely exponential but not constant in the tangential variables $x' = (x_1, \ldots, x_{n-1})$ must vanish if they satisfy the boundary conditions

$$dx_n \lrcorner u = 0, \quad dx_n \lrcorner du = 0, \quad \text{when } x_n = 0\,.$$

This means that $u = e^{i\langle x', \xi'\rangle - x_n|\xi'|} v$ where v is a p form with constant coefficients and $0 \neq \xi' \in \mathbf{R}^{n-1}$. With $\xi_n = i|\xi'|$ we have $du = i\langle \xi, dx\rangle \wedge u$, so the boundary conditions are

$$dx_n \lrcorner v = 0, \quad dx_n \lrcorner (\langle \xi, dx\rangle \wedge v) = 0\,.$$

The first condition means that dx_n is not a factor in any term in v, hence $\langle \xi, dx\rangle \wedge v - \xi_n dx_n \wedge v$ does not contain dx_n, so $dx_n \lrcorner (\langle \xi, dx\rangle \wedge v) = \xi_n v$. Since $\xi_n = i|\xi'| \neq 0$ this does not vanish unless $v = 0$, which proves ellipticity.

The theory of elliptic boundary problems now gives precise existence theorems: The set $\mathcal{H}^p$ of L^2 forms with $df = 0$ and $d_c^* f = 0$ is finite-dimensional and its elements are smooth in $\overline{\Omega}$. Every p form f which is smooth in $\overline{\Omega}$ has a unique decomposition

$$f = d_c^* \, dg + dd_c^* g + h$$

with $h \in \mathcal{H}^p$ and g smooth in $\overline{\Omega}$. If $df = 0$ then $dd_c^* \, dg = 0$, hence $\|d_c^* \, dg\|^2 = (dd_c^* \, dg, dg) = 0$, so $f = d(d_c^* g) + h$ which means that h is in the same cohomology class as f, and h is uniquely determined, for if $h \in \mathcal{H}^p$ and $h = du$ then $(h, h) = (du, h) = (u, d_c^* h) = 0$.

The cohomology of Ω $(\overline{\Omega})$ in degree p is thus isomorphic to $\mathcal{H}^p$. The cohomology with compact supports in Ω can similarly be identified with the harmonic forms corresponding to the Laplacian $d^* d_c + d_c d^*$, where d_c is the minimal operator defined by d. In Ω this is the same Laplacian as before, but the boundary conditions are now

$$d\varrho \wedge u = 0, \quad d\varrho \wedge d^* u = 0 \quad \text{on } \partial\Omega\,.$$

The verification of ellipticity is essentially the same as before.

For a p form f we have now obtained two orthogonal decompositions,

$$f = d_c^* \, dg_1 + dd_c^* g_1 + h_1, \quad \text{where } dh_1 = 0 \text{ and } d_c^* h_1 = 0\,, \tag{2.2}$$

$$f = d^* d_c g_2 + d_c d^* g_2 + h_2, \quad \text{where } d_c h_2 = 0 \text{ and } d^* h_2 = 0\,. \tag{2.3}$$

This gives a third orthogonal decomposition,

$$f = d_c^* \, dg_1 + d_c d^* g_2 + h, \quad \text{where } dh = 0 \text{ and } d^* h = 0\,, \tag{2.4}$$

for defining h by the first equality we obtain $dh = df - dd_c^* \, dg_1 = 0, d^* h = d^* f - d^* d_c d^* g_2 = 0$. The decomposition is orthogonal since the range of d_c^* is orthogonal to the kernel of d which contains the kernel of d_c, and the range of d_c is orthogonal to the kernel of d^* which contains the kernel of d_c^*. The orthogonal projection of f on the infinite-dimensional space of forms with $d^* h = 0$ and $dh = 0$ is thus $f - d_c^* \, dg_1 - d_c d^* g_2$. Since $dh = 0$ and $d^* h = 0$ imply $\Delta h = 0$ in Ω, hence that $h \in C^\infty$, it follows that the projection $f \mapsto h$ has a C^∞ kernel. The complex analogue will be discussed in the following section.

The preceding results on Hodge theory in a manifold with boundary were already obtained in the early and mid 1950's by classical integral equation methods. (See e.g., Conner [9] and the references therein.) The full theory of elliptic boundary problems was then not yet available in the literature in a systematic and easily applicable form, but the theory of singular integral equations due to Giraud and others had a comparable scope. A theory of elliptic boundary problems was presented in full generality in Agmon–Douglis–Nirenberg [1] and Hörmander [28, Chapter 10]. The introduction of pseudodifferential operators gave further simplifications in principle (see e.g., Hörmander [29, Chapter 20]). The theory of elliptic boundary problems also yields precise results on the regularity of the Hodge decomposition of a form f with given finite regularity. Roughly speaking, each term in the decomposition of f is at least as regular as f.

3. The Bergman kernel

Let $\Omega \subset \mathbf{C}^n$ be an open set, and denote by $\mathcal{H}(\Omega)$ the subspace of $L^2(\Omega)$ consisting of holomorphic functions. When $f \in \mathcal{H}(\Omega)$ we have by the mean value property of the harmonic function f that

$$|f(z)| \le C_n^{-\frac{1}{2}} \|f\|/d(z)^n, \quad z \in \Omega,$$

where $\|f\|$ is the norm in $L^2(\Omega)$, $d(z)$ is the distance from $z \in \Omega$ to $\partial\Omega$, and $C_n = \pi^n/n!$ is the volume of the unit ball in $\mathbf{C}^n = \mathbf{R}^{2n}$. Hence $\mathcal{H}(\Omega)$ is a closed subspace of $L^2(\Omega)$, thus a Hilbert space, and since the linear map $\mathcal{H}(\Omega) \ni f \mapsto f(z)$ is continuous for every $z \in \Omega$ we have

$$f(z) = \int_\Omega K(z,w) f(w)\, d\lambda(w), \quad f \in \mathcal{H}(\Omega),\ z \in \Omega, \tag{3.1}$$

where $w \mapsto \overline{K(z,w)}$ is in $\mathcal{H}(\Omega)$ and depends continuously on z. (Here $d\lambda$ is the Lebesgue measure.) For a general $f \in L^2(\Omega)$ the integral is equal to $g(z)$ where g is the orthogonal projection of f in $\mathcal{H}(\Omega)$. Since an orthogonal projection is selfadjoint, K has the hermitian property $K(z,w) = \overline{K(w,z)}$, so $z \mapsto K(z,w)$ is in $\mathcal{H}(\Omega)$, and $\Omega \times \Omega \ni (z,w) \mapsto K(z,w)$ is a harmonic function, hence real analytic. It is called the Bergman kernel.

Example. For the ball $\Omega = \{z \in \mathbf{C}^n; |z| < R\}$ the Bergman kernel is

$$K(z,w) = C_n^{-1} R^2 \big(R^2 - \langle z, \bar{w}\rangle\big)^{-n-1}, \quad z, w \in \Omega; \quad C_n = \pi^n/n!\,. \tag{3.2}$$

It suffices to verify this when $R = 1$. First we shall prove that

$$f(z) = \int_\Omega K(z,w) f(w)\, d\lambda(w), \quad z \in \Omega,$$

if f is a polynomial. By the unitary invariance we may assume that $z_2 = \cdots = z_n = 0$, and we may also assume that $f(z) = z^\alpha$ for some multiindex α. Since $K(z,w)$ is equal to $C_n^{-1} \sum_0^\infty \langle z, \bar{w}\rangle^k (k+n)!/n!k!$ we have then

$$\int_\Omega K(z,w) w^\alpha\, d\lambda(w) = \sum_{k=0}^\infty \int_\Omega z_1^k \bar{w}_1^k w^\alpha (k+n)!/n!k!\, d\lambda(w)/C_n\,.$$

The only non-vanishing term occurs when $\alpha_1 = k$ and $\alpha_2 = \cdots = \alpha_n = 0$. Since

$$\begin{aligned}\int_\Omega |w_1|^{2k}\, d\lambda(w) &= C_{n-1} \int_0^1 \left(1 - r^2\right)^{n-1} r^{2k} 2\pi r\, dr \\ &= \pi C_{n-1} \int_0^1 (1-t)^{n-1} t^k\, dt = \pi C_{n-1} (n-1)!k!/(n+k)! = C_n n!k!/(n+k)!\,,\end{aligned}$$

this proves (3.2) for polynomials f, hence for functions f which are holomorphic in a neighborhood of $\overline{\Omega}$. If φ is a continuous function with compact support in Ω we can apply this to $K\varphi$, which gives $K(K\varphi) = K\varphi$, hence using the Hermitian symmetry

$$\|K\varphi\|^2 = (K\varphi, K\varphi) = (K\varphi, \varphi) \le \|K\varphi\| \|\varphi\|\,,$$

so $\|K\varphi\| \le \|\varphi\|$, which proves that K defines a selfadjoint operator with norm ≤ 1 with range in $\mathcal{H}(\Omega)$ and equal to the identity on $\mathcal{H}(\Omega)$. Hence K is the orthogonal projection, so $\|K\| = 1$, and since $\int_\Omega K(z,w) K(w,\zeta)\, d\lambda(w) = K(z,\zeta)$ when $z, \zeta \in \Omega$, we have in particular

$$K(z,z) = \int_\Omega |K(z,w)|^2\, d\lambda(w) = \sup_{f \in \mathcal{H}(\Omega)} |f(z)|^2/\|f\|^2\,.$$

To study the Bergman kernel $K(z, w)$ of an arbitrary open bounded set $\Omega \subset \mathbf{C}$ with smooth boundary we shall first prove that if $f \in C_0^\infty(\Omega)$, then

$$Kf = f - \partial u/\partial z, \quad \text{where } \tfrac{1}{4}\Delta u = \partial f/\partial \bar{z} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{3.3}$$

Defining the map K for a moment by (3.3) we shall prove that the closure of K is in fact the Bergman projector. First note that Kf is holomorphic since $\partial(Kf)/\partial\bar{z} = \partial f/\partial\bar{z} - \partial^2 u/\partial z\partial\bar{z} = 0$. Secondly $f - Kf = \partial u/\partial z$ is orthogonal to every holomorphic function g, for $(\partial u/\partial z, g) = -(u, \partial g/\partial\bar{z}) = 0$, since $u = 0$ on $\partial\Omega$. Hence K is indeed the orthogonal projection on the holomorphic functions in Ω, restricted to a dense subset of $L^2(\Omega)$.

Let $G(z, w)$ be Green's function for the Laplacian with Dirichlet boundary condition in Ω, and write $G(z, w) = E(z - w) + G_0(z, w)$ where $E(z - w) = (2\pi)^{-1} \log|z - w|$ is the fundamental solution. Then G_0 is smooth in $\overline{\Omega} \times \overline{\Omega} \setminus \Delta_{\partial\Omega}$, and by (3.3) we have

$$(Kf)(z) - f(z) = -\partial/\partial z \int_\Omega 4G(z, w)\partial f(w)/\partial\bar{w}\, d\lambda(w) = \int_\Omega 4\partial^2 G(z, w)/\partial z\partial\bar{w} f(w)\, d\lambda(w)$$

in the sense of distributions. Since $\partial^2 E(z - w)/\partial z\partial\bar{w} = -\partial^2 E(z - w)/\partial z\partial\bar{z} = -\delta(z - w)/4$ it follows that

$$K(z, w) = 4\partial^2 G_0(z, w)/\partial z\partial\bar{w}\,. \tag{3.4}$$

This formula shows not only that the Bergman kernel can be obtained from Green's function, but in view of the boundary conditions $G_0(z, w) = -E(z - w)$ when z or w is in the boundary, one can recover Green's function from the Bergman kernel and two applications of Cauchy's integral formula.

The definition of the Bergman kernel uses only the most elementary facts from analytic function theory and nothing which is specific for the case of several variables. Nevertheless it was in the early 1950's by some mathematicians regarded as a promising approach to function theory in several variables. In a study of potential theory and conformal mappings in the one-dimensional case based on the Bergman kernel, Garabedian and Schiffer [19, p. 164] wrote: "Our new attack [...] has the advantage of generality, since it is applicable in the case of partial differential equations of elliptic type and since for functions of several complex variables the theory of the kernel function is well developed [...]."

As proposed in [19], Garabedian [17] took the kernel function as the starting point for a study of analytic functions of several complex variables, trying to connect the Cauchy–Riemann equations for functions of n complex variables with an associated system of n partial differential equations for n unknown functions, which he called generalized Laplace equations. Recall that if $K(z, w)$ is the Bergman kernel function of an open set $\Omega \subset \mathbf{C}^n$ and w is fixed, then $\varphi(z) = K(z, w)$ minimizes $\int_\Omega |\varphi(z)|^2\, d\lambda(z)$ among all $\varphi \in \mathcal{H}(\Omega)$ with $\varphi(w) = K(w, w)$; the minimum is $K(w, w) = \varphi(w)$. The minimizing property means that

$$\int \varphi(z)\overline{h(z)}\, d\lambda(z) = 0 \quad \text{if } \bar\partial h = 0 \text{ and } h(w) = 0\,,$$

so one is "led to deduce that there exist Lagrange multipliers" $g_0, g_1, \dots, g_n$ where $g_0 \in \mathbf{C}$ and $g_1, \dots, g_n$ are functions in Ω such that

$$\int \varphi(z)\overline{h(z)}\, d\lambda(z) + \sum_1^n \int g_j(z)\overline{\partial h(z)/\partial\bar{z}_j}\, d\lambda(z) + g_0\overline{h(w)} = 0\,.$$

This is a very questionable point in the argument. If a linear form in a Hilbert space H_1 vanishes in the kernel of a closed linear map T from H_1 to another Hilbert space H_2, that is, in the orthogonal space of the range of the adjoint $T^* : H_2 \to H_1$, one can only conclude that the linear form is the scalar product with an element in the *closure* of the range of T^*. The argument is therefore at best heuristic. In that spirit we assume as in [17] that the g_j exist and are regular up to the boundary. Taking $h \in C_0^\infty(\Omega \setminus \{w\})$ we find that $\varphi = \sum_1^n \partial g_j/\partial z_j$ in $\Omega \setminus \{w\}$, with $h = \varphi$ we obtain $g_0 = -1$, and allowing general $h \in C^\infty(\overline{\Omega})$ we get

$$\varphi = \sum \partial g_j/\partial z_j + \delta_w, \quad \sum_1^n g_j \partial \varrho/\partial z_j = 0 \text{ on } \partial\Omega$$

if ϱ is a defining function for Ω. Let

$$E(z, w) = -(n-2)!\pi^{-n}|z-w|^{2-2n}, \quad z, w \in \mathbf{C}^n ,$$

be the homogeneous fundamental solution of $\frac{1}{4}\Delta = \sum_1^n \partial^2/\partial z_j \partial \bar{z}_j$, thus

$$\delta_w = \sum_1^n \partial/\partial z_j \left(\partial E(z, w)/\partial \bar{z}_j\right) = \sum_1^n \partial/\partial z_j \left((n-1)!\pi^{-n}(z_j - w_j)|z-w|^{-2n}\right) ,$$

which gives

$$K(z, w) = \varphi(z) = \sum_1^n \partial G_j(z)/\partial z_j; \quad G_j(z) = g_j(z) + (n-1)!\pi^{-n}(z_j - w_j)|z-w|^{-2n} .$$

In [17, Theorem 1] it is stated without any justification that G_j is regular in $\overline{\Omega}$; we have of course

$$\sum_1^n G_j(z)\partial\varrho(z)/\partial z_j = \sum_1^n \partial E(z, w)/\partial \bar{z}_j \partial\varrho(z)/\partial z_j, \quad z \in \partial\Omega . \tag{3.5}$$

The Cauchy–Riemann equations for φ lead to the equations

$$\sum_{j=1}^n \partial^2 G_j(z)/\partial z_j \partial \bar{z}_k = 0, \quad k = 1, \dots, n , \tag{3.6}$$

called "generalized Laplace equations" in [17]. It was observed that they are the formal Euler equations for minimizing

$$\int_\Omega \left| \sum_1^n \partial G_j(z)/\partial z_j \right|^2 d\lambda(z)$$

but that the lack of ellipticity does not allow one to apply the Dirichlet principle. A minimizer would be far from unique anyway, and in [17] it is therefore required that $\int_\Omega \sum_1^n |G_j(z)|^2 \, d\lambda(z)$ should also be minimal, that is, that

$$\sum_1^n \int G_j(z)\overline{h_j(z)}\, d\lambda(z) = 0 \text{ if } \sum_1^n \partial h_j(z)/\partial z_j = 0 \text{ in } \Omega, \ \sum_1^n h_j(z)\partial\varrho(z)/\partial z_j = 0 \text{ in } \partial\Omega .$$

With the same heuristic argument as before this would mean that there is a function ϑ such that

$$\sum_1^n \int_\Omega G_j(z)\overline{h_j(z)}\, d\lambda(z) + \int_\Omega \vartheta(z) \sum_1^n \overline{\partial h_j(z)/\partial z_j}\, d\lambda(z) = 0$$

so $G_j(z) = \partial\vartheta(z)/\partial\bar{z}_j$ and $K(z, w) = \frac{1}{4}\Delta\vartheta$,

$$\sum_1^n \partial\vartheta(z)/\partial\bar{z}_j \partial\varrho(z)/\partial z_j = \sum_1^n \partial E(z, w)/\partial\bar{z}_j \partial\varrho(z)/\partial z_j \quad \text{on } \partial\Omega . \tag{3.5)$'$}$$

In [17, Theorem 2] it is claimed that minimizing $\|\Delta\vartheta\|$ under this boundary condition yields the Bergman kernel. Both "Theorems" 1 and 2 are mentioned as statements one is "led to", but at the end of Section 3 in [17] it is admitted that what has been given is only a "formal apparatus for a modified Dirichlet principle in the theory of analytic functions of several complex variables".

Two years later Garabedian [18] attacked the problem of finding the function ϑ above. Section 3 starts as follows, with slightly modified notation: "Given an arbitrary complex-valued function p on $\partial\Omega$, we let β denote a function defined in Ω with

$$\sum_{j=1}^n \partial\beta/\partial\bar{z}_j \partial\varrho/\partial z_j = p \quad \text{on } \partial\Omega ,$$

such that

$$\int_\Omega |\Delta\beta|^2 \, d\lambda = \text{minimum} .$$

We shall study the nature of the extremal function β by making elementary variations." No motivation at all is given for the *existence* of a (regular) minimizer which hides the real difficulty, so the article only provides the fairly simple verification that if there is such a minimizer, then it leads to a formula for the Bergman kernel as above. However the statements are labelled as theorems, although no valid proofs are given.

In functional analytic terms the approach of [17, 18] can be described as follows. (See also Appendix B.) Let T be the maximal operator defined by $\bar\partial$ from $L^2(\Omega)$ to $L^2_{(0,1)}(\Omega)$, the space of (0, 1) forms with coefficients in $L^2(\Omega)$. Thus $u \in L^2(\Omega)$ is in the domain of T if and only if $\partial u/\partial\bar{z}_j \in L^2(\Omega)$ for $j = 1, \dots, n$ in the sense of distribution theory, and then $Tu = \sum_1^n \partial u/\partial\bar{z}_j \, d\bar{z}_j$. The adjoint T^* is the closure of the operator defined by $\sum_1^n g_j(z)\, d\bar{z}_j \mapsto -\sum_1^n \partial g_j/\partial z_j$, where g is a smooth (0, 1) form with compact support in Ω. In a weak sense, $\sum g_j \partial\varrho/\partial z_j = 0$ on $\partial\Omega$ when g is in the domain of T^*. The Bergman operator K is the orthogonal projection on Ker T, which is the orthogonal space of the range of T^*. *If the range of T^* is closed* and $u \in L^2(\Omega)$, then $u - Ku = T^*g$ for some form g in the domain of T^*, and there is a unique choice of g orthogonal to Ker T^*, that is, in the range of T (which is closed if and only if the range of T^* is closed). Thus we can write $g = T\vartheta$ and obtain

$$Ku = u - T^*T\vartheta .$$

[When $n = 1$ then T^* is injective so this second step is not required; $TT^*g = Tu$ gives g as the solution of a Dirichlet problem which leads to (3.4).] Of course ϑ is not unique either unless we require that it is orthogonal to the kernel of T; T^*T is a densely defined, selfadjoint injective and surjective map in this space. Summing up, to carry out Garabedian's program one needs to prove that T, thus T^*, has a closed range and that smoothness is maintained in the preceding steps.

Garabedian's articles [17, 18] were followed by [20] in collaboration with Spencer. Motivated by Hodge's original definition of harmonic forms on a compact manifold they considered in an open set $\Omega \subset \mathbf{C}^n$ with smooth boundary the forms f of type $(p, 0)$ such that $\partial f = 0$ and $\delta f = 0$ where δ is the formal adjoint of ∂. If we write

$$f = \sum_{|I|=p}{}' f_I(z)\, dz^I, \quad \text{where } I = (i_1, \dots, i_p),\ dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} ,$$

f_I is antisymmetric in the indices, and $\sum'$ means summation only for $i_1 < \cdots < i_p$, then

$$\partial f = \sum_{j=1}^{n} \sum_{|I|=p}' \partial f_I(z)/\partial z_j \, dz_j \wedge dz^I, \quad \delta f = -\sum_{j=1}^{n} \sum_{|K|=p-1}' \partial f_{jK}(z)/\partial \bar{z}_j \, dz^K .$$

Since $\delta\partial + \partial\delta = -\Delta/4$, where Δ is the Laplacian, the equations $\partial f = 0$ and $\delta f = 0$ imply that f_I is a harmonic function. Hence the orthogonal projection in $L^2_{(p,0)}(\Omega)$ on the subspace $\mathcal{H}_{(p,0)}(\Omega)$ of "harmonic" forms has a kernel $K(z, w)$ such that $K(z, w)$ (resp. $\overline{K(z, w)}$) is in $\mathcal{H}_{(p,0)}(\Omega)$ as a form in z (resp. w). Denote by ∂ and δ the maximal operators defined in $L^2_{(p,0)}$ by ∂ and δ. The adjoint operators are the minimal operators δ_c and ∂_c defined as closures of δ and ∂ acting on (smooth) forms of compact support of type $(p+1, 0)$ and $(p-1, 0)$, respectively. Now Ker ∂ (resp. Ker δ) is the orthogonal space of the range of δ_c (resp. ∂_c), and these spaces are orthogonal since $\partial^2 = 0$. Hence the orthogonal space of $\mathcal{H}_{(p,0)}$ is the orthogonal sum of the closure of the range of δ_c and the closure of the range of ∂_c [cf. (2.4)]. *If these ranges are closed then*

$$Kf = f - \delta_c g - \partial_c h, \quad \text{where } \partial\delta_c g = \partial f, \ \delta\partial_c h = \delta f ,$$

when f is smooth and of compact support in Ω. [This is the analogue of (2.4).] With E now denoting the fundamental solution of $-\frac{1}{4}\Delta = \delta\partial + \partial\delta$ (note the change of sign) we have $f = (\delta\partial + \partial\delta)Ef$, hence

$$Kf = \delta G + \partial H \quad \text{where } G = \partial Ef - g, \ H = \delta Ef - h .$$

Then $\partial\delta G = \partial(\delta\partial + \partial\delta)Ef - \partial\delta g = 0$ and similarly $\delta\partial H = 0$. We should take g orthogonal to the kernel of δ_c, that is, in the closure of the range of ∂, which implies $\partial g = 0$, hence $\partial G = 0$, so G is harmonic. Similarly we should take h orthogonal to the kernel of ∂_c, which implies $\delta h = 0$ and $\delta H = 0$. Thus both G and H are harmonic, so boundary values are defined in a weak sense. That g is in the domain of δ_c and h in the domain of ∂_c gives $g \lrcorner \bar{\partial}\varrho = 0$ and $h \wedge \partial\varrho = 0$ on $\partial\Omega$, that is, the boundary conditions

$$G \lrcorner \bar{\partial}\varrho = (\partial Ef) \lrcorner \bar{\partial}\varrho, \quad H \wedge \partial\varrho = (\delta Ef) \wedge \partial\varrho ,$$

in addition to the equations

$$\partial\delta G = 0, \ \partial G = 0; \quad \delta\partial H = 0, \ \delta H = 0 .$$

Here $\lrcorner$ denotes the interior multiplication which is adjoint to exterior multiplication.

If the complex defined by the δ operator is exact, then $H = \delta Q$ where Q is a $(p, 0)$ form, and if Q is chosen orthogonal to the kernel then Q is in the closure of the range of ∂_c, hence $\partial Q = 0$, so Q is harmonic. As in [20, p. 227] one should then, as in the classical approach to the Neumann problem using integral equations, be able to write Q as a simple layer potential

$$Q = \int_{\partial\Omega} E(z, \zeta) u(\zeta) \, dS(\zeta)$$

where dS is the Euclidean surface measure on $\partial\Omega$ and $u(\zeta) = \sum'_{|I|=p} u_I(\zeta) \, dz^I$ is a form of type $(p, 0)$ depending on $\zeta \in \partial\Omega$. Then $H = \delta Q$ satisfies the equation $\delta\partial H = 0$ since $\delta H = 0$ and $\Delta H = 0$, so what remains is to satisfy the boundary condition

$$(\delta Q) \wedge \partial\varrho = (\delta Ef) \wedge \partial\varrho \quad \text{on } \partial\Omega . \tag{3.7}$$

The boundary value on $\partial\Omega$ of the potential of a simple layer σdS is a pseudodifferential operator of order -1 in $\partial\Omega$ acting on σ with principal symbol -2 times the reciprocal of the Euclidean length of a cotangent vector to $\partial\Omega$, and similarly for the derivatives. At a point on $\partial\Omega$ where the positive Im z_n axis is the interior normal, the principal symbol of the pseudodifferential operator $u \mapsto \delta Q|_{\partial\Omega}$ is interior multiplication by $-\zeta/\sqrt{|\xi|^2+|\eta'|^2}$ where $\zeta_j = \xi_j + i\eta_j$ and $(\xi, \eta') = (\xi_1, \eta_1, \ldots, \xi_{n-1}, \eta_{n-1}, \xi_n)$ is a real cotangent vector $\neq 0$ while $\eta_n = i\sqrt{|\xi|^2+|\eta'|^2}$. When $\xi' = \eta' = 0$ and $\xi_n > 0$ the principal symbol vanishes, so the pseudodifferential equation (3.7) is not elliptic. When [20] was written there was no theory of pseudodifferential operators so the operator was identified as a singular integral operator in the sense of Giraud, and it was claimed [20, p. 228] that his work implies solvability for all right-hand sides orthogonal to the solutions of the homogeneous adjoint equation. However, the crucial ellipticity condition, which occurs in the work of Giraud but is not as easily made explicit, was not examined so the argument is invalid. The same is true for the equation analogous to (3.7) which is required to find G. Just as in [17] and [18] the statements in [20] are therefore only justified on a formal level.

In [21] the formal calculations of [20] were extended to Kähler manifolds with boundary, which in view of the definition of Kähler manifolds given above is not surprising. Solvability of the analogue of (3.7) above was again claimed to follow "on the basis of the singular Fredholm theory," so no valid proofs were given. This seems to have been understood later on, for in an article by Kohn and Spencer [45, p. 89] discussed below the investigations in [20, 21] were described as "formal".

From the well-known expression above for the Bergman kernel for a ball and the fact that the Bergman kernel of Ω on the diagonal in $\Omega \times \Omega$ increases if Ω is decreased, it follows that the kernel is unbounded at every boundary point which can be reached by a ball containing Ω. On the other hand, at a boundary point z_0 where a deleted neighborhood of z_0 in some complex line through z_0 is contained in Ω, the kernel has an analytic extension to a neighborhood of (z_0, z_0) by a classical theorem of Hartogs. This suggests that the boundary behavior is highly dependent on some kind of curvature property of the boundary (cf. [8, Section 6]). However, no second order conditions on the boundary were assumed in the articles [17, 18, 20, 21]. In hindsight it is therefore evident that they must be seriously flawed.

4. Breakthrough, setback and success

For a compact Riemannian manifold Ω with boundary we saw in Section 2 that each residue class of closed p forms modulo derived p forms contains precisely one closed p form h such that $d_c^* h = 0$ where d_c^* is the minimal operator defined by the formal adjoint d^* of d. By de Rham's theorem it follows that the cohomology of Ω in degree p is isomorphic to the finite-dimensional vector space of such closed and coclosed forms.

In [45] Kohn and Spencer took up the analogous problem for a complex manifold Ω of dimension n, with boundary $\partial\Omega$. As already mentioned the exterior differential d is a sum $\partial + \bar\partial$ where ∂ (resp. $\bar\partial$) maps forms of type (p, q) to forms of type $(p+1, q)$ (resp. $(p, q+1)$), and we have

$$\partial^2 = 0, \quad \bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0\,.$$

For $\tau = (\tau_0, \tau_1) \in \mathbf{C}^2 \setminus \{0\}$, Kohn and Spencer studied more generally the linear combination

$$\partial_\tau = \tau_0\partial + \tau_1\bar\partial$$

which maps r forms to $r+1$ forms, with $\partial_\tau^2 = 0$. When $\tau_0\tau_1 \neq 0$ they established, using singular integral operator theory, results parallel to those of Hodge theory for the Riemannian case. This is not very surprising, for if T_τ denotes multiplication of forms of type (p, q) by $\tau_0^p \tau_1^q$, then $d_\tau = T_\tau \, d T_\tau^{-1}$. Hence the d_τ-closed (resp. d_τ derived) forms are precisely T_τ times the d-closed (resp. d derived forms). The only really new problem therefore concerns the operators ∂ and $\bar\partial$. Since they only differ by a conjugation it suffices to study the operator $\bar\partial$. For every fixed $p = 0, \dots, n$ it defines a complex where $\bar\partial$ acts from forms of type (p, q) to forms of type $(p, q+1)$. Without serious restriction we shall take $p = 0$. By the Dolbeault isomorphism [14] the cohomology is then isomorphic to the cohomology of Ω with values in the sheaf of germs of holomorphic functions, which was known to vanish when Ω is a Stein manifold, hence if $\Omega \subset \mathbf{C}^n$ and the Levi form of $\partial\Omega$ is strictly positive definite. It was therefore to be expected that if ϑ_c is the minimal operator defined by the formal adjoint ϑ of $\bar\partial$ with respect to the L^2 norms defined by a Hermitian metric in $\overline{\Omega}$, the corresponding "Laplace operator"

$$\Delta = \vartheta_c \bar\partial + \bar\partial \vartheta_c$$

should then be invertible in L^2. The study of this operator is called the *$\bar\partial$-Neumann problem.* Proving invertibility analytically would yield an alternative approach to the basic existence theorems in the theory of functions of several complex variables based on sheaf theory. However, little progress towards this goal was made in [45] where it was recognized that the problem leads to singular integral equations which are not "regularizable" in the sense of Giraud, that is, the boundary problem is non-elliptic in present terminology. (This is clear, for in the half space where $\operatorname{Im} z_n > 0$ a bounded solution of the homogeneous problem is given by $e^{iz_n} f$ where f is a constant (p, q) form not containing $d\bar z_n$.) In [45] only the case where Ω is a ball and the metric is the standard metric in $\mathbf{C}^n$ was successfully studied, by means of spherical harmonics expansions. Although such methods do not generalize it was stated optimistically on p. 132 that "there is little doubt that it is solvable and we hope to return to this question in another article."

The decisive step towards a solution of the $\bar\partial$-Neumann problem was taken by Morrey [47]. Whitney had proved two decades earlier that every C^∞ manifold M of dimension n admits a proper C^∞ embedding in $\mathbf{R}^{2n+1}$ (in fact in $\mathbf{R}^{2n}$). When M is real analytic and has a real analytic Riemannian metric, it had been proved by Bochner in the compact case and by Malgrange in general that there is a real analytic embedding. The existence of a real analytic embedding means precisely that real analytic functions on M are dense in $C^\infty(M)$. This could be proved using the analyticity of solutions of a real analytic Laplacian on M, but when M is compact Morrey gave a new proof which did not require such a tool. He first complexified M to a complex analytic manifold $\widetilde{M}$ of complex dimension n, where M is then embedded as a totally real submanifold of real dimension n. With respect to some C^∞ hermitian metric in $\widetilde{M}$, let M_R be the tubular neighborhood of M consisting of points at geodesic distance $< R$ to M. For small $R > 0$ this is a relatively compact open set in $\widetilde{M}$ with a C^∞ strictly pseudoconvex boundary. The key to the construction of global analytic functions on M was the estimate in L^2 norms on M_R given by

$$\|\varphi\|^2 \le CR^2\left(\|\vartheta_c\varphi\|^2 + \left\|\bar\partial\varphi\right\|^2\right), \tag{4.1}$$

for smooth (0, 1) forms φ in $\overline{M_R}$ in the domain of the minimal operator defined by ϑ. The proof, given in [47, Section 6], depends on an integration by parts and is quite elementary. However, it is the decisive discovery on which the L^2 techniques in the theory of functions of several complex variables have been built. From this estimate Morrey concluded that if K_R is the orthogonal projection in $L^2(M_R)$ on holomorphic functions then

$$\|u - K_R u\|^2 \le CR^2 \left\|\bar\partial u\right\|^2 . \tag{4.2}$$

Given a C^∞ function u on M one can by formal solution of the Cauchy–Riemann equations find an extension to $\widetilde{M}$ such that $\bar\partial u$ vanishes to infinite order on M, and combining (4.2) with simple standard estimates one concludes that $K_R u$ converges to u in $C^\infty(M)$ when $R \to 0$. (This is a somewhat oversimplified version of Morrey's arguments but the main points are in his article.) However, as we shall see later, there was a gap in Morrey's arguments.

Two months after Morrey's article in *Annals of Mathematics* Grauert [22] published an article in the same volume of the journal where he proved the embedding theorem without assuming compactness but using the full force of results from sheaf theory. Fortunately, this did not make everybody dismiss Morrey's article. J.J. Kohn extracted the crucial point in it, and in May 1961 he communicated a note [37] where he stated the following general result: Let Ω be an open, relatively compact subset of a complex manifold with a hermitian structure, and assume that the boundary $\partial\Omega$ is in C^∞ and strictly pseudoconvex. Then every (p, q) form $f \in C^\infty(\overline{\Omega})$ can be written

$$f = \left(\vartheta_c\bar\partial + \bar\partial\vartheta_c\right)u + h\,, \tag{4.3}$$

where u and h are (p, q) forms in $C^\infty(\overline{\Omega})$ and $\bar\partial h = 0$, $\vartheta_c h = 0$. The set of such forms h is finite-dimensional. The main point in the proof, generalizing Morrey's inequality (4.1), was the estimate

$$\|\partial f/\partial\bar z\|^2_\Omega + \|f\|^2_{\partial\Omega} \le C\left(\left\|\bar\partial f\right\|^2_\Omega + \|\vartheta_c f\|^2_\Omega + \|f\|^2_\Omega\right)\,. \tag{4.4}$$

Here $f \in C^\infty(\overline{\Omega})$ is assumed to be in the minimal domain of the adjoint operator ϑ, and the norms are L^2 norms in Ω or in $\partial\Omega$; $\partial f/\partial\bar z$ is defined in a cover with complex analytic coordinate patches. (Kohn assumed that the hermitian metric was Kähler close to the boundary but this hypothesis was removed in the later publications.) From this he concluded that f belongs to a compact set in the L^2 norm in Ω when the right-hand side is bounded.

Let us briefly sketch how (4.4) is proved, in the notationally simpler situation where $\Omega \subset \mathbf{C}^n$ with the standard Euclidean metric. We simplify notation further by considering only $(0, q)$ forms f, where $0 < q < n$,

$$f = \sum_{|J|=q}{}' f_J(z)\,d\bar z^J, \quad d\bar z^J = d\bar z_{j_1}\wedge\cdots\wedge d\bar z_{j_q},\ J = (j_1, \dots, j_q)\,.$$

The components f_J are defined for all q tuples J of indices between 1 and n and are antisymmetric in them; $\sum'$ means summation for increasing indices. Then

$$\bar\partial f = \sum_{j=1}^n\sum_{|J|=q}{}' \partial f_J/\partial\bar z_j\,d\bar z_j\wedge d\bar z^J, \quad \vartheta f = -\sum_{j=1}^n\sum_{|K|=q-1}{}' \partial f_{jK}/\partial z_j\,d\bar z^K\,,$$

and if f is in the domain of the minimal operator ϑ_c then

$$\sum_{j=1}^n \partial\varrho(z)/\partial z_j f_{jK}(z) = 0, \quad |K| = q-1,\ z \in \partial\Omega\,, \tag{4.5}$$

where ϱ is a defining function for Ω. Thus

$$|\bar\partial f|^2 = \sum_{j=1}^n\sum_{l=1}^n\sum_{|J|=|L|=q}{}' \varepsilon^{jJ}_{lL}\,\partial f_J/\partial\bar z_j\overline{\partial f_L/\partial\bar z_l}\,;$$

if $j \notin J$, $l \notin L$, and jJ is a permutation of lL, then ε^{jJ}_{lL} is the sign of the permutation but otherwise it is equal to 0. When $\varepsilon^{jJ}_{lL} \ne 0$ and $j = l$ then $J = L$, and if $j \ne l$ then removing l

from J gives the same multiindex K with length $q-1$ as removing j from L, thus

$$\varepsilon^{jJ}_{lL} = \varepsilon^{jJ}_{jlK}\varepsilon^{jlK}_{ljK}\varepsilon^{ljK}_{lL} = -\varepsilon^{J}_{lK}\varepsilon^{jK}_{L} ,$$

which gives the decisive identity

$$|\bar\partial f|^2 = \sum_{j=1}^n \sum_{|J|=q}' |\partial f_J/\partial \bar z_j|^2 - \sum_{j,l=1}^n \sum_{|K|=q-1}' \partial f_{lK}/\partial \bar z_j \overline{\partial f_{jK}/\partial \bar z_l} .$$

Since f is in the minimal domain of ϑ_c we obtain, if all derivatives are moved to the left by partial integration, that

$$\int_\Omega \left(|\bar\partial f|^2 + |\vartheta_c f|^2\right) d\lambda(z) = \sum_{j=1}^n \sum_{|J|=q}' \int_\Omega |\partial f_J/\partial \bar z_j|^2 \, d\lambda(z) - \sum_{j,l=1}^n \sum_{|K|=q-1}' \int_{\partial\Omega} \partial f_{lK}/\partial \bar z_j \overline{f_{jK}} \partial \varrho/\partial z_l \, dS ,$$

where dS is the Euclidean surface measure on $\partial\Omega$ and we have assumed that the Euclidean length of $d\varrho$ equals one there. Differentiation of (4.5) gives on $\partial\Omega$ that

$$\sum_{l=1}^n \sum_{j=1}^n \left(\partial^2 \varrho/\partial z_j \partial \bar z_l f_{jK} + \partial\varrho/\partial z_j \partial f_{jK}/\partial \bar z_l\right) \overline{g_l} = 0, \quad \text{if } \sum_1^n g_l \partial\varrho/\partial z_l = 0 .$$

By (4.5) we can take $g_l = f_{lK}$, which gives the fundamental identity

$$\int_\Omega \left(|\bar\partial f|^2 + |\vartheta_c f|^2\right) d\lambda(z) = \sum_{j=1}^n \sum_{|J|=q}' \int_\Omega |\partial f_J/\partial \bar z_j|^2 \, d\lambda(z) + \sum_{|K|=q-1}' \int_{\partial\Omega} \sum_{j,l=1}^n \partial^2\varrho/\partial z_j \partial \bar z_l f_{jK} \overline{f_{lK}} \, dS . \quad (4.6)$$

Here we have assumed that $\partial\Omega \in C^2$ and that $f \in C^2(\overline{\Omega})$ is a $(0, q)$ form satisfying (4.5), but with some care in the partial integration it suffices to assume that $f \in C^1(\overline{\Omega})$ satisfies (4.5). If $\partial\Omega$ is strictly pseudo-convex then $|f|^2$ is bounded by a constant times the Levi form in the last term in (4.6), since (4.5) is fulfilled, so we have proved (4.4) without the last term. The proof in the case of a manifold with hermitian metric is essentially the same if one works with a local orthonormal basis of $(1, 0)$ forms instead of dz_j. (See e.g., [26, Section 3.2].) Various error terms can be absorbed by the term $\|f\|^2_\Omega$ in the right-hand side of (4.4). Standard estimates for solutions of differential equations at a non-characteristic boundary show that (4.4) remains valid uniformly with the L^2 norm over $\partial\Omega$ replaced by the L^2 norm over a parallel surface close to $\partial\Omega$, and this implies that the set of smooth $(0, q)$ forms f in the domain of ϑ_c such that the right-hand side of (4.4) is bounded must be compact in $L^2(\Omega)$. However, it is not evident that this conclusion extends to all f in the intersection of the maximal domain of $\bar\partial$ and the minimal domain of ϑ, as required when one wants to use results from functional analysis.

At a CNRS meeting on partial differential equations in Paris in June 1962 Kohn [40] gave a lecture on his results. A manuscript for the first part of the article [38] containing the details was then available. Morrey, who was in Paris already before the meeting, was naturally eager to study it in detail, and it seemed to him that the preceding problem was not handled adequately. He went to the library to see what he had done in his article and discovered the same flaw there. As a result there was at the end of the meeting some doubt about all the results on the $\bar\partial$-Neumann problem in spite of the beautiful estimates. In [40] it was observed that all problems would be resolved if one could prove the following regularity theorem (for $\partial\Omega \in C^\infty$): Let $\mathcal{D}_1$ be the closure of $(0, q)$ forms in $C^\infty(\overline{\Omega})$ satisfying (4.5) in the norm $\|f\|_\Omega + \|\bar\partial f\|_\Omega + \|\vartheta f\|_\Omega$. Then, if $f \in \mathcal{D}_1$ and for some $(0, q)$ form $a \in C^\infty(\overline{\Omega})$ it holds that

$$(\bar\partial f, \bar\partial g) + (\vartheta_c f, \vartheta_c g) = (a, g), \quad g \in \mathcal{D}_1 , \tag{4.7}$$

it follows that $f \in C^\infty(\overline{\Omega})$, hence $\vartheta_c \bar\partial f + \bar\partial \vartheta_c f = a$. The difficulty in the proof is that even formally one can only expect f to have one derivative more than a (in the L^2 sense), as opposed to a gain of two derivatives for second order elliptic boundary problems. I had recently solved a similar difficulty for differential operators of principal type in [28, Chapter VIII] by working with norms involving powers of weight functions which help to make some constants small, and modifications of Sobolev norms involving a small parameter ε in such a way that they are very weak for a fixed ε but converge when $\varepsilon \to 0$ to the usual norm in a Sobolev space of high regularity. After the International Congress of Mathematicians in Stockholm in August 1962 Kohn stayed a while in Stockholm and during our discussions then I suggested to him that something like that might work for his problem. I was at the time too unfamiliar with the formal setup to be able to carry out such a project. However, Kohn succeeded in doing so. His proof of regularity was announced in [39] with all details provided in the second part of [38], which meant that the solution of the $\bar\partial$-Neumann problem was finally on solid ground. This justified Morrey's proof of the embedding theorem for compact real analytic manifolds, and gave a new proof of the Newlander–Nirenberg theorem that all integrable almost complex structures are actually genuinely complex analytic. The advantage of the new proof was that it follows with little additional effort from the existence theory for the $\bar\partial$ operator whereas the original proof relied on reduction to a non-linear problem solved by an iteration method. This potential application was already pointed out by Spencer before the successful solution of the $\bar\partial$-Neumann problem. The reason for the simplicity of the new approach is that the proof of (4.4) is applicable with little change for an integrable almost complex structure which leads to the existence of analytic functions defining the complex structure.

During the academic year 1962-1963 I gave a course in Stockholm on functions of several complex variables, mainly following the Tata Institute notes of Malgrange and the Cartan seminars 1951-1952. When I went to Stanford in April 1963, where I had been appointed to a part time professorship and was going to lecture on my book [28], I had therefore improved my knowledge of complex analysis in several variables a great deal, but felt that I had to understand the $\bar\partial$-Neumann technique better. The first observation I made at Stanford was that the difficulty which had caused Morrey and Kohn problems could be handled essentially by the old Friedrichs lemma, using only regularization along a flattened piece of the boundary. (See Appendix A.) When I told Peter Lax, who was also at Stanford then, he pointed out that he and Ralph Phillips had written an article [46] where they did just that in another context, so it was not so much a new discovery as an understanding that Morrey and Kohn had missed the simplicity of the problem. Kohn's estimates together with this density theorem suffice to prove the existence of L^2 solutions of the $\bar\partial$ equation, and standard elliptic theory then gives interior regularity. This suffices to complete the proof of Morrey's embedding theorem, for example.

My experience with Carleman estimates in [28, Chapter 8] made it natural to examine how the L^2 estimates would be affected by using L^2 norms with respect to weight functions. If in (4.6) one replaces the Lebesgue measure $d\lambda(z)$ by $e^{-\varphi}\, d\lambda(z)$ where $\varphi \in C^2(\overline{\Omega})$, keeping in mind that the adjoint of $\bar\partial$ with respect to the new L^2 norm becomes $e^{\varphi}\partial e^{-\varphi}$, one obtains

$$\int_\Omega \left(|\bar\partial f|^2 + |e^{\varphi}\partial e^{-\varphi} f|^2\right) e^{-\varphi}\, d\lambda(z) = \sum_{j=1}^{n} \sum_{|J|=q}' \int_\Omega |\partial f_J/\partial \bar z_j|^2 e^{-\varphi}\, d\lambda(z)$$
$$+ \sum_{|K|=q-1}' \int_\Omega \sum_{j,l=1}^{n} \partial^2\varphi/\partial z_j \partial \bar z_l f_{jK} \overline{f_{lK}} e^{-\varphi}\, d\lambda(z)$$
$$+ \sum_{|K|=q-1}' \int_{\partial\Omega} \sum_{j,l=1}^{n} \partial^2 \varrho/\partial z_j \partial \bar z_l f_{jK} \overline{f_{lK}} e^{-\varphi}\, dS\,, \qquad (4.6)'$$

when f is smooth in $\overline{\Omega}$ and satisfies (4.5). When φ is plurisubharmonic and $\partial\Omega$ is pseudoconvex, both terms on the right-hand side are non-negative. Using the fact that pseudoconvex domains are increasing limits of strictly pseudoconvex domains with smooth boundary, and that plurisubharmonic functions are decreasing limits of C^∞ plurisubharmonic functions, it is easy to deduce the following existence theorem [26, Theorem 2.2.1′]: If $\Omega \subset \mathbf{C}^n$ is pseudoconvex and φ is a plurisubharmonic function in Ω such that

$$\sum_{j,k=1}^{n} \partial^2\varphi(z)/\partial z_j \partial \bar z_k t_j \bar t_k \ge c(z) \sum_{1}^{n} |t_j|^2, \quad t \in \mathbf{C}^n\,,$$

in the sense of distribution theory for some positive continuous function $c(z)$ in Ω, and if f is a $(0, q)$ form in $L^2_{\mathrm{loc}}(\Omega)$ such that $\bar\partial f = 0$, then the equation $\bar\partial u = f$ has a solution $u \in L^2_{\mathrm{loc}}(\Omega)$ with

$$q \int_\Omega |u(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \le \int_\Omega |f(z)|^2 e^{-\varphi(z)} c(z)^{-1}\, d\lambda(z) \qquad (4.8)$$

if the right-hand side is finite. If φ is only assumed to be plurisubharmonic one can apply (4.8) with $\varphi(z)$ replaced by $\varphi(z) + 2\log(1 + |z|^2)$ and obtain a solution u with

$$2q \int_\Omega |u(z)|^2 e^{-\varphi(z)} \left(1 + |z|^2\right)^{-2} d\lambda(z) \le \int_\Omega |f(z)|^2 e^{-\varphi(z)}\, d\lambda(z)\,, \qquad (4.9)$$

if the right-hand side is finite. The uniformity of the estimate in both Ω and φ, without any regularity hypotheses, makes this a very flexible and useful tool.

At first sight the identity (4.6) suggests that the estimate (4.4) should be valid precisely when $\partial\Omega$ is strictly pseudoconvex. However, that is not the case. When $\partial\Omega$ is pseudoconcave in some directions it turns out that the boundary term in (4.6) can be improved by partial integration in the other term on the right-hand side. In [26, Section 3.2] I proved that (4.4) is valid for smooth $(0, q)$ forms satisfying (4.5) if and only if *at every boundary point the Levi form has either at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues.* Using also corresponding estimates involving weight functions it was proved in [26, Chapter 3] that the $\bar\partial$ cohomology is finite-dimensional in every open complex manifold Ω where there exists a function $\varphi \in C^2(\Omega)$ such that $\Omega_c = \{z \in \Omega;\ \varphi(z) < c\}$ is relatively compact, $d\varphi \ne 0$ on $\partial\Omega_c$ when c is large, and the Levi form of $\partial\Omega_c$ has at least $q + 1$ negative or at least $n - q$ positive eigenvalues then. This had already been proved by Andreotti and Grauert [2] using sheaf theory, but it was interesting to see that the approach using L^2 estimates led to the same conditions.

In [26] there was no discussion of the case where the Levi form is non-degenerate but has the excluded signature $(n - q - 1, q)$. Using (v) in Appendix B one can show that if there exists such a point on $\partial\Omega$ then the kernel of the $\bar\partial$-Neumann operator in Ω has infinite dimension if the range is closed. (See [32] and the references there.)

The reference [26] also contains a result on the asymptotic behavior of the Bergman kernel function at a C^2 strictly pseudoconvex boundary point. It has a special history. Stefan Bergman, for whom the kernel function is named, had been at Stanford for many years. He was a rather special person and had a reputation for cornering people to talk interminably about the kernel function for which his enthusiasm was unbounded. For quite a while I managed to avoid him, but at last I was cornered. What he wanted to talk to me about particularly was his article [5]. First he spoke at great length about how it had been rejected by Torsten Carleman when he had submitted it to *Acta Mathematica,* which still distressed him after more than 30 years. In order to convince me how wrong Carleman had been he then started to talk about the results in [5] on the boundary behavior of the kernel function for open sets in $\mathbf{C}^2$. They depended on approximation from the inside and outside with either balls or bidiscs, after an appropriate analytic change of variables. An obvious weakness was that he needed to have suitable new analytic coordinates defined over the whole set, and it is seldom possible to decide if such coordinates exist. However, at every C^2 strictly pseudoconvex boundary point one can choose local complex coordinates such that the boundary agrees with a ball up to higher order terms. When I walked home after being released by Bergman I realised that the new L^2 estimates were precisely what one needs to justify Bergman's asymptotic formula, extended to n complex variables, at arbitrary strictly pseudoconvex boundary points of a set for which the maximal $\bar\partial$ operator acting on scalars has a closed range, hence in particular for all pseudoconvex sets in $\mathbf{C}^n$. This was written down as Section 3.5 in my Acta article [26]. (A few years later Diederich [13] proved that the localization which is the main point in the proof could also have been based on the earlier sheaf theoretic methods.) Much more refined results have been obtained later on, and we shall come back to them in Section 5, but my result was the first general theorem of its kind as far as I know.

There was much activity concerning the $\bar\partial$ operator in 1963. On October 7, 1963 Morrey wrote to me in Stockholm:

Dear Professor Hormander:

As you know, I have been interested in the $\bar\partial$-Neumann problem. My interest arose from the fact that my proof of the analytic embedding theorem used my incorrect (in some respects) solution of that problem for forms of types (0, 0) and (0, 1). So when I heard at the Congress that Kohn still didn't have a proof of the results he had announced earlier, I spent a couple of months last fall and proved those results. I wrote up an article on this problem which I sent to Kohn with the idea that it would be a joint article since I used a great many of his ideas. But he returned it, stating that he had finally proved the smoothness results on which all of his results were based and stating that he would send me the material as soon as it was ready.

However, since I wanted my analytic embedding results cleared up, I went ahead and scheduled a talk at our colloquium on January 3. I received a reprint of Kohn's Proceedings note sketching his proof of the smoothness results just five days before I talked, which was not long enough for me to reconstruct the proof. However, I subsequently simplified his work considerably and wrote up a nice treatment of the whole problem for inclusion in my forthcoming Springer book.

I talked about this work at the Novosibirsk conference and have sent in an abstract to speak about it at the forthcoming Bombay conference. Now I have heard from several sources that you have also written up a treatment of this problem. I was too busy last spring trying to finish the Springer book and writing my part of a joint text with Protter to go to any colloquia here or at Stanford, and so I didn't know about your work. So, I would like to have a manuscript of your work to refer to in the main article for the Bombay talk and in the part of my book concerned with this problem. Of course, if you yourself are intending to speak on this problem at Bombay (I've heard that you're going), please let me know because I could change my talk to be about the problem of Plateau for higher-dimensional manifolds.

Please let me hear from you soon.

Sincerely yours,

Charles B. Morrey, Jr.

I answered Morrey on October 10:

Dear Professor Morrey:

Thank you for your letter. It was indeed a pity that we never got to talk about the $\bar\partial$-Neumann problem last summer. Of course I was aware of your old interest in this question since the main idea in the proof of the estimates is yours, but I did not know that you were working on it now. My own plan was at first only to get familiar with Kohn's work, but then it turned out that extensions were not hard to obtain by introducing methods from the theory of one partial differential equation. Let me describe the various results:

1) Let Ω be a smoothly bounded relatively compact subset of a complex manifold with a hermitian metric, let f be of type (p, q), let $\bar\partial$ and ϑ be defined as usual. Then in order that

$$\int_{\partial\Omega} |f|^2\, ds \le C \int_\Omega \left(|\bar\partial f|^2 + |\vartheta f|^2 + |f|^2\right) dx, \quad f \in \mathcal{D}_{\bar\partial} \cap \mathcal{D}_\vartheta ,$$

it is necessary and sufficient that at every point on $\partial\Omega$ the Levi form has either at least $n - q$ positive eigenvalues or at least $q + 1$ negative eigenvalues (n =dimension of the manifold). This leads to the results of Andreotti and Grauert, Bull. Soc. Math. France 90, 1962, for the sheaf $\mathcal{O}$.

2) In proving 1) I bypass the question of regularity on the boundary. It is enough to prove density of smooth forms in the space of forms considered, and this can be done by Friedrichs mollifiers. (Essentially, this result exists in an article of Lax and Phillips in Communications on Pure and Appl. Math.)

3) It is very useful to introduce weight functions in the estimates, in the same way as in the uniqueness proofs for the Cauchy problem in the case of one equation or a determined system. Thus, with a fixed function φ, I replace the density dx by $e^{-\tau\varphi}\, dx$ and consider the dependence of the estimates on τ. In doing so I change the definition of the adjoint operator ϑ to be the adjoint with respect to the new metrics. For example, for domains in C^n, estimates are then obtained for arbitrary domains with pseudoconvex boundaries if φ is strictly plurisubharmonic. It is not necessary to assume that Ω is bounded so one obtains global results which for example can be used in an essential part of the theory of Ehrenpreis–Malgrange–Palamodov of general overdetermined systems of constant coefficient partial differential operators. However, the main application is that by choosing φ large at the boundary of Ω one obtains in this way a direct and simple proof of the "Runge" approximation theorem. The manuscript is only written in part yet. I hope to have it completed in December and plan to use it in Bombay. However, I see no reason why we could not both talk about these questions there, for the overlap does not seem to be large, at least from what I understand by looking at your Novosibirsk conference.

I would of course be very interested if you could send me more details of your results. When my own article is written up, I will send you a copy right away.

Sincerely yours, /Lars Hörmander/

This letter is a fairly accurate summary of the article [25] I wrote for the proceedings of the Bombay colloquium on *Differential Analysis* held January 7-14, 1964, which also contains Morrey's article [48]. The complete manuscript of [26] was also ready just in time for that meeting. Kohn and Spencer were in Bombay too and lectured optimistically about more general overdetermined system, but that is still a quite unfinished story which cannot be discussed here.

Shortly before the Bombay meeting, on December 9, Louis Nirenberg wrote me about progress concerning the full boundary regularity:

There is very little mathematical news with me. In connection with my seminar I did succeed in working out the simplified proof of differentiability at the boundary for the $\bar\partial$-Neumann problem using just the *a priori* estimates of Kohn. In fact Kohn then indicated to me a shorter and better derivation of them which doesn't even need the exponential factor, and so the whole thing is rather clean now. We will write a short joint note on the simplified regularity proof. Let me describe it to you even though you don't need it in your approach,—which I look forward to seeing in detail.

The letter continued with an outline of the proof. The main point is to add a term $\varepsilon D(f, g)$ to the left-hand side of (4.7) where $D(f, f)$ is positive definite in all the first order derivatives of f. This gives a standard elliptic problem when $\varepsilon > 0$ and one gets a solution f_ε which is smooth up to the boundary. An adaptation of the proof of Kohn's estimates gives estimates for the derivatives which are bounded as $\varepsilon \to 0$, of course just for one more L^2 derivative of f than assumed for a. The details appeared later in [42], and this meant that the $\bar\partial$-Neumann problem was well understood at the time of the Bombay meeting, a year and a half after the confusion at the CNRS meeting in 1962. The work on the boundary regularity motivated Kohn and Nirenberg to develop the first generation of pseudo-differential operators published almost simultaneously in [43].

There is an essential difference between the bibliographies in [25] and in [26] which suggests that when the manuscript for the Bombay meeting was delivered I had not yet heard about an article by Andreotti and Vesentini [3] which appeared in 1965, just as [26]. Motivated by my first article on the uniqueness of the Cauchy problem they also used Carleman estimates to prove the results of [2]. An interesting feature of the proof is that they start from a complete hermitian metric, which makes the problems related to weak and strong extensions of differential operators disappear at the expense of error terms which are handled using the weight functions. Perhaps the overlap between [26] and [3] was a reason why the introduction of [26] concluded with the following modest statement:

"Apart from the results involving precise bounds, this article does not give any new existence theorems for functions of several complex variables. However, we believe that it is justified by the methods of proof."

5. Some applications

During the Spring and Summer quarters 1964 I lectured at Stanford on the theory of functions of several complex variables. The main aim was to show that the new L^2 techniques could give an attractive alternative approach to the theory of coherent analytic sheaves on a Stein manifold, and supplement it by useful estimates. A somewhat expanded version of the lecture notes was published as [27].

Ehrenpreis' fundamental principle, first announced (somewhat incorrectly) in 1960, states that if $P(D)$ is an arbitrary $J \times K$ matrix of constant coefficient differential operators in $\mathbf{R}^N$, and Ω is a convex open set in $\mathbf{R}^N$, then the system of differential equations $P(D)u = f$ has a (smooth) solution $u = (u_1, \dots, u_K)$ in Ω for every (smooth) $f = (f_1, \dots, f_J)$ in Ω such that $Q(D)f = 0$ for all $1 \times J$ systems $Q(D)$ with $Q(D)P(D) = 0$. (By Hilbert's basis theorem these are finitely generated so only finitely many compatibility conditions are required.) Furthermore, every solution of the homogeneous equation $P(D)u = 0$ can be decomposed as an absolutely convergent integral of exponential solutions. By duality and Laplace transformation one finds that the existence theorem will follow if one proves that given an entire function V in $\mathbf{C}^N$ with values in $\mathbf{C}^J$ such that

$$\left|{}^t P(\zeta)V(\zeta)\right| e^{-H_K(\operatorname{Im}\zeta)}(1+|\zeta|)^{-\nu} \le 1, \quad \zeta \in \mathbf{C}^N,$$

there exist another such function W with ${}^tP(\zeta)V(\zeta) = {}^tP(\zeta)W(\zeta)$ such that

$$|W(\zeta)|e^{-H_K(\operatorname{Im}\zeta)}(1+|\zeta|)^{-\nu'} \le C, \quad \zeta \in \mathbf{C}^N .$$

Here $\nu > 0$ and H_K is the supporting function of some compact set $K \subset \Omega$; the constants $\nu' \ge \nu$ and C are independent of V. Since $H_K(\operatorname{Im}\zeta) + \nu \log(1+|\zeta|)$ is plurisubharmonic this can be proved using the results on "$\bar\partial$ cohomology with bounds" which follow from (4.9), after a local construction of W has been made. (See [27, Section 7.6].) I tried to persuade Ehrenpreis to use this simple and flexible approach in his book [15], but he insisted that he had to use his original idea based on successively piecing together local solutions while keeping track of estimates. This is not easy to do. A sketch was given in Chapter IV of [15], but to give full details would undoubtedly be a formidable task. Palamodov's book [50], published the same year, did not use the L^2 existence theory either but followed an approach of Malgrange which involves the study of a real division problem which is in fact much harder. His motivation (see [50, p. 423]) was that the logarithm of the weight function $e^{-H_K(\operatorname{Im}\zeta)}(1+|\zeta|)^\nu$ which must be considered in the case of distribution solutions is not plurisuperharmonic if $\nu > 0$. This is of course true, but the strong pseudoconvexity of $H_K(\operatorname{Im}\zeta)$ near $\mathbf{R}^n$ can be smeared out so that one can find plurisuperharmonic functions which work just as well. This was discussed in great detail in [29, Chapter XV]. A proof of the existence theorem for smooth solutions was given in the last chapter of [27], but the decomposition theorem for solutions of the homogeneous equation was not included until the third edition since it requires additional preparation on local analytic function theory.

Approximation of functions on totally real submanifolds was the starting point for Morrey's article [47]. Questions from the theory of function algebras brought up by John Wermer led to our joint article [34]. Let Σ be a C^1 totally real submanifold of $\mathbf{C}^n$, which means that at every point in Σ the tangent plane T intersects iT only at the origin. Then it follows from Whitney's extension theorem (cf. [29, Theorem 2.3.6]) that there exists a function $\varphi \in C^2(\Omega)$ such that φ vanishes to second order on Σ and the second order term in the Taylor expansion at every point on Σ is the square of the Euclidean distance to the tangent plane. This implies that φ is strictly plurisubharmonic at Σ, so it follows that every compact set $K \subset \Sigma$ has a fundamental system of neighborhoods with strictly pseudoconvex boundary. When $\Sigma \in C^r$ and $2r \ge 2 + \dim\Sigma$ the argument outlined above in connection with Morrey's result proves that, on a compact subset K of Σ, functions analytic in a complex neighborhood are dense in the maximum norm. Such a high regularity assumption was needed in [34] to pass from L^2 estimates to estimates in the maximum norm, but it was soon reduced by various authors using integral formulas and/or hyperfunction theory. However, there is a better way. The classical proof of Weierstrass' approximation theorem, by convolution of a function with compact support in $\mathbf{R}^n$ with a Gaussian $e^{-\langle\zeta,\zeta\rangle/\varepsilon}(\varepsilon\pi)^{-n/2}$, gives at once when $\varepsilon \to 0$ that if $u \in C(\Sigma)$ has very small support, then one can find u_ε analytic in a complex neighborhood ω and converging to u uniformly in $\Sigma \cap \omega$ and to 0 uniformly in a complex neighborhood of a compact set $\subset (\Sigma \cap \omega) \setminus \operatorname{supp} u$. Multiplication by a suitable cutoff function which is equal to 1 near $\operatorname{supp} u$ gives v_ε converging to u uniformly on Σ such that $\bar\partial v_\varepsilon \to 0$ in C^∞ in a neighborhood of Σ. To approximate an arbitrary continuous function $u \in C(K)$ one first decomposes u by a partition of unity into a sum for which the preceding Weierstrass argument works and adds the results. This gives v_ε converging uniformly to u in a neighborhood of K on Σ such that $\bar\partial v_\varepsilon \to 0$ in C^∞ in a complex neighborhood of K. Using (4.9) we can then find w_ε with $\bar\partial w_\varepsilon = \bar\partial v_\varepsilon$ in a "tubular" neighborhood Ω of K converging to 0 on compact subsets, and $v_\varepsilon - w_\varepsilon$ then gives the required approximation. If $\Sigma \in C^r$ then the approximation works in C^{r-1} too. (Precise existence theorems for $\bar\partial$ in L^2 are no longer essential for this argument.)

That *solutions of convolution equations in convex domains can be approximated by sums of exponential solutions* was proved in [30] by means of (4.9). The analytical problem encountered there is the following: Let f and g be distributions with compact support in $\mathbf{R}^n$ such that the quotient $\hat{f}(\zeta)/\hat{g}(\zeta) = \psi(\zeta)$ of the Fourier–Laplace transforms is also an entire function. If H_f and H_g are the supporting functions of supp f and of supp g then $H = H_f - H_g$ is the supporting function of a convex compact set $K \subset \mathbf{R}^n$, and

$$|\psi(\zeta)| \leq C_\varepsilon e^{H(\operatorname{Im}\zeta)} e^{\varepsilon|\zeta|}, \quad \zeta \in \mathbf{C}^n ,$$

for every $\varepsilon > 0$. If ψ does not grow faster than a polynomial in $\mathbf{R}^n$ then $\psi = \hat{h}$ where $h \in \mathcal{E}'$ and supp $h \subset K$, hence $g * h = f$. Then an approximation theorem for solutions of the convolution equation $g * u = 0$ follows from the Hahn–Banach theorem. However, ψ may grow faster than any polynomial in $\mathbf{R}^n$ and the problem is to prove that in any case f is a limit in $\mathcal{E}'$ of convolutions $g * h$ where $h \in \mathcal{E}'$ has support close to K. This suffices for the application of the Hahn–Banach theorem. To approximate ψ by functions which are Fourier–Laplace transforms of distributions with support close to K we first define $\tilde{\psi}_\delta(\zeta) = e^{-\delta\langle\zeta,\zeta\rangle}\psi(\zeta)$ when $|\operatorname{Re}\zeta| > |\operatorname{Im}\zeta| + 1$ and $\tilde{\psi}_\delta(\zeta) = \psi(\zeta)$ when $|\operatorname{Im}\zeta| > |\operatorname{Re}\zeta|$ and make a smooth transition by a partition of unity so that

$$\left|\bar\partial\tilde{\psi}_\delta(\zeta)\right| \leq C'_\varepsilon\delta|\zeta|^2 e^{H(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|}, \quad \left|\hat{g}(\zeta)\tilde{\psi}_\delta(\zeta) - \hat{f}(\zeta)\right| \leq |\hat{f}(\zeta)|\delta|\zeta|^2 .$$

Passing to L^2 norms, using (4.9), and returning to pointwise estimates we conclude that we can find $u_\delta(\zeta)$ with $\bar\partial u_\delta(\zeta) = \bar\partial\tilde{\psi}_\delta(\zeta)$ and

$$|u_\delta(\zeta)| \leq C''_\varepsilon\delta(1 + |\zeta|)^N e^{H(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|}$$

for some N depending only on the dimension. Then $\psi_\delta(\zeta) = \tilde{\psi}_\delta(\zeta) - u_\delta(\zeta)$ is entire and

$$\left|\hat{g}(\zeta)\psi_\delta(\zeta) - \hat{f}(\zeta)\right| \leq \delta\left(|\hat{f}(\zeta)||\zeta|^2 + C'''_\varepsilon e^{H_f(\operatorname{Im}\zeta)+\varepsilon|\operatorname{Im}\zeta|}(1 + |\zeta|)^N\right) .$$

Thus ψ_δ is the Fourier–Laplace transform of a distribution h_δ with support in an ε-neighborhood of K, and $g * h_\delta \to f$ in $\mathcal{E}'$ when $\delta \to 0$.

A construction of analytic functions with prescribed zeros using (4.9) was first achieved by Bombieri [6] in solving a problem from algebraic number theory. The point is that, given a plurisubharmonic function φ in a pseudoconvex open set $\Omega \subset \mathbf{C}^n$, one can by repeated use of (4.9) construct analytic functions f in Ω with

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)}\left(1 + |z|^2\right)^{-3n} d\lambda(z) < \infty \tag{5.1}$$

and $f(z_0) \neq 0$ for a given point z_0 such that $e^{-\varphi}$ is integrable in some neighborhood. Such points always exist. On the other hand, (5.1) implies that $f(z) = 0$ if $e^{-\varphi}$ is not integrable in any neighborhood of z. (In the first edition of [27] I had tried to avoid the technical difficulties dealt with in Appendix A by a device using weight functions converging to 0 with different speeds at the boundary, perhaps motivated by [3]. This did not affect the conclusions on the $\bar\partial$ operator acting on C^∞ forms. However, the full force of (4.9) was only given for $\Omega = \mathbf{C}^n$ where there is no boundary. Using a minor additional technical device the full result was restored in the second edition of [27], where the result of Bombieri was included.) The idea was further developed by Skoda [53] in a study of analytic submanifolds of $\mathbf{C}^n$; he also noted that the passage from (4.8) to (4.9) can be made so that one does not lose a factor $(1+|z|^2)^2$ in the estimates but only $(1+|z|^2)$ times a power of the logarithm. Finally, Bombieri's idea is indispensable for the proof of Siu's theorem on the Lelong numbers of plurisubharmonic functions, or more generally closed positive currents. For these matters and references see e.g., [31].

Approximation of plurisubharmonic functions. Every subharmonic function p in an open set $\Omega \subset \mathbf{C}$ is a limit of subharmonic functions of the form $N^{-1}\log|f(z)|$ where f is analytic in Ω

and N is a positive integer. This can be proved by approximating the positive measure $(N/2\pi)\Delta p$ with a discrete measure having integer point masses and constructing suitable analytic functions with the corresponding zeros and multiplicities. If instead p is plurisubharmonic in an open set $\Omega \subset \mathbf{C}^n$ with $n \geq 2$ such an argument is impossible since the zeros are surfaces and not discrete. However, after a preliminary approximation showing that one may assume that p is strictly pseudoconvex, the same conclusion can be obtained as follows. First one chooses a dense sequence $z_1, z_2, \dots \in \Omega$ and for fixed ν an analytic function g_N in a neighborhood ω_ν of $z_1, \dots, z_\nu$ such that $|g_N(z)|$ is equal to $\exp(Np(z))$ when $z = z_1, \dots, z_\nu$ but is smaller by a factor $\exp(-Nc_\nu d(z)^2)$ in ω_ν where $c_\nu > 0$ and $d(z) = \min_{j \leq \nu} |z - z_j|$. After multiplication of g_N by a suitable cutoff function one can then use the existence theorems for the $\bar\partial$ operator with weight function e^{-2Np} to change g_N to an analytic function f_N with $f_N - g_N$ much smaller than e^{Np} for large N. When ν and N tend to infinity in a suitable way then $N^{-1}\log|f_N| \to p$. (See e.g., [31, Theorem 4.2.13].)

A much deeper result for entire functions of order $\varrho > 0$ was proved with similar methods by Sigurdsson [52, Theorem 1.3.1]: If p is a plurisubharmonic function in $\mathbf{C}^n$ of order ϱ and finite type, then there exists an entire function f in $\mathbf{C}^n$ such that

$$t^{-\varrho}(p(tz) - \log|f(tz)|) \to 0 \quad \text{in } L^1_{\text{loc}}\left(\mathbf{C}^n\right) \text{ when } t \to +\infty\,.$$

This leads to very complete results on the possible behavior at infinity for entire functions of order ϱ. However, in the particularly interesting case $\varrho = 1$ when one imposes boundedness conditions in $\mathbf{R}^n$ the results are still incomplete (see [33]).

The Bergman kernel for open sets $\Omega \subset \mathbf{C}^n$ with strictly pseudoconvex boundary is now well understood. If K is the Bergman kernel and $f \in C_0^\infty(\Omega)$ then the projection $f - Kf$ on the orthogonal space of the holomorphic functions is equal to $\vartheta_c(\vartheta_c\bar\partial + \bar\partial\vartheta_c)^{-1}\bar\partial f$, hence

$$Kf = f - \vartheta_c\left(\vartheta_c\bar\partial + \bar\partial\vartheta_c\right)^{-1}\bar\partial f$$

which is the analogue of the classical formula (3.3). By Kohn's theorem on regularity at the boundary, the right-hand side is in $C^\infty(\overline{\Omega})$ if $\partial\Omega \in C^\infty$. If we take f depending only on the distance to a point $w \in \Omega$ and with integral one, this means that $z \mapsto K(z, w)$ is smooth up to the boundary. Hence $K(z, w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \partial\Omega \times \partial\Omega)$. As proved by Kerzman [36], one can replace $\partial\Omega \times \partial\Omega$ by the diagonal. A complete asymptotic expansion of $K(z, z)$ at the boundary was given by Fefferman [16]: If $-\varrho$ is a defining function of Ω then there are functions $a, b \in C^\infty(\overline{\Omega})$ such that $K(z, z) = a(z)/\varrho(z)^{n+1} + b(z)\log\varrho(z)$. Here $a(z)$ is given for $z \in \partial\Omega$ by [26, Theorem 3.5.1]. Kashiwara [35] proved that when the boundary is real analytic then this is true with a and b real analytic close to the boundary. More complicated asymptotics of $K(z, w)$ when z and w approach the same boundary point in an arbitrary way were obtained by Boutet de Monvel and Sjöstrand [7], but discussing their work would take us too far afield here. It seems clear that the methods used in these articles should suffice to determine the asymptotics for the reproducing kernels of "harmonic" (p, q) forms in a strictly pseudoconvex domain, as intended in [20, 21], but as far as I know this has not yet been done.

We can also get an analogue of (3.4), for the self-adjoint operator $(\vartheta_c\bar\partial + \bar\partial\vartheta_c)^{-1}$ is a right inverse of $-\frac{1}{4}\Delta = \vartheta\bar\partial + \bar\partial\vartheta = -\sum_1^n \partial^2/\partial z_j\partial\bar z_j$, so the kernel can be written

$$-E(z - w)\delta_{jk} + R_{jk}(z, \bar w)\,, \quad \text{where } E(z - w) = -(n-2)!\pi^{-n}|z - w|^{2-2n}$$

is the classical fundamental solution of $\frac{1}{4}\Delta$, and $R_{jk} \in C^\infty(\Omega \times \Omega \setminus \partial\Omega \times \partial\Omega)$, $\Delta_z R_{jk}(z, \bar w) =$

$\Delta_w R_{jk}(z, \bar{w}) = 0$. We have the boundary condition

$$\sum_{j=1}^{n} \partial\varrho(z)/\partial z_j \left(R_{jk}(z, \bar{w}) - E(z-w)\delta_{jk}\right) = 0, \quad z \in \partial\Omega,\ w \in \Omega,\ k = 1, \dots, n\,, \tag{5.2}$$

and a similar condition for $\sum_{j=1}^{n} \bar{\partial}_z R_{jk}(z, \bar{w}) d\bar{z}_j$. Hence

$$\mathfrak{d}_c \left(\mathfrak{d}_c\bar{\partial} + \bar{\partial}\mathfrak{d}_c\right)^{-1} \bar{\partial} f = f - \sum_{j,k} \int \partial R_{jk}(z, \bar{w})/\partial z_j \partial f(w)/\partial \bar{w}_k \, d\lambda(w)\,,$$

which means that we have an analogue of (3.4),

$$K(z, w) = -\sum_{j,k=1}^{n} \partial^2 R_{jk}(z, \bar{w})/\partial z_j \partial \bar{w}_k\,. \tag{5.3}$$

With $G_j(z) = -\sum_k \partial R_{jk}(z, \bar{w})/\partial \bar{w}_k$, the formulas (3.5), (3.6) of Garabedian are justified for strictly pseudoconvex domains by (5.3) and application of $\partial/\partial\bar{w}_k$ to (5.2). For the form $u = (\mathfrak{d}_c\bar{\partial} + \bar{\partial}\mathfrak{d}_c)^{-1}\bar{\partial} f$, $f \in C_0^\infty(\Omega)$, we have $\mathfrak{d}_c\bar{\partial}u + \bar{\partial}(\mathfrak{d}_c u - f) = 0$, and since the two terms are orthogonal it follows that $\mathfrak{d}_c\bar{\partial}u = 0$. Hence $(\mathfrak{d}_c\bar{\partial}u, u) = \|\bar{\partial}u\|^2 = 0$, that is,

$$\bar{\partial}\left(\partial \int E(z-w) f(w)\, d\lambda(w) - \sum_{j,k} \int R_{jk}(z, \bar{w})\, \partial f(w)/\partial\bar{w}_k \, d\lambda(w)\, d\bar{z}_j\right) = 0\,,$$

which means that $\sum_{j,k} \int \partial R_{jk}(z, \bar{w})/\partial\bar{w}_k f(w)\, d\lambda(w)\, d\bar{z}_j$ is $\bar{\partial}$ closed. Since $f \in C_0^\infty$ is arbitrary this implies that $\bar{\partial} \sum_1^n G_j(z)\, d\bar{z}_j = 0$. For strictly pseudoconvex domains it follows that there exists a smooth function ϑ such that $G_j(z) = \partial\vartheta(z)/\partial\bar{z}_j$, as claimed by Garabedian in general.

Appendix A

In this appendix we shall review some basic facts concerning weak and strong definitions of maximal and minimal differential operators. (See also [26, Section 1.2].) Let $\Omega \subset \mathbf{R}^N$ be an open set, and let $L(x, D) = \sum_1^N L_j(x) D_j + L_0(x)$ be a first-order differential operator where $L_j \in C^1(\Omega, \mathbf{C}^{K\times J})$, $j = 1, \dots, N$, $L_0 \in C^0(\Omega, \mathbf{C}^{K\times J})$. Here $\mathbf{C}^{K\times J}$ denotes the space of $K \times J$ matrices with complex coefficients. (The extension to manifolds and vector bundles poses no difficulties, and this setup will simplify the notation.) The graph of the *maximal differential operator* defined by $L(x, D)$ in L^2 consists of all pairs $(u, f) \in L^2(\Omega, \mathbf{C}^J) \times L^2(\Omega, \mathbf{C}^K)$ such that $L(x, D)u = f$ in the sense of distribution theory. This means, that with L^2 scalar products,

$$\left(u, L^*(x, D)v\right) = (f, v), \quad v \in C_0^\infty\left(\Omega, \mathbf{C}^K\right), \quad \text{where}$$
$$L^*(x, D) = \sum_1^N L_j^*(x) D_j + L_0^* + \sum_1^N D_j L_j^*(x)$$

is the formal adjoint of $L(x, D)$. Thus the maximal operator defined by $L(x, D)$ is the adjoint of the formal adjoint $L^*(x, D)$ with domain $C_0^\infty(\Omega, \mathbf{C}^K)$, so the maximal operator is closed and its adjoint is the closure of $L^*(x, D)$, first defined with domain $C_0^\infty(\Omega, \mathbf{C}^K)$; it is called the *minimal operator* defined by $L^*(x, D)$. Similarly, the adjoint of the maximal operator defined by $L^*(x, D)$ is the minimal operator defined by $L(x, D)$, that is, the closure of $L(x, D)$ with domain $C_0^\infty(\Omega, \mathbf{C}^J)$. If $u \in L^2(\Omega, \mathbf{C}^J)$ is in the minimal domain of $L(x, D)$ and $L(x, D)u = f$, and if $\tilde{u} = u$, $\tilde{f} = f$ in Ω, $\tilde{u} = 0$, $\tilde{f} = 0$ in $\complement\Omega$, then $L(x, D)\tilde{u} = \tilde{f}$ in $\mathbf{R}^N$ if L_j are defined in a neighborhood of $\overline{\Omega}$ with the regularity properties assumed above in Ω. This is obvious if $u \in C_0^\infty(\Omega, \mathbf{C}^J)$ and follows in general since $L(x, D)$ is continuous from $L^2(\mathbf{R}^N)$ to $\mathcal{D}'(\mathbf{R}^N)$.

Proposition A.1. *If Ω is bounded and $\partial\Omega \in C^1$, $L_j \in C^1$ when $j \neq 0$ and $L_0 \in C^0$ in a neighborhood of $\overline{\Omega}$, then the maximal operator defined by $L(x, D)$ is the closure of its restriction to functions which are in C^∞ in a neighborhood of $\overline{\Omega}$. The minimal domain of $L(x, D)$ consists precisely of the functions $u \in L^2(\Omega, \mathbf{C}^J)$ such that $L(x, D)\tilde{u} \in L^2(\mathbf{R}^N, \mathbf{C}^K)$ if $\tilde{u} = u$ in Ω and $\tilde{u} = 0$ in $\complement\Omega$. Then u is said to have vanishing Cauchy data with respect to $L(x, D)$.*

Proof. Application of a partition of unity shows that when proving the statements it suffices to consider functions u with support close to a boundary point, with exterior normal ν say. Choose a function $\chi \in C_0^\infty(\mathbf{R}^N)$ with integral 1 and support in the half space where $\langle x, \nu\rangle > 0$, and set $\chi_\varepsilon(x) = \chi(x/\varepsilon)/\varepsilon^N$. Then

$$u_\varepsilon(x) = u * \chi_\varepsilon(x) = \int_\Omega u(y)\chi_\varepsilon(x-y)\,dx$$

is for small $\varepsilon > 0$ in C^∞ in a neighborhood of $\overline{\Omega}$ if $\langle x, \mu\rangle > 0$ for $x \in \operatorname{supp}\chi$ when μ is the exterior normal of $\partial\Omega$ at a point in the closure of $\operatorname{supp} u$. If $L(x, D)u = f$ then $u_\varepsilon \to u$ and $f_\varepsilon \to f$ in $L^2(\Omega)$ when $\varepsilon \to 0$, and

$$L(x, D)u_\varepsilon - f_\varepsilon = L(x, D)(u * \chi_\varepsilon) - (L(x, D)u) * \chi_\varepsilon$$

converges to 0 in $L^2(\Omega)$ by the classical Friedrichs' lemma. (See e.g., [26, Lemma 1.2.1].) This proves the first statement. To prove the second one we argue similarly. With $L(x, D)u = f$ in Ω and $\tilde{u} = u$, $\tilde{f} = f$ in Ω, $\tilde{u} = 0$, $\tilde{f} = 0$ in $\complement\Omega$ we have $L(x, D)\tilde{u} = \tilde{f}$ by hypothesis. If we set $\check{\chi}(x) = \chi(-x)$ and define $u_\varepsilon = \tilde{u} * \check{\chi}_\varepsilon$, $f_\varepsilon = \tilde{f} * \check{\chi}_\varepsilon$, then $u_\varepsilon \in C_0^\infty(\Omega)$ for small $\varepsilon > 0$, and $L(x, D)u_\varepsilon - f_\varepsilon \to 0$ in L^2 when $\varepsilon \to 0$, again by Friedrichs' lemma, so $u_\varepsilon \to u$ and $L(x, D)u_\varepsilon \to f$ in L^2 when $\varepsilon \to 0$, which proves the second statement. □

A technical difficulty in the study of the $\bar\partial$-Neumann problem is that one has to approximate at the same time within the domain of one minimal and one maximal differential operator. The following proposition covers the local arguments required after a change of variables which flattens the boundary $\partial\Omega$, which is possible if $\partial\Omega \in C^2$.

Thus we consider now in an open neighborhood U of $0 \in \mathbf{R}^N$ two differential operators

$$L(x, D) = \sum_1^N L_j(x)D_j + L_0(x), \quad M(x, D) = \sum_1^N M_j(x)D_j + M_0(x), \quad \text{where}$$
$$L_j \in C^1\left(U, \mathbf{C}^{K_0\times J}\right),\ M_j \in C^1\left(U, \mathbf{C}^{K_1\times J}\right),\ j = 1, \dots, N,$$

and $L_0 \in C^0(U, \mathbf{C}^{K_0\times J})$, $M_0 \in C^0(U, \mathbf{C}^{K_1\times J})$. Let $U_- = \{x \in U; x_N < 0\}$.

Proposition A.2. *Assume that* $\operatorname{Ker} L_N(x)$ *and* $\operatorname{Ker} L_N(x) \cap \operatorname{Ker} M_N(x)$ *have constant dimension when $x \in U$. If $u \in L^2(U_-, \mathbf{C}^J)$ is in the minimal domain of $L(x, D)$ and the maximal domain of $M(x, D)$ in U_-, and if* $\operatorname{supp} u$ *is sufficiently close to the origin, then there exists a sequence $u_\nu \in C_0^1(U)$ such that u_ν restricted to U_- is in the minimal domain of $L(x, D)$ and $u_\nu \to u$, $L(x, D)u_\nu \to L(x, D)u$, $M(x, D)u_\nu \to M(x, D)u$ in $L^2(U_-)$.*

Proof. Let $r_0 = J - \dim \operatorname{Ker} L_N(x)$ and $r = J - \dim(\operatorname{Ker} L_N(x) \cap \operatorname{Ker} M_N(x))$, when $x \in U$. Since $\operatorname{Ker} L_N(x) \cap \operatorname{Ker} M_N(x) \subset \operatorname{Ker} L_N(x) \subset \mathbf{C}^J$ we can choose a basis $e_{r+1}(x), \dots, e_J(x)$ for $\operatorname{Ker} L_N(x) \cap \operatorname{Ker} M_N(x)$, and extend first to a basis $e_{r_0+1}(x), \dots, e_J(x)$ for $\operatorname{Ker} L_N(x)$ and then a basis $e_1(x), \dots, e_J(x)$ for $\mathbf{C}^J$. Shrinking U if necessary we can choose e_j as a C^1 function of x, for if we first choose $e_1(0), \dots, e_J(0)$ we can define $e_j(x)$ by projecting $e_j(0)$ along the space spanned by the other vectors $e_i(0)$ on the appropriate space. Writing $u(x) = \sum_1^J u_\nu(x) e_\nu(x)$ we have

$$L(x, D)u(x) = \sum_{\nu=1}^{J} \sum_{j=1}^{N} L_j(x) e_\nu(x) D_j u_\nu(x) + \sum_{\nu=1}^{J} \Big(\sum_{j=1}^{N} L_j(x) D_j e_\nu(x) + L_0(x) e_\nu(x) \Big) u_\nu(x).$$

Here $L_N(x)e_\nu(x) = 0$ when $\nu > r_0$, and $L_N(x)e_1(x), \dots, L_N(x)e_{r_0}(x)$ are linearly independent. By left multiplication of L by an appropriate C^1 matrix we can attain that

$$L_N(x)e_\nu(x) = (\delta_{\nu k})_{k=1}^{K_0}, \ \nu \le r_0, \quad L_N(x)e_\nu(x) = 0, \ \nu > r_0 .$$

Since $M_N(x)e_\nu(x) = 0$ when $\nu > r$ and these vectors are linearly independent when $r_0 < \nu \le r$ we can similarly attain that

$$M_N(x)e_\nu(x) = (\delta_{\nu - r_0, k})_{k=1}^{K_1}, \ r_0 < \nu \le r, \quad M_N(x)e_\nu(x) = 0, \ \nu > r .$$

Finally we can choose a $K_1 \times K_0$ matrix $A(x)$ in C^1 such that we have $M_N(x)e_\nu(x) = A(x)L_N(x)\, e_\nu(x)$, $\nu = 1, \dots, r_0$, for this only defines the first r_0 columns in $A(x)$. Replacing $M(x, D)$ by $M(x, D) - A(x)L(x, D)$, we have attained that for the new operators, still denoted by $L(x, D)$ and $M(x, D)$, with the basis $\{e_\nu\}$ in $\mathbf{C}^J$, we have

$$L_N(x)_{kj} = 1 \ \text{when} \ 1 \le j = k \le r_0, \quad M_N(x)_{kj} = 1 \ \text{when} \ r_0 < j = k + r_0 \le r ,$$

while all other elements in $L_N(x)$ and in $M_N(x)$ vanish.

The proof of the proposition is now easy, for the only terms in $L(x, D)$ and $M(x, D)$ which contain derivatives with respect to x_N have constant coefficients, so they commute with convolution with respect to the variables $x' = (x_1, \dots, x_{N-1})$. Choosing $\chi \in C_0^\infty(\mathbf{R}^{n-1})$ with integral 1, we set

$$u_\varepsilon(x) = \int u\left(x' - \varepsilon y', x_N\right) \chi\left(y'\right) dy' .$$

Then u_ε is in the domain of the minimal operator defined by $L(x, D)$, u_ε and all its derivatives with respect to the x' variables are in $L^2(U_-)$, and $u_\varepsilon \to u$, $L(x, D)u_\varepsilon \to L(x, D)$ and $M(x, D)u_\varepsilon \to M(x, D)u$ when $\varepsilon \to 0$, by Friedrichs' lemma. Hence also the derivatives with respect to x_N of the first r components of u_ε are in L^2, so they are continuous functions in $\overline{U_-}$. The first r_0 vanish when $x_N = 0$ since u_ε is in the minimal domain of $L(x, D)$. Making a small translation of the first r_0 components of u_ε in the negative x_N direction we do not change u_ε or $L(x, D)u_\varepsilon$ very much, and then the support of these components becomes compact in U_-. We can make a final approximation of u_ε by convolving with $\tilde{\chi}(x/\delta)/\delta^N$ where $\tilde{\chi} \in C_0^\infty(\mathbf{R}^N)$ has integral 1 and $x_N > 0$ in supp $\tilde{\chi}$, and letting $\delta \to 0$. Friedrichs' lemma is not needed in this final step. □

Phrased for two first order operators L and M between two vector bundles over an open subset Ω of a manifold, with C^2 boundary having defining function ϱ, the hypothesis required to apply Proposition A.2 at every boundary point is that the kernel of the principal symbol of L at $(x, d\varrho(x))$ and its intersection with that of M have constant dimension in a neighborhood of $\partial\Omega$. This condition depends on the choice of defining function and it is natural to expect that it should be sufficient to assume that it is valid on $\partial\Omega$. Under a stronger regularity condition this is in fact true but we omit the proof, for the stronger hypotheses in Proposition A.2 are fulfilled by the $\bar{\partial}$-Neumann problem for any choice of defining function.

Appendix B

For the convenience of the reader we shall here recall some basic facts from functional analysis used in this article. Proofs or references can be found in [26, Section 1.1] for example.

(i) If H_1 and H_2 are Hilbert spaces and $T : H_1 \to H_2$ is a linear closed densely defined operator, then the adjoint $T^* : H_2 \to H_1$ is also closed and densely defined, and $(T^*)^* = T$.

(ii) The orthogonal complement of the kernel Ker T of T is the closure of the range $\mathcal{R}_{T^*}$ of T^*, thus the closure of $\mathcal{R}_T$ is the orthogonal complement of Ker T^*.

(iii) The range of T is closed if and only if the range of T^* is closed, and then

$$\|f\|_{H_1} \le C\|Tf\|_{H_2},\ f \in \mathcal{D}_T \cap \mathcal{R}_{T^*}; \quad \|g\|_{H_2} \le C\|T^*g\|_{H_1},\ g \in \mathcal{D}_{T^*} \cap \mathcal{R}_T\ ;$$

moreover, the range of the selfadjoint operator T^*T (resp. TT^*) is equal to that of T^* (resp. T).

(iv) If H_3 is another Hilbert space and $S : H_2 \to H_3$ is a linear closed densely defined operator with $ST = 0$, then $\mathcal{R}_T$ and $\mathcal{R}_S$ are both closed if and only if

$$\|g\|_{H_2}^2 \le C^2\left(\|T^*g\|_{H_1}^2 + \|Sg\|_{H_3}^2\right);\ g \in \mathcal{D}_{T^*} \cap \mathcal{D}_S,\ g \perp N\ ,$$
$$N = \operatorname{Ker} T^* \cap \operatorname{Ker} S = \operatorname{Ker}\left(TT^* + S^*S\right)\ .$$

Then $H_2 = \mathcal{R}_T \oplus \mathcal{R}_{S^*} \oplus N = \mathcal{R}_{TT^*} \oplus \mathcal{R}_{S^*S} \oplus N$ and the range of $TT^* + S^*S$ is closed. Conversely, the ranges of T and S are closed if the range of $TT^* + S^*S$ is closed.

(v) $\mathcal{R}_T$ and $\mathcal{R}_S$ are both closed and $N = \operatorname{Ker} T^* \cap \operatorname{Ker} S$ is finite-dimensional if and only if every bounded sequence $g_k \in \mathcal{D}_{T^*} \cap \mathcal{D}_S$ such that $T^*g_k \to 0$ in H_1 and $Sg_k \to 0$ in H_3 has a convergent subsequence. In particular this is true if

$$\left\{g \in \mathcal{D}_{T^*} \cap \mathcal{D}_S;\ \|T^*g\|_{H_1}^2 + \|Sg\|_{H_3}^2 + \|g\|_{H_2}^2 \le 1\right\} \quad \text{is precompact in } H_2\ .$$

The condition in (v) means that T and S have closed range and that Ker $S/\mathcal{R}_T$ is finite-dimensional so it is not affected if the norms in H_1, H_2, H_3 are replaced by equivalent norms. However, the operator T^* changes a great deal, and this is exploited in the estimates involving weight functions discussed in Section 4.

Appendix C

As mentioned in the introduction, I have found in my files a three page manuscript with the heading *$\bar\partial$-Neumann Problem* (History). I believe that it was written by D.C. Spencer and reproduce it here with the only change that the keys to the references are changed to fit the more extensive bibliography in this article.

"The problem was first formulated by D.C. Spencer in 1954, as a modification of an earlier problem considered by P.R. Garabedian and D.C. Spencer [20, 21]. The relationship between the $\bar\partial$-Neumann problem and the Garabedian–Spencer problem was described by Spencer in lectures at the Collège de France in January-February, 1955, entitled "Les opérateurs de Green et de Neumann sur les variétés ouvertes riemanniennes et hermitiennes" (copy in the library of the Institut Henri Poincaré). The relationship is described in the real case (with the exterior differential operator d replacing $\bar\partial$) by P.E. Conner in his thesis [9] where the $\bar\partial$-Neumann problem corresponds to the laplacian Δ_M and the Garabedian–Spencer problem to the laplacian Δ_N.

The solution of the $\bar\partial$-Neumann [problem] on strongly pseudo-convex manifolds with smooth boundary was first obtained by J.J. Kohn [38] who, as a by-product, obtained new proofs of the theorem of Newlander–Nirenberg [49] and of the Levi problem. Kohn's solution used, in particular, a basic estimate which, in the case of (0, 1) forms on strongly pseudo-convex manifolds, was first given by C.B. Morrey [47] (to whom Spencer had communicated the $\bar{\partial}$-Neumann problem in 1956).

Later, L. Hörmander [26, 27] introduced Carleman type estimates in the context of this problem and was able to generalize many of the above results and simplify their proofs. However, his methods do not yield the boundary regularity.

A. Andreotti and E. Vesentini [3] also used Carleman-type estimates, but on a complete manifold; their results are parallel to those of Hörmander.

Kohn and H. Rossi [44] treated the $\bar{\partial}$-operator restricted to the boundary of a domain (or to submanifolds of higher codimension), the so-called $\bar{\partial}_b$-problem. The abstract $\bar{\partial}_b$-Neumann problem was solved by Kohn [41]."

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